Substitution Tilings and Substitution Tiling Spaces

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Outline

1. The chair tiling
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2. 1D, where a word is worth 1000 pictures
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3. Tiling spaces
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What are $n$-supertiles?

- Traditional view: $\sigma^n(t) = \sigma(\sigma^{n-1}(t))$ is what you get when you apply the substitution to every tile in an $n-1$-supertile.
What are $n$-supertiles?

- Traditional view: $\sigma^n(t) = \sigma(\sigma^{n-1}(t))$ is what you get when you apply the substitution to every tile in an $n-1$-supertile.
- Fusion perspective: $\sigma^n(t) = \sigma^{n-1}(\sigma(t))$ is what you get when you assemble several $n-1$-supertiles according to the pattern of $\sigma(t)$. 
A smaller patch
Counting chairs

- How many ways are there to extend a chair tile to a tiling?
Counting chairs

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- 4 ways to make a 1-supertile including the base tile.
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- $4 \times 4$ ways to make a 2-supertile.
- $4^n$ ways to make an $n$—supertile.
Counting chairs

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- 4 ways to make a 1-supertile including the base tile.
- $4 \times 4$ ways to make a 2-supertile.
- $4^n$ ways to make an $n$—supertile.
- Uncountably many ways to make a tiling.
- Only countably many of those are translates of a given tiling, so there are uncountably many chair tilings, up to translation.
How different are different chair tilings?

Let $T_1$ and $T_2$ be different chair tilings.

- Every patch $P$ of $T_1$ lives in an $n$-supertile $S_n$. 
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Let $T_1$ and $T_2$ be different chair tilings.

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Let $T_1$ and $T_2$ be different chair tilings.

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- $T_2$ contains lots of copies of $P$, separated by bounded gaps, so
- $T_1$ and $T_2$ have exactly the same local patterns. Same “LI class’.”
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Live is easier without shape

- In 1D, the geometry takes care of itself.
- Only need to specify the combinatorics.
- E.g., Fibonacci: $\sigma(a) = ab$, $\sigma(b) = a$.
- Up to overall scale, tile lengths are determined by substitution matrix.
The substitution matrix

- $M_{ij}$ counts the number of $t_i$ tiles in $\sigma(t_j)$.
- For Fibonacci, $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Don't confuse $M$ with $M^T$!
The substitution matrix

- $M_{ij}$ counts the number of $t_i$ tiles in $\sigma(t_j)$.
- For Fibonacci, $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.
- For period-doubling, $\sigma(a) = ab$, $\sigma(b) = aa$, $M = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$.
- Don’t confuse $M$ with $M^T$!
Populations and eigenvectors

- $j$-th column of $M$ gives population of 1-supertile $\sigma(t_j)$.
- $j$-th column of $M^n$ gives population of $\sigma^n(t_j)$. 
Populations and eigenvectors

- $j$-th column of $M$ gives population of 1-supertile $\sigma(t_j)$.
- $j$-th column of $M^n$ gives population of $\sigma^n(t_j)$.
- Population scales as $\lambda^n$, where $\lambda$ is Perron-Frobenius eigenvalue of $M$.
- Relative density of different tiles is given by right-eigenvector.
Let $L = (L_1, L_2, \ldots, L_m)$ be the length of the tiles.
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- $LM^n e_j$ is the length of an $n$-supertile of type $j$.
- $LM^n$ is the row vector of $n$-supertile lengths.
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- $LM^n e_j$ is the length of an $n$-supertile of type $j$.
- $LM^n$ is the row vector of $n$-supertile lengths.
- $LM = \lambda L$.

Relative length of tiles is given by leading left-eigenvector of $M$. 
Example: Fibonacci

- $\sigma(a) = ab$, $\sigma(b) = a$
- $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, eigenvalues $\lambda_1 = \phi$, $\lambda_2 = 1 - \phi$. 
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- $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, eigenvalues $\lambda_1 = \phi$, $\lambda_2 = 1 - \phi$.
- Leading right-eigenvector $\begin{pmatrix} \phi \\ 1 \end{pmatrix}$. “a” tiles outnumber “b” tiles $\phi : 1$. 
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Example: Fibonacci

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- Leading right-eigenvector $\begin{pmatrix} \phi \\ 1 \end{pmatrix}$. “a” tiles outnumber “b” tiles $\phi : 1$.
- Leading left-eigenvector $(\phi, 1)$. “a” tiles are $\phi$ times longer than “b” tiles.
Example: Period-doubling

- $\sigma(a) = ab$, $\sigma(b) = aa$.
- $M = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$. Eigenvalues 2 and $-1$.\[\text{Leading right-eigenvector} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{. "a" tiles outnumber "b" tiles 2:1.}\]

- Leading left-eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Both tiles have the same length.
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Example: Period-doubling

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- $M = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$. Eigenvalues 2 and $-1$.
- Leading right-eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. “a” tiles outnumber “b” tiles 2:1.
- Leading left-eigenvector (1, 1). Both tiles have the same length.
Non-periodicity

- Fibonacci is non-periodic since $\phi$ is irrational.
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Period-doubling is non-periodic because it has structures at all scales $2^n$

... $abaabababaaabaa$...

