Substitution Tilings and Substitution Tiling Spaces

Lorenzo Sadun

University of Texas

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Outline

1. The chair tiling

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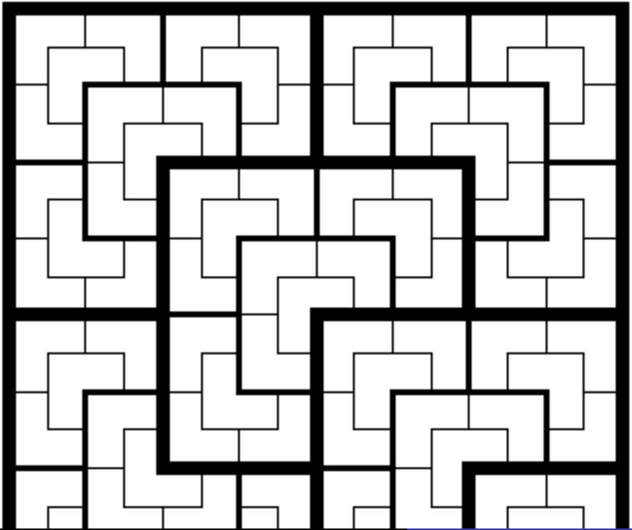
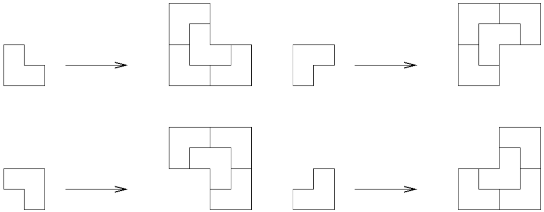
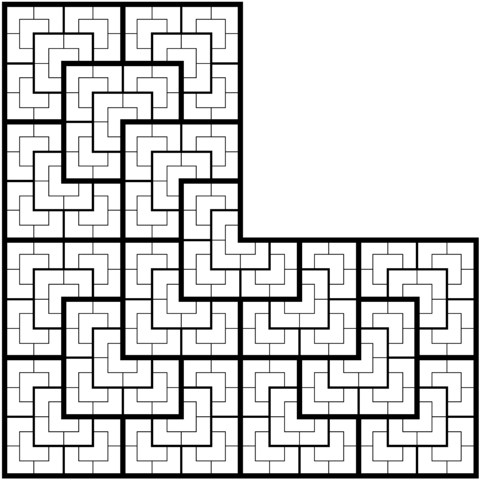
A big patch



Musical chairs



A smaller patch



What are n-supertiles?

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- Fusion perspective: $\sigma^n(t) = \sigma^{n-1}(\sigma(t))$ is what you get when you assemble several $n-1$-supertiles according to the pattern of $\sigma(t)$.

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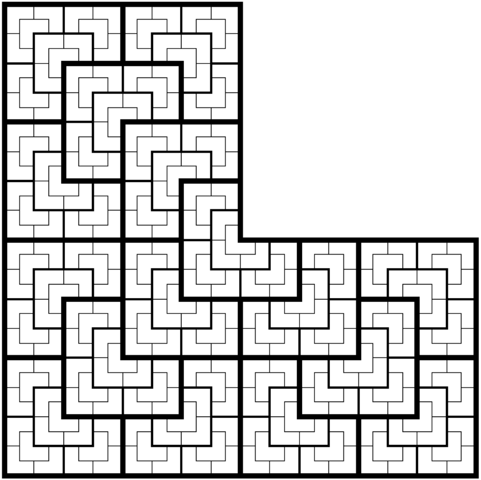

Counting chairs

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- 4 ways to make a 1-supertile including the base tile.
- 4×4 ways to make a 2-supertile.
- 4^n ways to make an $n-$supertile.
- Uncountably many ways to make a tiling.
- Only countably many of those are translates of a given tiling, so there are uncountably many chair tilings, up to translation.

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- T_2 contains lots of copies of P, separated by bounded gaps, so
- T_1 and T_2 have exactly the same local patterns. Same "LI class'.

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Live is easier without shape

- In 1D, the geometry takes care of itself.
- Only need to specify the combinatorics.
- E.g., Fibonacci: $\sigma(a) = ab$, $\sigma(b) = a$.
- Up to overall scale, tile lengths are determined by substitution matrix.

The substitution matrix

- M_{ij} counts the number of t_i tiles in $\sigma(t_j)$.
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- For period-doubling, $\sigma(a) = ab$, $\sigma(b) = aa$, $M = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$.
- Don't confuse M with M^T!

Populations and eigenvectors

- j-th column of M gives population of 1-supertile $\sigma(t_j)$.
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- Population scales as λ^n, where λ is Perron-Frobenius eigenvalue of M
- Relative density of different tiles is given by right-eigenvector.

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- $LM^n e_j$ is the length of an n-supertile of type j.
- LM^n is the row vector of n-supertile lengths.
- $LM = \lambda L$.
- Relative length of tiles is given by leading left-eigenvector of M.

Example: Fibonacci

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- Leading left-eigenvector $(\phi, 1)$. "a" tiles are ϕ times longer than "b" tiles.

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- Leading left-eigenvector $(1, 1)$. Both tiles have the same length.

