Renormalisation in Spectral Theory: Lyapunov Exponents

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Outline

1. Method: Lyapunov Exponents
2. Objects: Substitutions and Diffraction
3. Result: Absence of Absolutely Continuous Diffraction
4. Examples
Lyapunov Exponents
What are Lyapunov exponents?

- $\{M_j\}_{j \geq 0}$: sequence of matrices in $\text{Mat}(n, \mathbb{C})$
- $M^{(N)} := M_{N-1} M_{N-2} \cdots M_1 M_0$

Its **Lyapunov exponent** $\chi : \mathbb{C}^d \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$
\chi(v) := \limsup_{N \to \infty} \frac{1}{N} \log \| M^{(N)} v \|.
$$

Main references: Barreira and Pesin, Viana, Duarte and Klein
Extremal exponents

Under some invertibility assumptions, one can define the **extremal Lyapunov exponents**

\[ \chi_{\text{max}} := \limsup_{N \to \infty} \frac{1}{N} \log \| M^{(N)} \| \quad \chi_{\text{min}} := \liminf_{N \to \infty} \frac{1}{N} \log \| (M^{(N)})^{-1} \|^{-1} \]
Lyapunov Exponents
Substitutions and Diffraction
Absence of AC Diffraction
Examples

Two (easy) examples

\[ M^{(N)} = M_{N-1} M_{N-2} \ldots M_1 M_0 \nu \quad \text{if} \quad M_j \text{ are unitary} \]

- \( M_j = M \) for all \( j \geq 0 \), \( M \) is normal
  \[ \Rightarrow \chi_{\text{max}} = \log |\lambda_{\text{max}}| \]

- \( M_j \) is a unitary matrix for all \( j \geq 0 \)
  \[ \Rightarrow \chi_{\text{max}} = 0 \]

\[ \| M_0 \nu \| = \| \nu \| \]

\[ \| M^{(N)} \nu \| = \| \nu \| \]
Matrix cocycles

How to get sequences of matrices:

- Random matrix products

- **Matrix cocycles**

- $(X, f, \mu)$: measure-preserving dynamical system

- $A: X \to \text{Mat}(d, \mathbb{C})$: measurable matrix-valued map
Matrix cocycles

How to get sequences of matrices:

- Random matrix products
- Matrix cocycles
- \((X, f, \mu)\): measure-preserving dynamical system
- \(A: X \to \text{Mat}(d, \mathbb{C})\): measurable matrix-valued map

The map \(F: X \times \mathbb{C}^d \to X \times \mathbb{C}^d\) given by

\[
(x, v) \mapsto (f(x), A(x)v)
\]

is called a **linear cocycle** over \(f\).
Matrix cocycles

\[ f^n(x, v) \]

\[
(x, v) \xrightarrow{F} (f(x), A(x)v) \xrightarrow{F} (f^2(x), A(f(x))A(x)v) \\
\vdots
\]

\[
f^n(x, v) = (f^n(x), A(f^{n-1}x)A(f^{n-2}x) \ldots A(f(x))A(x)v)
\]

\[ A: \text{generator} \]
\[ f: \text{base dynamics} \]

\[ \text{matrix cocycle} \]
Kingman’s subadditive ergodic theorem

- \((X, f, \mu)\): ergodic m.p.d.s.
- \(A^{(N)}(x)\): cocycle with base dynamics \(f\)
- \(\log^+ \|A(x)\| \in L^1(\mu)\)

Then

\[
\chi_{\text{max}}(x) = \lim_{N \to \infty} \frac{1}{N} \log \|A^{(N)}(x)\| = \inf \frac{1}{N} \int_X \log \|A^{(N)}(\xi)\| d\mu(\xi)
\]

for \(\mu\)-a.e. \(x \in X\).
Lyapunov exponents in aperiodic order

- Diffraction and dynamical spectra
- Schrödinger operators
- $S$-adic systems
- Continued fraction algorithms
Main texts

- Baake, Gähler (2016): Pair correlations
- Baake, Frank, Grimm, Robinson (2019): Non-Pisot example
- Bufetov, Solomyak (2020): Singular $\mathbb{Z}$-actions
- Bufetov, Solomyak (2018): Spectral cocycle
Substitutions and Diffraction
Optical diffraction
X-ray diffraction

Schechtman’s discovery (1982)

A quenched state of an AlMn-alloy with icosahedral symmetry that displayed isolated bright spots in its diffraction pattern.

Mathematical diffraction

Motto: Mathematical diffraction theory of infinite point sets in $\mathbb{R}^d$ is the harmonic analysis of unbounded Radon measures.

