Renormalisation in Spectral Theory: Lyapunov Exponents

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Outline

1. Method: Lyapunov Exponents
2. Objects: Substitutions and Diffraction
3. Result: Absence of Absolutely Continuous Diffraction
4. Examples

Lyapunov Exponents

What are Lyapunov exponents?

- $\{M_j\}_{j \geqslant 0}$: sequence of matrices in $\mathrm{Mat}(n, \mathbb{C})$
- $M^{(N)} := M_{N-1} M_{N-2} \cdots M_1 M_0$

Its **Lyapunov exponent** $\chi \colon \mathbb{C}^d \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$\chi(v) := \limsup_{N \to \infty} \frac{1}{N} \log \|M^{(N)} v\|.$$

Main references: Barreira and Pesin, Viana, Duarte and Klein

Extremal exponents

Under some invertibility assumptions, one can define the **extremal Lyapunov exponents**

$$\chi_{\max} := \limsup_{N\to\infty} \frac{1}{N} \log \|M^{(N)}\| \qquad \chi_{\min} := \liminf_{N\to\infty} \frac{1}{N} \log \|\left(M^{(N)}\right)^{-1}\|^{-1}$$

$$\underline{\chi_{\min}} \leq \chi(v) \leq \underline{\chi_{\max}}$$

Two (easy) examples

$M^{(N)}v = M_{N-1} M_{N-2} \cdots M_1 \underbrace{M_0 v}$; M_j are unitary

$\|M_0 v\| = \|v\|$

$\|M^{(N)} v\| = \|v\|$

- $M_j = M$ for all $j \geqslant 0$, M is normal
  $\Longrightarrow \chi_{\max} = \log|\lambda_{\max}|$
- M_j is a unitary matrix for all $j \geqslant 0$
  $\Longrightarrow \chi_{\max} = 0$

Matrix cocycles

How to get sequences of matrices:

- Random matrix products $S = \{M_1, \ldots, M_n\}$
- Matrix cocycles
- (X, f, μ): measure-preserving dynamical system
- $A\colon X \to \mathrm{Mat}(d, \mathbb{C})$: measurable matrix-valued map

Matrix cocycles

How to get sequences of matrices:

- Random matrix products
- Matrix cocycles

- (X, f, μ): measure-preserving dynamical system
- $A\colon X \to \mathrm{Mat}(d, \mathbb{C})$: measurable matrix-valued map

The map $F\colon X \times \mathbb{C}^d \to X \times \mathbb{C}^d$ given by

$\Downarrow$
(x, v)

$$(x, v) \mapsto (f(x), \underline{A(x)}v)$$

is called a **linear cocycle** over f.

Matrix cocycles

$F^n(x,v)$

$$(x,v) \xrightarrow{F} (f(x), \underbrace{A(x)v}_{v'}) \xrightarrow{F} (f^2(x), \underbrace{A(f(x))}\,\underline{A(x)v})$$

$$\cdots\; F^n(x,v) = (f^n(x), \underbrace{A(f^{n-1}x)A(f^{n-2}x)\cdots A(f(x))A(x)}_{\text{matrix cocycle}}v)$$

A : generator

f : base dynamics

Kingman's subadditive ergodic theorem

C

N 1 2 3 4 . .

- (X, f, μ): ergodic m.p.d.s.
- $A^{(N)}(x)$: cocycle with base dynamics f
- $\log^+ \|A(x)\| \in L^1(\mu)$

Then

exists as a limit!   constant for μ-a.e. x

$$\chi_{\max}(x) \overset{\checkmark}{=} \lim_{N\to\infty} \frac{1}{N} \log \|A^{(N)}(x)\| \overset{\checkmark}{=} \underline{\underline{\inf_N \frac{1}{N} \int_X \underline{\log \|A^{(N)}(\xi)\|} \mathrm{d}\mu(\xi)}}$$

for μ-a.e. $x \in X$.