... $XbXaXbXbXaXbXa$...

... $XYXaXYXbXYXaYXYa$...

... $XYZXYYbXYZXYYXa$...

... $XYXaXbXYXaYXYa$...
Define metric on set of all tilings with given tile set.

- \( d(T, T') \leq \epsilon \) if \( T \) and \( T' \) agree on \( B_{1/\epsilon}(0) \), up to \( \epsilon \)-translation.
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- \((d(T, T') = \min(1, \inf\{\epsilon | T \text{ and } T' \text{ agree} \ldots\})\).)

Metric depends on choice of origin, but topology doesn't. 

\[ \lim_{i \to \infty} d(T_i, T_\infty) = 0 \] means \( T_i \cap K \Rightarrow T_\infty \cap K \) for all compact \( K \subset \mathbb{R}^n \).
Define metric on set of all tilings with given tile set.

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- Metric depends on choice of origin, but topology doesn’t.
Tiling metric

Define metric on set of all tilings with given tile set.

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- Metric depends on choice of origin, but topology doesn’t.
- $\lim d(T_i, T_\infty) = 0$ means $T_i \cap K \Rightarrow T_\infty \cap K$ for all compact $K \subset \mathbb{R}^n$. 
Group action

$G = \mathbb{R}^n$ acts on tilings. $T - x$ is what you get by translating all the tiles in $T$ by $-x$. (Equivalently, moving the origin in $T$ by $+x$.)
A tiling space is a set $\Omega$ of tilings on a fixed tile set with fixed adjacency rules such that

- $\Omega$ is translation invariant. (If $T \in \Omega$ and $x \in \mathbb{R}^n$, then $T - x \in \Omega$.)
- $\Omega$ is closed in the tiling metric.
Definition

A tiling space is a set \( \Omega \) of tilings on a fixed tile set with fixed adjacency rules such that

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Continuous Hulls

Simplest way to build a tiling space:

1. Start with an FLC tiling $T$. 

2. Consider the set \( \Omega_T = \{ T - x \} \) of translates of $T$.

Orbit closure of $T$ = Tiling space of $T$ = Continuous hull of $T$. 

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Substitution Tilings and Substitution Tiling Spaces
Continuous Hulls

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3. $\Omega_T = \overline{\{T - x\}}$.
4. Orbit closure of $T = \text{Tiling space of } T = \text{Continuous hull of } T$. 
Hulls of periodic tilings

What is $\Omega_T$?
Hulls of periodic tilings

What is $\Omega_T$?

A torus!
A non-periodic example

\[ T = \ldots AAAAA.BBBB \ldots \triangleq A^\infty . B^\infty . \]

What is \( \Omega_T \)?
A non-periodic example

\[ T = \ldots AAAAA.BBBB\ldots \overset{\equiv}{=} A^\infty.B^\infty. \]

What is \( \Omega_T \)?

- Orbit of \( T \) is copy of \( \mathbb{R} \).
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What is \( \Omega_T \)?

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- As \( x \to \infty \), \( T - x \) approaches periodic \( \ldots BBBBB \ldots \) tiling. Limiting circle.
- Hull = slinky! Connected but not path-connected.
If $T$ is a tiling, what does an $\epsilon$-neighborhood of $T$ in $\Omega_T$ look like?
Local topology

If $T$ is a tiling, what does an $\epsilon$-neighborhood of $T$ in $\Omega_T$ look like?

- Restrict $T$ to $B_{1/\epsilon}$.
- Move $T$ by up to $\epsilon$: continuous degrees of freedom.
- Fill out near $\infty$. Discrete choices.
- Neighborhood $\sim B_\epsilon \times C$. 
Theorem

A tiling $T'$ is in $\Omega_T$ if and only if every patch of $T'$ is found somewhere in $T$. 
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Dynamical vs. combinatorial properties

- If $T$ has FLC, $\Omega_T$ is compact.
- If $T$ has uniform patch frequencies, $\Omega_T$ is uniquely ergodic.
- If $T$ is repetitive, $\Omega_T$ is minimal.
- Path components of $\Omega_T$ are orbits. Typically uncountably many.
Theorem (Mossé, Solomyak)

Let $\sigma$ be a primitive substitution, and let $T$ be a tiling built from $\sigma$. The substitution induces a surjective map $\sigma : \Omega_T \to \Omega_T$. This map is injective if and only if $T$ is non-periodic.