A Radon measure $\mu$ on $\mathbb{R}^d$ is a (complex-valued) linear functional on $C_c(\mathbb{R}^d)$. 
From tilings ($\mathcal{T}$) to point sets ($\Lambda$)

Vertices of (possibly) different tile types $\rightarrow$ Atoms of (possibly) distinct scattering strength
Let $\Lambda \subset \mathbb{R}^d$ be a point set in $\mathbb{R}^d$ and $w(x) \in \mathbb{C}$ be the weight associated to the point $x \in \Lambda$.

**Atomic distrib.:**

$$M = \sum_{x \in \Lambda} w(x) \delta_x$$

**Autocorrelation:**

$$\gamma = M \cdot \widetilde{m}$$

**Diffraction:**

$\hat{\gamma}$
Let $\Lambda \subset \mathbb{R}^d$ be a point set in $\mathbb{R}^d$ and $w(x) \in \mathbb{C}$ be the weight associated to the point $x \in \Lambda$.

**Atomic distrib.:**  
**Autocorrelation:**  
**Diffraction:**

The diffraction $\widehat{\gamma}$ admits the unique Lebesgue decomposition

$$\widehat{\gamma} = \widehat{\gamma}_{pp} + \widehat{\gamma}_{ac} + \widehat{\gamma}_{sc}$$
Absolutely continuous component

\[ \hat{\gamma} = \hat{\gamma}_{pp} + \hat{\gamma}_{ac} + \hat{\gamma}_{sc}. \]

1. Supported on sets of positive Lebesgue measure
2. Analytic counterpart of diffuse diffraction
3. Presence suggests randomness
4. \[ \hat{\gamma}_{ac} = h(k)\mu_{\text{Leb}} \]

Quintas: \[ a \rightarrow ab \]
\[ b \rightarrow \overline{a} \]
\[ \overline{a} \rightarrow \overline{ab} \]
\[ \overline{b} \rightarrow \overline{ab} \]

\( W_+ \): unbarred
\( W_- \): barred
Absolutely continuous component

\[ \widehat{\gamma} = \widehat{\gamma}_{pp} + \widehat{\gamma}_{ac} + \widehat{\gamma}_{sc}. \]

1. Supported on sets of positive Lebesgue measure

2. Analytic counterpart of diffuse diffraction

3. Presence suggests randomness

4. \[ \widehat{\gamma}_{ac} = h(k) \mu_{\text{Leb}} \]

\[ h(k) = \left| \frac{w_+ - w_-}{2} \right|^2 \]

for the Rudin–Shapiro substitution.
The Radon-Nikodym density of $f_{\text{ac}}$ is denoted by $h(k) \sim v(k)$.

For $k \in \mathbb{R}^d$,

$$v(\lambda k) = \sqrt{X} B^{-1}(k) v(k)$$

$$v(\lambda^n k) = \lambda^{n/2} B^{-1}(\lambda^{n-1} k) \cdots B^{-1}(\lambda k) B^{-1}(k) v(k)$$

Matrix cocycle: $B(k)$, Fourier matrix

$f: k \mapsto \lambda k$
A measure $\mu$ in $\mathbb{R}^d$ is called **translation-bounded** if for every compact $K \subset \mathbb{R}^d$, there exists a constant $C_K$ such that

$$\sup_{x \in \mathbb{R}^d} |\mu|(x + K) < C_K.$$ 

- Each $\hat{\gamma}_{ij}$ is a translation-bounded measure.
- Translation-boundedness and exponential growth of the Radon–Nikodym density are incompatible.
Fourier matrix from inflation rules

\[ \rho_F^2 : \]

\[ \begin{array}{ccc}
0 & \tau & \tau + 1 \\
0 & \tau &
\end{array} \]
Fourier matrix from inflation rules

\[ B(0) = M \]

\[ B(k) = \left( 1 + \frac{e^{2\pi i(1+\tau)k}}{e^{2\pi i\tau k}} \right) \begin{pmatrix} 1 \\ e^{2\pi i\tau k} \end{pmatrix} \]

\[ B(k) = \delta_\tau(k) \] (same)
Diffraction of systems arising from inflations

Question: When is $\hat{\gamma} = \hat{\gamma}_{pp} + \hat{\gamma}_{ac} + \hat{\gamma}_{sc}$?