Lyapunov exponents in aperiodic order

- Diffraction and dynamical spectra
- Schrödinger operators
- S-adic systems
- Continued fraction algorithms

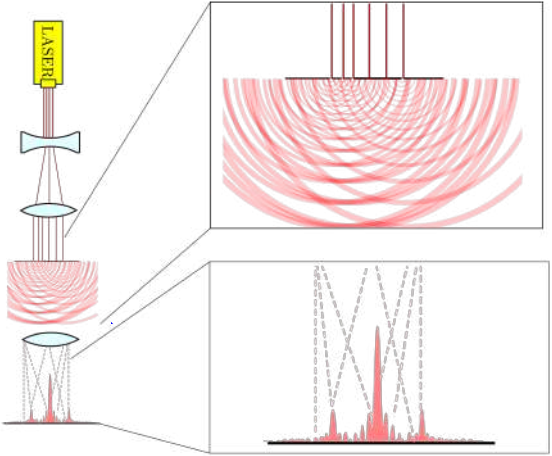
Main texts

- Baake, Gähler (2016): Pair correlations
- Baake, Frank, Grimm, Robinson (2019): Non-Pisot example
  Baake, Grimm, M. (2018): Family of generalisations
- Baake, Gähler, M. (2019): General treatment and extension to HD

- Bufetov, Solomyak (2020): Singular $\mathbb{Z}$-actions
- Bufetov, Solomyak (2018): Spectral cocycle

Substitutions and Diffraction

Optical diffraction



X-ray diffraction

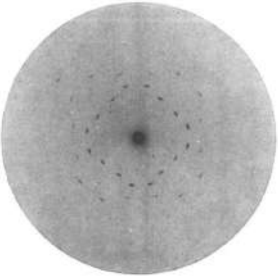
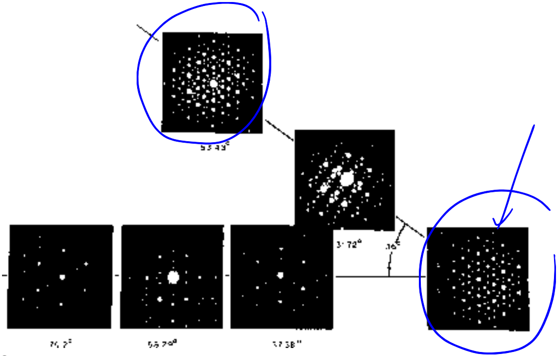


Fig. 5

Friedrich, Knipping and von Laue, Interferenzerscheinungen bei Röntgenstrahlen, *Ann. d. Phys.* (1913).

Schechtman's discovery (1982)

A quenched state of an AlMn-alloy with icosahedral symmetry that displayed isolated bright spots in its diffraction pattern.



(c) Schechtman, et.al., Metallic phase with long-range orientational order and no translational symmetry, *Phys. Rev. Lett.* (1984)

Mathematical diffraction

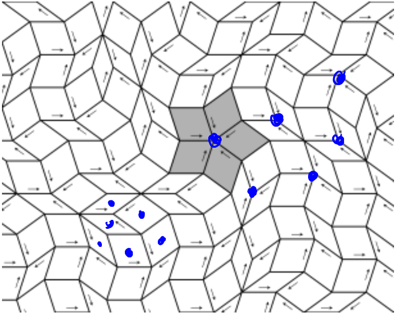
Motto: Mathematical diffraction theory of infinite point sets in $\mathbb{R}^d$ is the harmonic analysis of unbounded Radon measures.

$\mu(\mathbb{R}^d) = \infty$

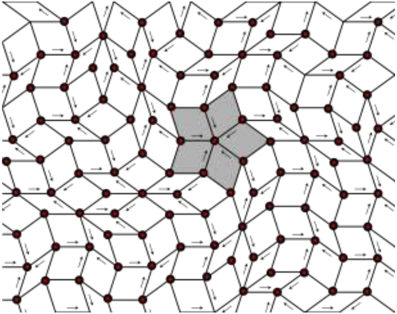
A **Radon measure** μ on $\mathbb{R}^d$ is a (complex-valued) linear functional on $C_c(\mathbb{R}^d)$.