If $\sigma$ is invertible, we say that the substitution is recognizable, or that it has the unique decomposition property.
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Inverse limits in general

If $X_0, X_1, \ldots$ are spaces and $\rho_n : X_n \rightarrow X_{n-1}$ are continuous maps,

$$X = \lim \leftarrow X_i := \{(x_0, x_1, \ldots) \in \prod X_n | \rho_n(x_n) = x_{n-1} \forall n \}.$$
Inverse limits in general

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$X_n$ is called $n$-th approximant to $X$, since $x_n$ determines $(x_0, \ldots, x_n)$.
Inverse limits in general

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$X_n$ is called the $n$-th approximant to $X$, since $x_n$ determines $(x_0, \ldots, x_n)$.

$X$ has the product topology. $(x_0, x_1, \ldots)$ is close to $(y_0, y_1, \ldots)$ if $x_i \approx y_i$ for all $i \leq N$. I.e. if $x_N \approx y_N$. 
Dyadic Solenoid

Example of inverse limit space. Take

$$X_n = \mathbb{R}/(2^n \mathbb{Z}) \simeq S^1.$$
Dyadic Solenoid

Example of inverse limit space. Take

- $X_n = \mathbb{R}/(2^n \mathbb{Z}) \cong S^1$.
- $\rho_n$ induced by identity on $\mathbb{R}$. Winds $X_n$ twice around $X_{n-1}$. 
Dyadic Solenoid

Example of inverse limit space. Take

- \( X_n = \mathbb{R}/(2^n \mathbb{Z}) \cong S^1 \).
- \( \rho_n \) induced by identity on \( \mathbb{R} \). Winds \( X_n \) twice around \( X_{n-1} \).
Tiling spaces are inverse limits

- CW complex $\Gamma_n$ describes tiling out to distance that grows with $n$. 
Tiling spaces are inverse limits

- CW complex $\Gamma_n$ describes tiling out to distance that grows with $n$.
- $\rho_n$ is forgetful map.
Tiling spaces are inverse limits

- CW complex $\Gamma_n$ describes tiling out to distance that grows with $n$.
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- Many different schemes: different details, same strategy.
Tiling spaces are inverse limits

- CW complex $\Gamma_n$ describes tiling out to distance that grows with $n$.
- $\rho_n$ is forgetful map.
- Many different schemes: different details, same strategy.
- $\lim \leftarrow \Gamma_n =$ consistent instructions for tiling bigger and bigger regions, i.e. instructions for a complete tiling.
Tiling spaces are inverse limits

- CW complex $\Gamma_n$ describes tiling out to distance that grows with $n$.
- $\rho_n$ is forgetful map.
- Many different schemes: different details, same strategy.
- $\lim \Gamma_n = \text{consistent instructions for tiling bigger and bigger regions, i.e. instructions for a complete tiling.}$
- So how do instructions for partial tilings turn into a CW complex?!
To place a tile at the origin, need:
To place a tile at the origin, need:

- Choice of tile type $t_i$. 

**Anderson-Putnam Complex**

The chair tiling

1D, where a word is worth 1000 pictures

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Substitution Tilings and Substitution Tiling Spaces
Anderson-Putnam Complex

To place a tile at the origin, need:

- Choice of tile type \( t_i \).
- Choice of point in \( t_i \) to associate with origin.
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- What if origin is on boundary of 2 (or more tiles)? Identify!
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- Choice of tile type $t_i$.
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- What if origin is on boundary of 2 (or more) supertiles? Identify!
- $\Gamma_n = \bigsqcup \sigma^n(t_i)/\sim$ is the Anderson-Putnam complex.
- $\Gamma_n$ looks just like $\Gamma_0$, only a factor of $\lambda^n$ bigger.
Theorem

If $\sigma$ is a primitive and recognizable substitution that “forces the border”, then

- $\Omega_\sigma$ is homeomorphic to $\lim_{\leftarrow}(\Gamma_0, \sigma)$.
- $\tilde{H}^k(\Omega_\sigma) = \lim_{\to}(H^k(\Gamma_0), \sigma^*)$. 
Living without border forcing

**Theorem (Anderson-Putnam)**

*If* \( \sigma \) *doesn’t force the border, then we can get an equivalent substitution that does force the border by “collaring”.*

**Theorem (Barge-Diamond)**

*If a 1D substitution meets some conditions that are a lot weaker than forcing the border, then* \( \Omega_\sigma \) *has the same cohomology as* \( \lim_{\leftarrow} (\Gamma_0, \sigma) \), *even though they aren’t homeomorphic.*

The BD theorem applies to Fibonacci and period-doubling.
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Dyadic solenoid

\[ H^1(S^1) = \mathbb{Z}. \]  Substitution map induces multiplication by 2.

\[ \tilde{H}^1(S) = \mathbb{Z}[1/2] \]
Fibonacci

[Switch to doc-cam]
Fibonacci results

- $\Gamma_0$ is the wedge of 2 circles, obtained by identifying all of the endpoints of the $a$ and $b$ tiles.
- $H^1(\Gamma_0) = \mathbb{Z}^2$
- $\sigma^*$ expressed by invertible matrix $M^T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.
- $\check{H}^1(\Omega_{Fib}) = \mathbb{Z}^2$. 