$\mathbb{L}(k) = \left| \frac{w_+ - w_-}{2} \right|^2$

Fibonacci
$\hat{\delta} = \hat{\delta}_{pp}$

Rudin-Shapiro
$\hat{\delta} = \hat{\delta}_{pp} + \hat{\delta}_{ac}$

Thue-Morse
$\hat{\delta} = \hat{\delta}_{pp} + \hat{\delta}_{sc}$
Absence of Absolutely Continuous Diffraction
Lyapunov exponents from inflation rules

Squared Fibonacci inflation:

\[ \varphi_F^2 : \quad \begin{array}{c}
\begin{array}{c}
\text{red} \\
\text{blue}
\end{array}
\end{array} \rightarrow \begin{array}{c}
\begin{array}{c}
\text{red} \\
\text{blue} \\
\text{red}
\end{array}
\end{array} \]

For a fixed \( k \in \mathbb{R} \), consider the cocycle

\[ B^{(N)}(k) = B(k)B(\lambda k) \cdots B(\lambda^{N-1} k). \]

Fourier matrix:

\[ B(k) = \begin{pmatrix}
1 + e^{2\pi i (1+\tau)k} & 1 \\
e^{2\pi i \tau k} & e^{2\pi i \tau k}
\end{pmatrix} \]

\[ f : k \mapsto \lambda^k \quad (\mathbb{R}) \]

\[ k \mapsto Q^T k \quad (\mathbb{R}^d) \]

For this cocycle, consider the (maximal) Lyapunov exponent given by

\[ \chi^B(k) = \limsup_{N \to \infty} \frac{1}{N} \log \| B^{(N)}(k) \|. \]
Sufficient condition for absence of ac diffraction

Theorem (Baake–Gähler–M.)

Let \( \varrho \) be a primitive inflation rule, with inflation multiplier \( \lambda \), and let \( B(k) \) be the corresponding Fourier matrix. Under some mild assumptions, if there is an \( \varepsilon > 0 \) such that

\[
\chi^B(k) \leq \log \sqrt{\lambda} - \varepsilon
\]

for Lebesgue-a.e. \( k \in \mathbb{R}^d \), then the diffraction \( \hat{\gamma} \) does not contain an absolutely continuous component.
Examples

- **Singular examples**
  - Abelian bijective
  - Fibonacci
  - Godrèche–Lançon–Billard tiling

- **Examples with ac**
  - Rudin–Shapiro
  - Chan–Grimm–Short’s 9-letter example
Singular examples
Abelian bijective substitutions

Example

Let $\mathcal{A} = \{0, 1, 2\}$ and consider

\[
\begin{align*}
\varrho_{ab} : & \begin{cases}
0 & \mapsto \begin{pmatrix} 0 & 2 & 1 & 1 & 2 \\
1 & \mapsto & 1 & 0 & 2 & 2 & 0 \\
2 & \mapsto & 2 & 1 & 0 & 0 & 1 \\
\end{pmatrix}
\end{cases}
\end{align*}
\]

This is an example of a **bijective substitution**. The zeroth and the third column are $C_0 = (0, 1, 2)^T$ and $C_3 = (1, 2, 0)^T$, respectively.
Abelian bijective substitutions

\[ \varrho_{ab} : \begin{cases} 
0 \mapsto 0 2 1 1 2 \\
1 \mapsto 1 0 2 2 0 \\
2 \mapsto 2 1 0 0 1 
\end{cases} \]

Example

From \( \varrho_{ab} \), one automatically gets the inflation

\[ \varrho_{ab} : \]
Abelian bijective substitutions

\[
\varphi : \begin{cases}
0 &\mapsto 0 \ 2 \ 1 \ 1 \ 2 \\
1 &\mapsto 1 \ 0 \ 2 \ 2 \ 0 \\
2 &\mapsto 2 \ 1 \ 0 \ 0 \ 1 
\end{cases}
\]

One can consider the columns of \( \varphi \) as permutations in \( S_{|A|} \) via

\[
\sigma_{C_\ell}(0, 1, 2, \ldots, |A| - 1)^T = C_\ell.
\]

A bijective substitution \( \varphi \) is Abelian if the group \( G \) “generated by the columns”, i.e., \( G = \langle \sigma_{C_\ell} \rangle \) is Abelian.
Proposition

Let \( \varphi \) be primitive, bijective, Abelian. Then

\[
B(k) = \sum_{\ell=0}^{L-1} e^{2\pi i \ell k} D_{\ell}
\]

where \( \{D_{\ell}\} \) are commuting permutation matrices.
Proposition

Let \( \varrho \) be primitive, bijective, Abelian. Then the Lyapunov exponents of \( B(k) \) are given by \( \chi_j = m(P_j) \), where \( m(P_j) \) is the logarithmic Mahler measure of the polynomial

\[
P_j(z) = \sum_{\ell=0}^{L-1} \rho_j(\sigma_C^\ell) z^\ell,
\]

where \( \rho_j \) is the \( j \)th irrep of \( G \).
Absence of ac Diffraction

**Proposition**

Let $\varphi$ be primitive, bijective, Abelian. Then the Lyapunov exponents of $B(k)$ are given by $\chi_j = m(P_j)$, where $m(P_j)$ is the logarithmic Mahler measure of the polynomial

$$P_j(z) = \sum_{\ell=0}^{L-1} \rho_j(\sigma_{C_\ell})z^\ell,$$

where $\rho_j$ is the $j$th irrep of $G$.