From tilings ($\mathcal{T}$) to point sets (Λ)

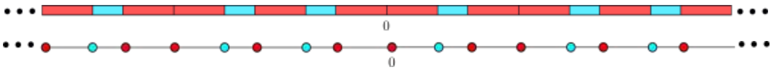
Vertices of (possibly) different tile types $\Longrightarrow$



Atoms of (possibly) distinct scattering strength



Autocorrelation and diffraction



- Let $\Lambda \subset \mathbb{R}^d$ be a point set in $\mathbb{R}^d$ and $w(x) \in \mathbb{C}$ be the weight associated to the point $x \in \Lambda$.

Atomic distrib.:

$$\mu = \sum_{x \in \Lambda} w(x)\,\delta_x$$

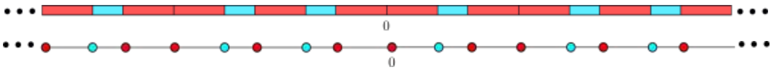
Autocorrelation:

$$\gamma = \mu \circledast \tilde{\mu}$$

Diffraction:

$$\hat{\gamma}$$

Autocorrelation and diffraction



- Let $\Lambda \subset \mathbb{R}^d$ be a point set in $\mathbb{R}^d$ and $w(x) \in \mathbb{C}$ be the weight associated to the point $x \in \Lambda$.

Atomic distrib.: **Autocorrelation:** **Diffraction:**

- The diffraction $\widehat{\gamma}$ admits the unique Lebesgue decomposition

$$\Longrightarrow \quad \widehat{\gamma} = \widehat{\gamma}_{pp} + \widehat{\gamma}_{ac} + \widehat{\gamma}_{sc}$$

Absolutely continuous component

$$\widehat{\gamma} = \widehat{\gamma}_{\mathrm{pp}} + \widehat{\gamma}_{\mathrm{ac}} + \widehat{\gamma}_{\mathrm{sc}}.$$

1. Supported on sets of positive Lebesgue measure
2. Analytic counterpart of **diffuse diffraction**
3. Presence suggests **randomness**
4. $\widehat{\gamma}_{\mathrm{ac}} = h(k)\mu_{\mathrm{Leb}}$

Q_{RS} : $a \mapsto ab$, $b \mapsto a\bar{b}$

$\bar{a} \mapsto \bar{a}\bar{b}$

$\bar{b} \mapsto \bar{a}b$

W_+: unbarred

W_-: barred

Absolutely continuous component

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1. Supported on sets of positive Lebesgue measure
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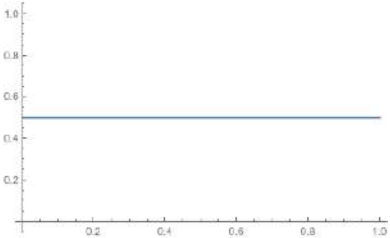


$$h(k) = \left| \frac{w_+ - w_-}{2} \right|^2$$

for the Rudin–Shapiro substitution

Renormalisation equations

$k \in \mathbb{R}^d$

$h(k)$: Radon-Nikodym density of $\hat{\gamma}_{ac}$

$h(k) \sim v(k)$

$$v(\lambda k) = \sqrt{\lambda}\, B^{-1}(k)\, v(k)$$

$$v(\lambda^n k) = \lambda^{n/2}\, \underline{B^{-1}(\lambda^{n-1} k) \cdots B^{-1}(\lambda k)\, B^{-1}(k)}\, v(k)$$

matrix cocycle: $\rightarrow B(k)$: Fourier matrix

$f : k \mapsto \lambda k$

Remarks

A measure μ in $\mathbb{R}^d$ is called **translation-bounded** if for every compact $K \subset \mathbb{R}^d$, there exists a constant C_K such that

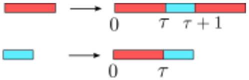
$$\sup_{x \in \mathbb{R}^d} |\mu|(x + K) < C_K.$$

$$\hat{\gamma} = \mathrm{dens}(\Lambda) \sum_{i,j=1}^{s} w_i \widehat{\Upsilon_{ij}} \overline{w_j}$$

- Each $\widehat{\Upsilon_{ij}}$ is a translation-bounded measure.
- Translation-boundedness and exponential growth of the Radon–Nikodym density are incompatible.

