Seminar on Hodge locus – SS2024

1. Overview (Floccari) – 10.04

General outline and distribution of the talks.

2. The Noether-Lefschetz theorem (Mazzanti) – 10.04

The goal of the talk is to introduce the topic of Hodge locus starting with the classical Noether-Lefschetz theorem for surfaces in $\mathbb{P}^3$. References: [17, II, Chapter 1] and [16, Section 4.3], and the more exhaustive [8].

3. Hodge structures and Mumford-Tate groups – 24.04

Recall the Hodge decomposition on the cohomology of a smooth and projective complex variety (a possible reference is [17, I]). Define abstract Hodge structures and their formal properties; in particular, the language of Mumford-Tate groups and tannakian categories should be introduced. A good (and concise) reference are the lecture notes [14], another reference is [11].

4. Period maps and domains – 24.04

Explain how the Hodge decomposition varies in a smooth family of projective varieties. Define the associated period domain and period map; prove that the period map is holomorphic and discuss the transversality property. References: [17, I, Chapters 9 and 10].

5. Variations of Hodge structures – 08.05

Give the definition of an abstract variation of Hodge structures. Introduce the algebraic monodromy group and discuss results of Deligne and André showing that the monodromy group is semisimple, and its connected component of the identity is a normal subgroup of the Mumford-Tate group of a very general Hodge structure in the variation. References: [17, I, Chapter 10], [10, Section 4], [1].

6. The Hodge locus – 08.05

Explain the behaviour of Mumford-Tate groups in a variation of Hodge structures; introduce and dicuss in particular the generic Mumford-Tate group (see [11]; see also [15, Section 4.1]). Define the Hodge locus and show that it is union of countably many analytic subvarieties; state the theorem of Cattani-Deligne-Kaplan ([9]) and explain its relation with the Hodge conjecture, see [16].

7. Mumford-Tate domains, Hodge varieties and special subvarieties – 22.05

Discuss the period map in the language of [4], which is taken from [11]. Thus, discuss Mumford-Tate domains and Hodge varieties. Introduce and discuss (weakly) special subvarieties of Hodge varieties, as in [4, Section 4.4] and [13, Section 4]. In particular give the statement of [13, Proposition 4.8]. If time permits, discuss the level of a Hodge structure as in [4, Section 4.6]

8. The Zilber-Pink conjecture – 22.05

Define atypical and typical subvarieties as in [4, Sections 2 and 5]. State and disucss the Zilber-Pink conjecture for typical and atypical Hodge locus [4, Conjectures 2.5 and 2.7]. You may also wish to look at the article [12]. This seems a good point to discuss the case of variations of Hodge structures of level 1, in which case the associated Hodge variety is a Shimura variety; the study of special subvarieties of Shimura varieties is more classical. See [13, Section 1.3] and references therein.

9. Results on the Zilber-Pink conjecture – 05.06

Explain the main results of [4]: Theorem 3.1 which establishes the Zilber-Pink conjecture for the atypical Hodge locus of positive period dimension, Theorems 3.3 and 3.9 on the typical Hodge locus, and their consequence Corollary 3.12. You could then give a sketch of the proof of Theorem 3.3 (proven in Section 7) or Theorem 3.9 (proven in Section 10).

10. Functional transcendence – 05.06

The goal of the talk is to discuss functional transcendence and in particular to explain the Ax-Schanuel theorem [4, Theorem 4.16]. This result is proven in [2] generalizing the proof from the case of Shimura varieties, which prominently uses o-minimality. The speaker may survey this approach, or, alternatively, discuss the results of [7] and explain how they imply a Ax-Schanuel theorem following [3].

11. The Zilber-Pink conjecture for the atypical Hodge locus – 26.06

Present the proof of [4, Theorem 3.1] given in Section 6. I suggest to focus in particular on how the Ax-Schanuel theorem is used, and not much on o-minimality and its consequences.

12. Applications – 26.06

Return to the Noether-Lefschetz locus following [6]. Alternatively, this talk may discuss the arithmetic application [5].

References

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