**Theorem (Baake–Gähler–M.)**

Let $\varphi$ be a primitive, bijective, Abelian substitution. Then, for a.e. $k \in \mathbb{R}$, all Lyapunov exponents of $B(k)$ are strictly less than $\log \sqrt{\lambda}$. Consequently, the corresponding diffraction $\hat{\gamma}$ does not contain an absolutely continuous component.
Squared-Fibonacci inflation

Before proceeding, we need to introduce a proposition about the expansion factor $\lambda$ and the algebraic degree $r$. Under some mild assumptions, there exists an ergodic cocycle $B$ on $\mathbb{T}^r$, such that

$$B^{(n)}(k) = \tilde{B}^{(n)}(x_1, \ldots, x_r) \bigg|_{x_1=k, x_2=\theta_1 k, \ldots, x_r=\theta_{r-1} k}$$

where $\{1, \theta_1, \ldots, \theta_{r-1}\}$ are rationally independent.
A Kingman-type estimate

\[ B(k) = \begin{pmatrix} 1 + e^{2\pi i(1+\tau)k} & 1 \\ e^{2\pi i\tau k} & e^{2\pi i\tau k} \end{pmatrix} \]

**Proposition**

Let \( \varrho \) be a primitive one-dimensional inflation. Under some mild assumptions,

\[ \chi^B(k) \leq \frac{1}{N} \int_{\mathbb{T}^1} \log \| \tilde{B}(N)(x) \| d\mu_H(x), \]

for a.e. \( k \in \mathbb{R} \) and for all \( N \geq 1 \).
Strategy

1. Compute the Fourier matrix $B(k)$
2. Check if $\det(B(k)) \neq 0$
3. Compute the finite approximants of the Kingman-type bound via numerical integration
4. Wait until it goes below the threshold $(\log \sqrt{\lambda})$ for some $N$

If there is such an $N \implies$ no ac diffraction
Squared-Fibonacci inflation

\[ \varrho_F^2 : \quad \text{block inflation} \]

\[ B(k) = \begin{pmatrix} 1 + e^{2\pi i (1+\tau)k} & 1 \\ e^{2\pi i \tau k} & e^{2\pi i \tau k} \end{pmatrix} \]

<table>
<thead>
<tr>
<th>( N )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{N} \int_{\mathbb{T}^2} \log | \hat{B}^{(N)}(x) |_F^2 )</td>
<td>1.5668</td>
<td>1.1091</td>
<td>0.8776</td>
<td>0.7409</td>
</tr>
</tbody>
</table>

Table: Numerical values for upper bounds for \( 2\chi^B(k) \) for \( \varrho_F^2 \). Here, \( \| \cdot \|_F \) stands for the Frobenius norm and \( \log \lambda \approx 0.9624 \).

\[ \Rightarrow \hat{\delta} \text{ does not have an a.c. component!} \]
Higher-dimensional example

Godrèche–Lançon–Billard tiling

Here, $\lambda = \frac{5 + \sqrt{5}}{2}$ is non-Pisot.
Higher-dimensional example

Godrèche–Lançon–Billard tiling

Higher-dimensional example

Set of control points: $\Lambda \subset \mathbb{Z}[\zeta_5]$
Higher-dimensional example

Figure: Level-1 supertiles of the GLB tiling
Higher-dimensional example

Godrèche–Lançon–Billard tiling

\[ \hat{\sigma} = I_0 \delta_0 + \hat{\sigma}_{56} \]

<table>
<thead>
<tr>
<th>( N )</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{N} \int_{T^4} \log | \tilde{B}^{(N)}(\cdot) |_F^2 )</td>
<td>2.571</td>
<td>2.517</td>
<td>2.474</td>
<td>2.440</td>
<td>2.411</td>
<td>2.387</td>
</tr>
</tbody>
</table>

Table: Numerical upper bounds for \( \chi^B \) for the GLB inflation \( \rho_{\text{GLB}} \). Here \( \log \lambda \approx 2.571862 \).
Examples with AC
Necessary condition for presence of ac diffraction

**Theorem (Berlinkov–Solomyak, 2019)**
Let \( \varphi \) be a primitive substitution of constant length \( L \). If it admits a Lebesgue component in its dynamical spectrum, then its substitution matrix must have an eigenvalue of modulus \( \sqrt{L} \).
Theorem (Baake–Gähler-M., 2019)

Let \( \varrho \) be a primitive inflation rule. Under some mild assumptions, one has \( \chi^B(k) \leq \log \sqrt{\lambda} \) for a.e. \( k \in \mathbb{R}^d \).