Fourier matrix from inflation rules

ϱ_{F}^{2} :



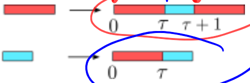
Fourier matrix from inflation rules

$t \longmapsto e^{2\pi i t k}$

$B(0) = M$

subs. matrix

ϱ_{F}^2 :



$$B(k) = \begin{pmatrix} 1 + e^{2\pi i(1+\tau)k} & 1 \\ e^{2\pi i\tau k} & e^{2\pi i\tau k} \end{pmatrix}$$

$\longrightarrow B(k) = \check{\delta}_T(k)$

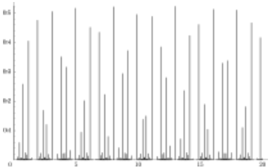
same!

Diffraction of systems arising from inflations

$w_+ - w_- = \sqrt{2}$

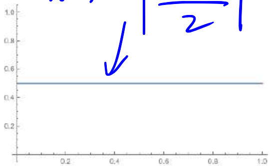
Question: When is $\widehat{\gamma} = \widehat{\gamma}_{\mathsf{pp}} \cancel{+ \widehat{\gamma}_{\mathsf{ac}}} + \widehat{\gamma}_{\mathsf{sc}}$?

$h(k) = \left| \frac{w_+ - w_-}{2} \right|^2$



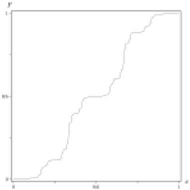
Fibonacci

$\hat{\gamma} = \hat{\gamma}_{pp}$



Rudin-Shapiro

$\hat{\gamma} = \hat{\gamma}_{pp} + \hat{\gamma}_{ac}$



Thue-Morse

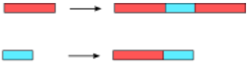
$\hat{\gamma} = \hat{\gamma}_{pp} + \hat{\gamma}_{sc}$

Absence of Absolutely Continuous Diffraction

Lyapunov exponents from inflation rules

Squared Fibonacci inflation:

ϱ_{F}^{2} :



Fourier matrix:

$$B(k) = \begin{pmatrix} 1 + \mathrm{e}^{2\pi\mathrm{i}(1+\tau)k} & 1 \\ \mathrm{e}^{2\pi\mathrm{i}\tau k} & \mathrm{e}^{2\pi\mathrm{i}\tau k} \end{pmatrix}$$

$f: k \mapsto \lambda k \quad (\mathbb{R})$

$k \mapsto Q^{T} k \quad (\mathbb{R}^{d})$

For a fixed $k \in \mathbb{R}$, consider the cocycle

$$B^{(N)}(k) = B(k)B(\lambda k)\cdots B(\lambda^{N-1}k).$$

For this cocycle, consider the (maximal) Lyapunov exponent given by

$$\chi^{B}(k) = \limsup_{N\to\infty} \frac{1}{N} \log \| B^{(N)}(k) \|.$$

Sufficient condition for absence of ac diffraction

Theorem (Baake–Gähler–M.)

Let ϱ be a primitive inflation rule, with inflation multiplier λ, and let $B(k)$ be the corresponding Fourier matrix. Under some mild assumptions, if there is an $\varepsilon > 0$ such that

$$\chi^{B}(k) \leqslant \log \sqrt{\lambda} - \varepsilon$$

for Lebesgue-a.e. $k \in \mathbb{R}^{d}$, then the diffraction $\widehat{\gamma}$ does not contain an absolutely continuous component.

Examples

- Singular examples
  - Abelian bijective
  - Fibonacci
  - Godrèche–Lançon–Billard tiling
- Examples with ac
  - Rudin–Shapiro
  - Chan–Grimm–Short's 9-letter example

Singular examples

Abelian bijective substitutions

Example

Let $\mathcal{A} = \{0, 1, 2\}$ and consider

$$\varrho_{\mathsf{ab}} : \begin{cases} 0 \mapsto \boxed{0}\,2\;1\,\boxed{1}\,2 \\ 1 \mapsto \boxed{1}\,0\;2\,\boxed{2}\,0 \\ 2 \mapsto \boxed{2}\,1\;0\,\boxed{0}\,1 \end{cases}$$

This is an example of a **bijective substitution**.
The zeroth and the third column are $\mathcal{C}_0 = (0, 1, 2)^{\mathsf{T}}$ and $\mathcal{C}_3 = (1, 2, 0)^{\mathsf{T}}$, respectively.