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- Period-doubling is non-periodic because it has structures at all scales 2^n
- $\ldots abaaabababaaabaa \ldots$
- $\ldots XbXaXbXbXbXaXbXa \ldots$
- $\ldots XYXaXYXbXYXaYXYa \ldots$
- $\ldots XYXZXYXbXYXZXYXa \ldots$

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- Metric depends on choice of origin, but topology doesn't.
- $\lim d(T_i, T_\infty) = 0$ means $T_i \cap K \Rightarrow T_\infty \cap K$ for all compact $K \subset \mathbb{R}^n$.

Group action

$G = \mathbb{R}^n$ acts on tilings. $T - x$ is what you get by translating all the tiles in T by $-x$. (Equivalently, moving the origin in T by $+x$.)

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- Ω is translation invariant. (If $T \in \Omega$ and $x \in \mathbb{R}^n$, then $T - x \in \Omega$.)
- Ω is closed in the tiling metric.

Continuous Hulls

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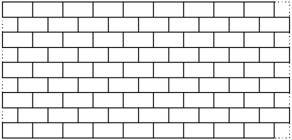
- $\Omega_T = \overline{\{T - x\}}$.
- Orbit closure of T = Tiling space of T = Continuous hull of T.

Hulls of periodic tilings



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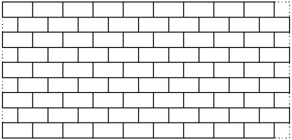

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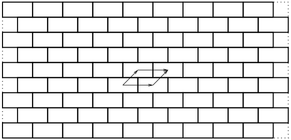

A torus!

A non-periodic example

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- Hull $=$ slinky! Connected but not path-connected.

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- Restrict T to $B_{1/\epsilon}$.
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- Fill out near ∞. Discrete choices.
- Neighborhood $\sim B_\epsilon \times C$.

Description of Ω_T

Theorem

A tiling T' is in Ω_T if and only if every patch of T' is found somewhere in T.

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Dynamical vs. combinatorial properties

- If T has FLC, Ω_T is compact.
- If T has uniform patch frequencies, Ω_T is uniquely ergodic.
- If T is repetitive, Ω_T is minimal.
- Path components of Ω_T are orbits. Typically uncountably many.

Recognizability

Theorem (Mossé, Solomyak)

Let σ be a primitive substitution, and let T be a tiling build from σ. The substitution induces a surjective map $\sigma : \Omega_T \to \Omega_T$. This map is injective if and only if T is non-periodic.

If σ is invertible, we say that the substitution is *recognizable*, or that it has the *unique decomposition property*.

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Inverse limits in general

If $X_0, X_1, \ldots$ are spaces and $\rho_n : X_n \to X_{n-1}$ are continuous maps,

$$X = \varprojlim X_i := \{(x_0, x_1, \ldots) \in \prod X_n | \rho_n(x_n) = x_{n-1} \forall n\}.$$

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X has the product topology. $(x_0, x_1, \ldots)$ is close to $(y_0, y_1, \ldots)$ if $x_i \approx y_i$ for all $i \leq N$. I.e. if $x_N \approx y_N$.

Dyadic Solenoid

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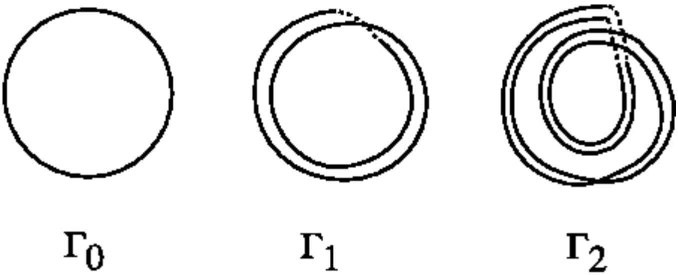

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- Many different schemes: different details, same strategy.
- $\varprojlim \Gamma_n$ = consistent instructions for tiling bigger and bigger regions, i.e. instructions for a complete tiling.
- So how do instructions for partial tilings turn into a CW complex?!

Anderson-Putnam Complex

To place a tile at the origin, need:

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- $\Gamma_n = \coprod \sigma^n(t_i)/\sim$ is the Anderson-Putnam complex.
- Γ_n looks just like Γ_0, only a factor of λ^n bigger.

Anderson-Putnam Theorem

Theorem

If σ is a primitive and recognizable substitution that "forces the border", then

- Ω_σ *is homeomorphic to* $\varprojlim(\Gamma_0, \sigma)$.
- $\check{H}^k(\Omega_\sigma) = \varinjlim(H^k(\Gamma_0), \sigma^*)$.

Living without border forcing

Theorem (Anderson-Putnam)

If σ doesn't force the border, then we can get an equivalen substitution that does force the border by "collaring".

Theorem (Barge-Diamond)

If a 1D substitution meets some conditions that are a lot weaker than forcing the border, then Ω_σ has the same cohomology as $\varprojlim(\Gamma_0, \sigma)$, even though they aren't homeomorphic.

The BD theorem applies to Fibonacci and period-doubling.

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Dyadic solenoid

$H^1(S^1) = \mathbb{Z}$. Substitution map induces multiplication by 2.
$\check{H}^1(S) = \mathbb{Z}[1/2]$

Fibonacci

[Switch to doc-cam]

Fibonacci results

- Γ_0 is the wedge of 2 circles, obtained by identifying all of the endpoints of the a and b tiles.
- $H^1(\Gamma_0) = \mathbb{Z}^2$
- σ^* expressed by invertible matrix $M^T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.
- $\check{H}^1(\Omega_{Fib}) = \mathbb{Z}^2$.