Corollary (Baake–Gähler-M., 2019)

Let \( \varrho \) be a primitive inflation rule. Under some mild assumptions, if its diffraction admits a Lebesgue component, then its Fourier cocycle must have \( \chi^B(k) = \log \sqrt{\lambda} \) for a set of \( k \in \mathbb{R}^d \) of positive Lebesgue measure.
Examples with AC

Rudin–Shapiro substitution

\[ \varphi_{RS} : a \mapsto ab, b \mapsto a\bar{b}, \bar{b} \mapsto \bar{a}b, \bar{a} \mapsto \bar{a}\bar{b} \]

\[ B(k) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ e^{2\pi i k} & 0 & e^{2\pi i k} & 0 \\ 0 & e^{2\pi i k} & 0 & e^{2\pi i k} \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \text{det}(B(k)) = 0 ! \]

\[ SB(k)S^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \]
Examples with AC

Rudin–Shapiro substitution

\[ \varrho_{RS} : a \mapsto ab, b \mapsto a\bar{b}, \bar{a} \mapsto \bar{a}b, \bar{b} \mapsto \bar{a} \bar{b} \]

\[ B(k) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ e^{2\pi ik} & 0 & e^{2\pi ik} & 0 \\ 0 & e^{2\pi ik} & 0 & e^{2\pi ik} \\ 0 & 0 & 1 & 1 \end{pmatrix} \]

\[ B'(k) = \begin{pmatrix} 1 + e^{2\pi ik} & 0 & \ldots & 0 \\ -e^{2\pi ik} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 + e^{2\pi ik} \\ 0 & 0 & \ldots & e^{2\pi ik} - e^{2\pi ik} \end{pmatrix} \]

\[ x^{B_2} = \log \sqrt{2} \text{ for all } k \in \mathbb{R} \]

\[ \sqrt{2} \mu(k) \text{ unitary!} \]
Example (Chan–Grimm–Short, 2018)

A nine-letter example: \( q_9 \)

\[
\begin{align*}
0 & \mapsto 012, & \bar{0} & \mapsto \bar{0}1\bar{2}, & \bar{\bar{0}} & \mapsto \bar{\bar{0}}\bar{1}\bar{2}, \\
1 & \mapsto 0\bar{1}\bar{2}, & \bar{1} & \mapsto \bar{0}\bar{1}\bar{2}, & \bar{\bar{1}} & \mapsto \bar{\bar{0}}1\bar{2}, \\
2 & \mapsto 0\bar{1}\bar{2}, & \bar{2} & \mapsto \bar{0}\bar{1}\bar{2}, & \bar{\bar{2}} & \mapsto \bar{\bar{0}}1\bar{2}
\end{align*}
\]
Proposition (M., 2019)

Let $B(k)$ be the Fourier matrix of $\varphi_9$. One has

1. For $k \in \mathbb{R}$, all $B(k)$ are simultaneously block diagonalisable into

$$B'(k) = \begin{pmatrix}
Z'_1 & 0 & 0 \\
0 & Z'_2 & 0 \\
0 & 0 & Z'_3
\end{pmatrix},$$

where $Z'_2$ and $Z'_3$ are constant multiples of unitary matrices, with multiplier $c = \sqrt{3}$.

2. The cocycles defined by the blocks $Z'_2$ and $Z'_3$ have degenerate Lyapunov spectrum $\chi^{Z'_2} = \chi^{Z'_3} = \log(\sqrt{3}) = \log \sqrt{\lambda}$. 

There exists an infinite family of primitive inflation tilings in $\mathbb{R}^d$ with non-trivial absolutely continuous spectral component such that the corresponding Lebesgue multiplicity is exactly the number of Lyapunov exponents of the Fourier cocycle $B(k)$ equal to $\log \sqrt{\lambda}$. 

Theorem (Frank–M., 202X)
Summary

\[ b = b_{pp} + b_{ac} + b_{sc} \]

\[ B(k) \to \chi^B(k) \leq \log \lambda - \varepsilon \text{ for a.e. } k \in \mathbb{R}^d \]
Thank you for your attention!