Abelian bijective substitutions

$$\varrho_{\mathrm{ab}} : \begin{cases} 0 \mapsto 0\,2\,1\,1\,2 \\ 1 \mapsto 1\,0\,2\,2\,0 \\ 2 \mapsto 2\,1\,0\,0\,1 \end{cases}$$

Example

From ϱ_{ab}, one automatically gets the inflation

ϱ_{ab} :

Abelian bijective substitutions

$$\varrho : \begin{cases} 0 \mapsto \boxed{0}\,2\,1\,\boxed{1}\,2 \\ 1 \mapsto \boxed{1}\,0\,2\,\boxed{2}\,0 \\ 2 \mapsto \boxed{2}\,1\,0\,\boxed{0}\,1 \end{cases}$$

One can consider the columns of ϱ as permutations in $S_{|\mathcal{A}|}$ via

$$\sigma_{\mathcal{C}_\ell}(0, 1, 2, \ldots, |\mathcal{A}| - 1)^{\mathsf{T}} = \mathcal{C}_\ell.$$

A bijective substitution ϱ is **Abelian** if the group G "generated by the columns", i.e., $G = \langle \sigma_{\mathcal{C}_\ell} \rangle$ is Abelian.

Fourier matrices

Proposition

Let ϱ be primitive, bijective, Abelian. Then

$$B(k) = \sum_{\ell=0}^{L-1} \mathrm{e}^{2\pi\mathrm{i}\ell k} D_\ell$$

where $\{D_\ell\}$ are commuting permutation matrices.

Absence of ac Diffraction

Proposition

Let ϱ be primitive, bijective, Abelian. Then the Lyapunov exponents of $B(k)$ are given by $\chi_j = \mathfrak{m}(P_j)$, where $\mathfrak{m}(P_j)$ is the logarithmic Mahler measure of the polynomial

$$P_j(z) = \sum_{\ell=0}^{L-1} \overline{\rho_j(\sigma_{\mathcal{C}_\ell})} z^\ell, \quad \text{where } \rho_j \text{ is the } j\text{th irrep of } G.$$

Absence of ac Diffraction

Proposition

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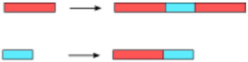
Theorem (Baake–Gähler–M.)

Let ϱ be a primitive, bijective, Abelian substitution. Then, for a.e. $k \in \mathbb{R}$, all Lyapunov exponents of $B(k)$ are strictly less than $\log \sqrt{\lambda}$. Consequently, the corresponding diffraction $\widehat{\gamma}$ does not contain an absolutely continuous component.

Squared-Fibonacci inflation

ϱ_{F}^2 :



$$B(k) = \begin{pmatrix} 1 + \mathrm{e}^{2\pi\mathrm{i}(1+\tau)k} & 1 \\ \mathrm{e}^{2\pi\mathrm{i}\tau k} & \mathrm{e}^{2\pi\mathrm{i}\tau k} \end{pmatrix}$$

$f : k \mapsto \lambda k$ $\quad X = \mathbb{R}$ $\quad \log^{+} \| B(k) \|$

Proposition

Let ϱ be a primitive one-dimensional inflation rule whose expansion factor λ has algebraic degree r. Under some mild assumptions, there exists an ergodic cocycle $\widetilde{B}$ on $\mathbb{T}^r$, such that

$f \longrightarrow Q_f : \mathbb{T}^r \circlearrowleft$

$$B^{(n)}(k) = \widetilde{B}^{(n)}(x_1, \ldots, x_r)\big|_{x_1=k,\, x_2=\theta_1 k,\, \ldots,\, x_r=\theta_{r-1} k}$$

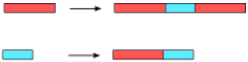
ergodic w.r.t.

where $\{1, \theta_1, \ldots, \theta_{r-1}\}$ are rationally independent.

