Oberseminar WiSe 2023: GZ Formula via Borcherds' lifts

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1 Plan of the seminar

We prove the Gross-Zagier formula following [6].

1. Overview over the classical Gross-Zagier formula [10][8] (22.11.23, Gabriele Bogo):

   Introduce the modular curve $X_0(N)$ and its Heegner points. Introduce the Neron-Tate pairing and state the Gross-Zagier formula. See [10] for a survey and [8] for the original article.

2. Quadratic spaces, orthogonal groups, special divisors [5][2][3][7] (22.11.23, Sören Sprehe):

   Introduce the orthogonal upper half plane corresponding to the orthogonal group of a quadratic space V over $\mathbb{Q}$ of signature $(n, 2)$. Define the discriminant kernel attached to an even lattice $L \subseteq V$ of full rank. Define special divisors and modular forms.

3. Theta and Eisenstein series, Siegel-Weil formula [6, Section 2] (29.11.23, Simon Paege):

   Introduce the Weil-representation acting on Schwartz functions. For a Schwartz function define the attached theta function and show its modular properties. Introduce Eisenstein series corresponding to a standard section. Mention its meromorphic continuation and functional equation relating the values at s with the values at $-s$. State the Siegel-Weil formula. Give an overview of Section 2.2 of [6] as an example.

4. Vector-valued modular forms and harmonic weak Maass forms [4, Section 3][6, Section 3] (29.11.23, Rebekka Strathausen):

   Introduce vector-valued modular forms and harmonic weak Maass forms. Show that they have a Fourier expansion involving certain special functions. Define the ξ-operator and state the short exact sequence of [4, Corollary 3.8]. Show the existence of harmonic weak Maass forms with prescribed principal part [6, Lemma 3.4].

5. Regularized Theta Integrals [1][4][3] (13.12.23, Annika Burmester):

   Define the regularized theta lift following [1] and [3] and prove [6, Theorem 4.2].

6. CM values of automorphic Green functions [6] (10.01.24, Manuel Hoff):

   Prove [6, Theorem 4.8].

7. Faltings' heights of CM cycles [6, Section 5][9, Chapter III] (10.01.24, Paul Kiefer):

   Introduce Arakelov geometry and the notion of an arithmetic divisor [9, Chapter II, Chapter III]. Define the Faltings height. Mention the relation to the calculations of the previous talk. If time permits, state [6, Conjecture 5.1, 5.2, 5.3]

8. Height pairings on modular curves: Modular curves as orthogonal Shimura varieties and the Shimura lift [6, Section 7.1, 7.2] (24.01.24, Patrick Bieker, Lennart Gehrmann):

   Show that the modular curves $X(\Gamma_0(N))$ are orthogonal Shimura varieties by choosing a certain lattice L of signature $(1, 2)$. Recall the Shimura lift and prove [6, Lemma 7.3].

9. Height pairings on modular curves: Gross-Zagier formula [6, Section 7.3] (24.01.24, Patrick Bieker, Lennart Gehrmann):

   Prove the Gross-Zagier formula [6, Corollary 7.8].

References

[1] Richard E. Borcherds. Automorphic forms with singularities on Grassmannians. *Invent. Math.*, 132(3):491–562, 1998.

[2] Bruinier. Barbados notes.

[3] Jan H. Bruinier. *Borcherds products on O(2, l) and Chern classes of Heegner divisors*, volume 1780 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002.

[4] Jan Hendrik Bruinier and Jens Funke. On two geometric theta lifts. *Duke Math. J.*, 125(1):45–90, 2004.

[5] Jan Hendrik Bruinier, Gerard van der Geer, Günter Harder, and Don Zagier. *The 1-2-3 of modular forms*. Universitext. Springer-Verlag, Berlin, 2008. Lectures from the Summer School on Modular Forms and their Applications held in Nordfjordeid, June 2004.

[6] Jan Hendrik Bruinier and Tonghai Yang. Faltings heights of CM cycles and derivatives of L-functions. *Invent. Math.*, 177(3):631–681, 2009.

[7] Freitag. https://www.mathi.uni-heidelberg.de/ freitag/.

[8] Benedict H. Gross and Don B. Zagier. Heegner points and derivatives of L-series. *Invent. Math.*, 84(2):225–320, 1986.

[9] C. Soulé. *Lectures on Arakelov geometry*, volume 33 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1992. With the collaboration of D. Abramovich, J.-F. Burnol and J. Kramer.

[10] Wei Zhang. The Birch–Swinnerton-Dyer conjecture and Heegner points: a survey. In *Current developments in mathematics 2013*, pages 169–203. Int. Press, Somerville, MA, 2014.