OBERSEMINAR WINTER TERM 2024/25:
K-THEORY OF NUMBER RINGS

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1 **Locally symmetric spaces** **December 4**
Suppose that a group Γ acts freely and properly on a contractible topological space X. Show that the group (co-)homology of Γ can be computed by the singular (co-)homology of $\Gamma\backslash X$ [Wei94][Theorem 6.10.5]. Let G be a Lie group (with finite component group). State that all maximal compact subgroups K of G are conjugated and that the symmetric space $X = G/K$ is contractible [Hoc65][Theorem XV.3.1]. Discuss the case $G = GL_n(\mathbb{R})$, $K = O_n$. In particular, show that G/K is contractible by Gram–Schmidt. Introduce arithmetic subgroups. Show that they contain torsion-free subgroups of finite index [Mil][Appendix, Lemma 8.2] and that they act properly on the associated symmetric space [Har23][Chapter 1.2.3]. Sketch the proof of the de Rham theorem ([Lee13][Chapter 18]) and use it to deduce that for an torsion-free arithmetic group $\Gamma \subset G$ one has $H^*(\Gamma, \mathbb{R}) = H^*(\Omega(X)^\Gamma)$. The existence of a torsion-free normal subgroup then implies the same equality for all arithmetic groups $\Gamma \subseteq G$.

2 **The main theorem and injectivity** **December 4**
Define semi-simple algebraic groups and give $\mathrm{SL}_n(\mathbb{R})$ as an example. Introduce the subcomplex $I_G \subseteq \Omega(X)$ [Bor74][Section 3]. Discuss that I_G can be related to Lie algebra cohomology and discuss basic results, i.e., forms in I_G are closed and harmonic (see [BW00], Chapter II). Use the unitarian trick and Bott periodicity to compute $I_{\mathrm{SL}_n(\mathbb{R})}$ (see [Sou] and the references therein). Assuming that $I_{\mathrm{SL}_n(\mathbb{R})} \to \Omega(X)^\Gamma$ is an isomorphism in small degrees, compute the rational K-groups of $\mathbb{Z}$. State Proposition 3.6 of [Bor74] and prove the injectivity statement. In particular, discuss [Bor74][Section 1 and 2].

3 **Duality groups and Borel–Serre compactification** **December 18**
Introduce FL groups and how they arise from actions on nice topological spaces. Describe the cohomology $H^*(\Gamma, \mathbb{Z}[\Gamma])$ in terms of the boundary of the universal covering. [Bro94][Chapter VIII, Lemma 7.4, Propositions 7.5, 8.1 and 8.2]. Prove the implication $(iii) \Rightarrow (iv)$ of [Bro94][Chapter VIII, Theorem 10.1]. Discuss the Borel–Serre compactification $\Gamma\backslash\overline{X}$ of $\Gamma\backslash X$ following [Har23][Chapter 1.2.7 and 1.2.8]. Focus on the case $G = \mathrm{GL}_n(\mathbb{R})$. Relate the boundary of $\overline{X}$ to the spherical building of G [BS73][Section 2.8].

4 **Finite generation of K-groups** **December 18**
Explain Quillen's proof of the Solomon-Tits theorem for GL_n [Qui73a][Theorem 2]. Show that K-groups of number rings are finitely generated. You can either follow Quillen's original proof by first explaining Quillen's Q-construction of K-theory given in [Qui73b] and deduce the statement from the rank filtration long exact sequence [Qui73a][Theorem 3]. Or you can deduce it from homological stability as in [vdK80].

5 **Matsushima's theorem and surjectivity** **January 22**
Matsushima's theorem states that in small degrees the homomorphism $I_G^\Gamma \to H^*(\Omega(X)^\Gamma)$ is an isomorphism as long as $\Gamma\backslash X$ is compact. Prove Matsushima's theorem [Mat62] and say some words on Borel's generalization to the non-compact case [Bor74][Theorem 3.5].

6 **Borel's theorem** **January 22**
Sketch the construction of Borel's subcomplex $C \subseteq \Omega(X)^\Gamma$ [Bor74][Section 4-7, in particular Theorem 7.4].

References

[Bor74] A. Borel. Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup. (4)*, 7:235–272, 1974.

[Bro94] K. S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

[BS73] A. Borel and J.-P. Serre. Corners and arithmetic groups. *Comment. Math. Helv.*, 48:436–491, 1973.

[BW00] A. Borel and N. Wallach. *Continuous cohomology, discrete subgroups, and representations of reductive groups*, volume 67 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2000.

[Har23] G. Harder. Lectures on Cohomology of Arithmetic groups I. book project, 2023.

[Hoc65] G. Hochschild. *The structure of Lie groups*. Holden-Day, Inc., San Francisco-London-Amsterdam, 1965.

[Lee13] J. M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[Mat62] Y. Matsushima. On Betti numbers of compact, locally sysmmetric Riemannian manifolds. *Osaka Math. J.*, 14:1–20, 1962.

[Mil] J. S.. Milne. Lie Algebras, Algebraic Groups, and Lie Groups. book project.

[Qui73a] D. Quillen. Finite generation of the groups K_i of rings of algebraic integers. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, volume Vol. 341 of *Lecture Notes in Math.*, pages 179–198. Springer, Berlin-New York, 1973.

[Qui73b] D. Quillen. Higher algebraic K-theory I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, volume Vol. 341 of *Lecture Notes in Math.*, pages 179–198. Springer, Berlin-New York, 1973.

[Sou] G. Soule. Higher algebriac K-theory of algebraic integers and cohomology of arithmetic groups. lecture notes.

[vdK80] W. van der Kallen. Homology stability for linear groups. *Invent. Math.*, 60:269–295, 1980.

[Wei94] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.