M_f

A Kingman-type estimate

ϱ_{F}^{2} :



$$B(k) = \begin{pmatrix} 1 + \mathrm{e}^{2\pi \mathrm{i}(1+\tau)k} & 1 \\ \mathrm{e}^{2\pi \mathrm{i}\tau k} & \mathrm{e}^{2\pi \mathrm{i}\tau k} \end{pmatrix}$$

Proposition

Let ϱ be a primitive one-dimensional inflation. Under some mild assumptions,

$$\chi^{B}(k) \leqslant \frac{1}{N} \int_{\mathbb{T}^r} \log \| \widetilde{B}^{(N)}(x) \| \mathrm{d}\mu_{\mathsf{H}}(x),$$

for a.e. $k \in \mathbb{R}$ and for all $N \geqslant 1$.

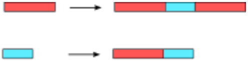
Strategy

1. Compute the Fourier matrix $B(k)$
2. Check if $\det(B(k)) \neq 0$
3. Compute the finite approximants of the Kingman-type bound via numerical integration
4. Wait until it goes below the threshold $(\log\sqrt{\lambda})$ for some N

If there IS such an $N \Longrightarrow$ no ac diffraction

Squared-Fibonacci inflation

ϱ_{F}^2 :



$$B(k) = \begin{pmatrix} 1 + \mathrm{e}^{2\pi\mathrm{i}(1+\tau)k} & 1 \\ \mathrm{e}^{2\pi\mathrm{i}\tau k} & \mathrm{e}^{2\pi\mathrm{i}\tau k} \end{pmatrix}$$

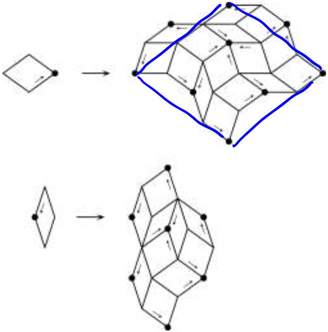
N	1	2	3	4
$\frac{1}{N}\int_{\mathbb{T}^2} \log \lVert \widetilde{B}^{(N)}(x) \rVert_{\mathrm{F}}^2$	1.5668	1.1091	0.8776	0.7409

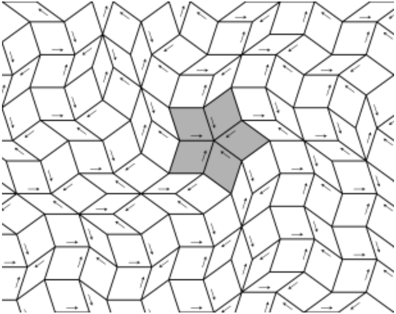
Table: Numerical values for upper bounds for $2\chi^B(k)$ for ϱ_{F}^2. Here, $\lVert \cdot \rVert_{\mathrm{F}}$ stands for the Frobenius norm and $\log \lambda \approx 0.9624$.

$\Longrightarrow$ $\hat{\gamma}$ does not have an a.c. component!

Higher-dimensional example

Godrèche–Lançon–Billard tiling

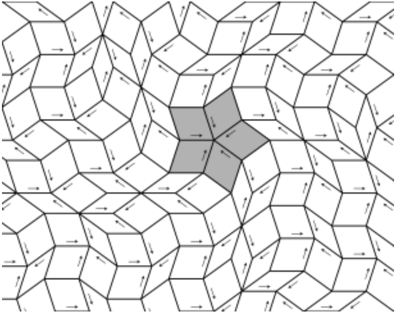


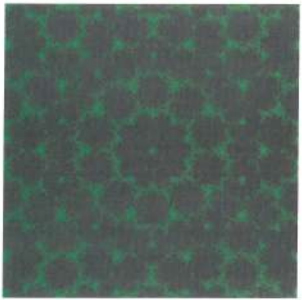


Here, $\lambda = \frac{5+\sqrt{5}}{2}$ is non-Pisot.

Higher-dimensional example

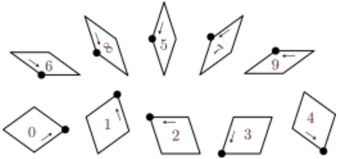
Godrèche–Lançon–Billard tiling

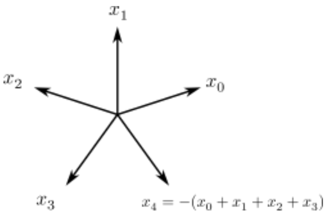




Godrèche C and Lançon F, A simple example of a non-Pisot tiling with five-fold symmetry, *J. Phys. I. France* **2** (1992)

Higher-dimensional example





Set of control points: $\Lambda \subset \mathbb{Z}[\zeta_5]$

Higher-dimensional example

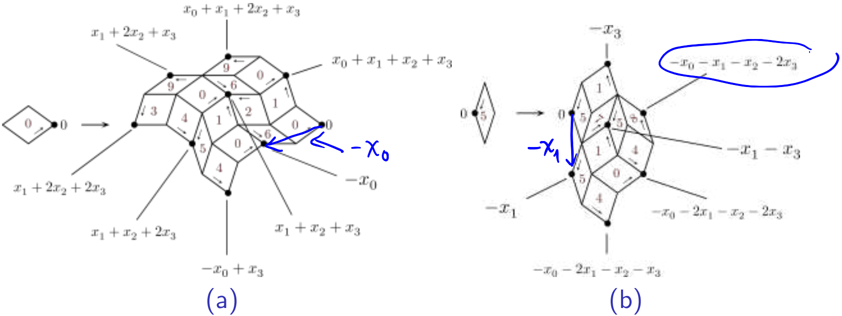
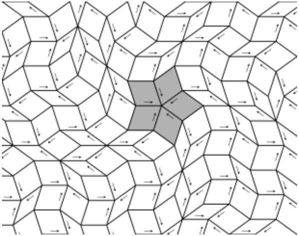


Figure: Level-1 supertiles of the GLB tiling

Higher-dimensional example

Godrèche–Lançon–Billard tiling



$\hat{\gamma} = I_0 \delta_0 + \hat{\gamma}_{sc}$

N	7	8	9	10	11	12
$\frac{1}{N}\int_{\mathbb{T}^4} \log \lVert \widetilde{B}^{(N)}(.) \rVert_{\mathrm{F}}^2$	2.571	2.517	2.474	2.440	2.411	2.387

Table: Numerical upper bounds for χ^B for the GLB inflation ϱ_{GLB}. Here $\log\lambda \approx 2.571862$.

Examples with AC

Necessary condition for presence of ac diffraction

Theorem (Berlinkov–Solomyak, 2019)

Let ϱ be a primitive substitution of constant length L. If it admits a Lebesgue component in its dynamical spectrum, then its substitution matrix must have an eigenvalue of modulus $\sqrt{L}$.

Necessary condition for presence of ac diffraction

Theorem (Baake–Gähler-M., 2019)

Let ϱ be a primitive inflation rule. Under some mild assumptions, one has $\chi^B(k) \leqslant \log\sqrt{\lambda}$ for a.e. $k \in \mathbb{R}^d$.

Corollary (Baake–Gähler-M., 2019)

Let ϱ be a primitive inflation rule. Under some mild assumptions, if its diffraction admits a Lebesgue component, then its Fourier cocycle must have $\chi^B(k) = \log\sqrt{\lambda}$ for a set of $k \in \mathbb{R}^d$ of positive Lebesgue measure.

Examples with AC

Rudin–Shapiro substitution

$$\varrho_{\mathrm{RS}} : a \mapsto ab,\ b \mapsto a\bar{b},\ \bar{b} \mapsto \bar{a}b,\ \bar{a} \mapsto \bar{a}\bar{b}$$

$$B(k) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ \mathrm{e}^{2\pi \mathrm{i} k} & 0 & \mathrm{e}^{2\pi \mathrm{i} k} & 0 \\ 0 & \mathrm{e}^{2\pi \mathrm{i} k} & 0 & \mathrm{e}^{2\pi \mathrm{i} k} \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$\det(B(k)) = 0$!

$S B(k) S^{-1} =$

Examples with AC

Rudin–Shapiro substitution

$\varrho_{\mathrm{RS}} : a \mapsto ab,\ b \mapsto a\bar{b},\ \bar{b} \mapsto \bar{a}b,\ \bar{a} \mapsto \bar{a}\bar{b}$

$$B(k) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ \mathrm{e}^{2\pi i k} & 0 & \mathrm{e}^{2\pi i k} & 0 \\ 0 & \mathrm{e}^{2\pi i k} & 0 & \mathrm{e}^{2\pi i k} \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\boxed{\chi^{B_2} = \log\sqrt{2} \text{ for all } k \in \mathbb{R}}$$

$$B'(k) = \left(\begin{array}{cc|cc} 1+\mathrm{e}^{2\pi i k} & 0 & \multicolumn{2}{c}{\boldsymbol{0}} \\ -\mathrm{e}^{2\pi i k} & 0 & & \\ \hline \multicolumn{2}{c|}{\boldsymbol{0}} & 1 & 1 \\ & & \mathrm{e}^{2\pi i k} & -\mathrm{e}^{2\pi i k} \end{array}\right)$$

B_2

$\sqrt{2}\,\underbrace{U(k)}_{\text{unitary!}}$

Examples with AC

Example (Chan–Grimm–Short, 2018)

A nine-letter example: ϱ_9

$$
\begin{array}{lll}
0 \mapsto 012, & \bar{0} \mapsto \bar{0}\bar{1}\bar{2}, & \bar{\bar{0}} \mapsto \bar{\bar{0}}\bar{\bar{1}}\bar{\bar{2}}, \\
1 \mapsto 0\bar{1}\bar{\bar{2}}, & \bar{1} \mapsto \bar{0}\bar{\bar{1}}2, & \bar{\bar{1}} \mapsto \bar{\bar{0}}1\bar{2}, \\
2 \mapsto 0\bar{\bar{1}}\bar{2}, & \bar{2} \mapsto \bar{0}1\bar{\bar{2}}, & \bar{\bar{2}} \mapsto \bar{\bar{0}}\bar{1}2
\end{array}
$$

Examples with AC

Proposition (M., 2019)

Let $B(k)$ be the Fourier matrix of ϱ_9. One has

(1) For $k \in \mathbb{R}$, all $B(k)$ are simultaneously block diagonalisable into

$$B'(k) = \left(\begin{array}{c|c|c} Z_1' & \boldsymbol{O} & \boldsymbol{O} \\ \hline \boldsymbol{O} & Z_2' & \boldsymbol{O} \\ \hline \boldsymbol{O} & \boldsymbol{O} & Z_3'. \end{array}\right), \quad \log\sqrt{\lambda}$$

where Z_2' and Z_3' are constant multiples of unitary matrices, with multiplier $c = \sqrt{3}$.

(2) The cocycles defined by the blocks Z_2' and Z_3' have degenerate Lyapunov spectrum $\chi^{Z_2'} = \chi^{Z_3'} = \log(\sqrt{3}) = \log\sqrt{\lambda}$.

Examples with AC

Theorem (Frank–M., 202X)

There exists an infinite family of primitive inflation tilings in $\mathbb{R}^d$ with non-trivial absolutely continuous spectral component such that the corresponding Lebesgue multiplicity is exactly the number of Lyapunov exponents of the Fourier cocycle $B(k)$ equal to $\log\sqrt{\lambda}$.

Summary

$$\begin{tikzcd}
\underset{}{\overset{\textbf{Inflation Rule}}{\varrho}} \arrow[rr] \arrow[d] & & \overset{\textbf{Point Set}}{\varLambda} \arrow[r] & \overset{\textbf{Diffraction}}{\widehat{\gamma} = \widehat{\gamma}_{\mathsf{pp}} + \cancel{\widehat{\gamma}_{\mathsf{ac}}} + \widehat{\gamma}_{\mathsf{sc}}} \\
\underset{\textbf{Fourier Matrix}}{B(k)} \arrow[rrr] & & & \underset{\textbf{Lyapunov Exponent}}{\chi^{B}(k)} \arrow[u, Rightarrow]
\end{tikzcd}$$

$\le \log\sqrt{\lambda} - \varepsilon$ for a.e. $k \in \mathbb{R}^d$

Thank you for your attention!