A geometric approach to $q$-Painlevé equations and their hypergeometric solutions

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Topic of this talk:
- Painlevé and discrete Painlevé equations
- Affine Weyl group symmetries
- Formulations of discrete Painlevé equations:
  Point configuration space and geometry of plane curves on $\mathbb{P}^2$
- Hypergeometric solutions:
  beyond the Gauss hypergeometric function?
Painlevé and Discrete Painlevé Equations (1)

List of Painlevé Equations

(P_1) \[ y'' = 6y^2 + t, \]

(P_II) \[ y'' = 2y^3 + ty + \alpha, \]

(P_III) \[ y'' = \frac{1}{y}(y')^2 - \frac{1}{t}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}, \]

(P_IV) \[ y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}, \]

(P_V) \[ y'' = \left( \frac{1}{2y} + \frac{1}{y-1} \right)(y')^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2} \left( \alpha y + \frac{\beta}{y} \right) + \gamma \frac{y}{t} + \delta \frac{y(y+1)}{y-1}, \]

(P_VI) \[ y'' = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right)(y')^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right)y' + \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left[ \alpha + \beta \frac{t}{y^2} + \gamma \frac{t-1}{(y-1)^2} + \delta \frac{t(t-1)}{(y-t)^2} \right]. \]
**Painlevé and Discrete Painlevé Equations (2)**

**Painlevé Equations:**

✔ 2nd order nonlinear ODE with the “Painlevé property”
   = No movable branch points
   (6(8) equations)

✔ “Space of initial values” :
   defining manifold (Okamoto, Takano) ← blow-up of $\mathbb{P}^2$

✔ Symmetries:
   Affine Weyl group = reflection + translation

✔ Solutions:
   ☀ Transcendental in general
      (Nishioka, Umemura, Noumi, Okamoto,...)
   ☀ Particular solutions for special values of parameters
      (1) *Hypergeometric solutions*
         (classification finished: Okamoto, Umemura, Noumi, Watanabe)
      (2) *Algebraic solutions* (most cases rational solutions)
         (classification finished except for $P_{VI}$: Murata, Umemura, Kitaev, Watanabe, Dubrovin-Mazocco)
Painlevé and Discrete Painlevé Equations (3)

List of Some Discrete Painlevé Equations:

\[ \text{dP}_1: \quad x_{n+1} + x_n + x_{n-1} = \frac{an + b}{x_n} + c \]
\[ x_{n+1}x_\sigma x_{n-1} = aq^n x_n + b, \quad \sigma = 0, 1, 2 \]

\[ \text{dP}_II: \quad x_{n+1} + x_{n-1} = \frac{(an + b)x_n + c}{1 - x_n^2} \]
\[ (x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = aq^{2n} \frac{x_n}{x_n + q^n} \]

\[ \text{dP}_{III}: \quad x_{n+1}x_{n-1} = \frac{ab(x_n + cq^n)(x_n + dq^n)}{(x_n + a)(x_n + b)} \]

**Type \( E_7^{(1)} \)**

\[
\begin{align*}
\frac{(\mathcal{g}f - t)(gf - t^2)}{(gf - 1)(gf - 1)} &= \frac{(f - b_1 t)(f - b_2 t)(f - b_3 t)(f - b_4 t)}{(f - b_5)(f - b_6)(f - b_7)(f - b_8)} , \\
\frac{(gf - 1)(gf - 1)}{(gf + 1)(gf + 1)} &= \frac{(g - t/b_1)(g - t/b_2)(g - t/b_3)(g - t/b_4)}{(g - 1/b_5)(g - 1/b_6)(g - 1/b_7)(g - 1/b_8)} , \\
\tilde{t} &= qt, \quad b_1b_2b_3b_4 = q, \quad b_5b_6b_7b_8 = 1.
\end{align*}
\]

**Type \( E_6^{(1)} \):**

\[
\begin{align*}
\frac{(\mathcal{g}f - 1)(gf - 1)}{gf - 1} &= t^2 \frac{(f - b_1)(f - b_2)(f - b_3)(f - b_4)}{(f - b_5)(f - t/b_6)}, \\
\frac{gf - 1}{gf - 1} &= t^2 \frac{(g - 1/b_1)(g - 1/b_2)(g - 1/b_3)(g - 1/b_4)}{(g - b_5 t)(g - t/b_6)} , \\
\tilde{t} &= qt, \quad b_1b_2b_3b_4 = 1.
\end{align*}
\]

**Type \((A_2 + A_1)^{(1)}(qP_{III})\)**

\[
\begin{align*}
\bar{g}gf &= b_0 \frac{1 + a_0 f}{a_0 t + f} , \\
\bar{g}f &= b_0 \frac{a_1 / t + g}{1 + ga_1 / t} , \\
\tilde{t} &= qt.
\end{align*}
\]
Discrete Painlevé Equations:

✔ 2nd order nonlinear ordinary DIFFERENCE equations with the “singularity confinement property” (MANY equations)

✔ Space of initial values” :
  defining manifold (Sakai) ← blow-up of $\mathbb{P}^2$

✔ Symmetries:
  Affine Weyl group = reflection + translation

✔ Solutions:
  ☛ Transcendence?
  ☛ Particular solutions for special values of parameters

  (1) Hypergeometric solutions

  (2) Algebraic solutions: almost nothing has been done
**Particular solutions: hypergeometric solutions**

\[
P_{II} : \quad \frac{d^2 y}{dt^2} = 2y^3 + ty + \alpha
\]

\[
dP_{II} : \quad x_{n+1} + x_{n-1} = \frac{(\alpha n + \beta)x_n + \gamma}{1 - x_n^2}
\]

**Hypergeometric solution (Riccati solution)**

Riccati equation \(\rightarrow\) linearization

\[
y' = a(t)y^2 + b(t)y + c(t)
\]

\[
\frac{dy}{dt} = -y^2 + \frac{t}{2} \quad \rightarrow \quad \alpha = -\frac{1}{2}
\]

\[
y = \frac{d}{dt} \log f, \quad \frac{d^2 f}{dt^2} = -\frac{t}{2}f
\]

\[
x_{n+1} = \frac{a_n x_n + b_n}{c_n x_n + d_n}
\]

\[
x_{n+1} = \frac{x_n - (pn + q)}{1 + x_n} \quad \rightarrow \quad \alpha = 2p, \quad \beta = -p + 2q + 2, \quad \gamma = -p
\]

\[
x_n = \frac{g_{n+1} - g_n}{g_n}, \quad g_{n+2} - 2g_{n+1} + g_n = -(pn + q)g_n
\]
Affine Weyl group symmetry: $P_{IV}$

$$P_{IV} : \quad y'' = \frac{y'^2}{2y} + \frac{3}{2}y^3 - 2ty + \left(\frac{t^2}{2} - \alpha_0 + \alpha_2\right)y - \frac{\alpha_1^2}{2y}$$

\[\uparrow\]

"symmetric form" of $P_{IV}$ (Noumi-Yamada, 1998)

\[
\begin{align*}
\varphi_0' &= \varphi_0(\varphi_1 - \varphi_2) + \alpha_0 \\
\varphi_1' &= \varphi_1(\varphi_2 - \varphi_0) + \alpha_1 \\
\varphi_2' &= \varphi_2(\varphi_0 - \varphi_1) + \alpha_2
\end{align*}
\]

$y = \varphi_1$, normalization: $\alpha_0 + \alpha_1 + \alpha_2 = 1$, $\varphi_0 + \varphi_1 + \varphi_2 = t$

Bäcklund transformations $s_0, s_1, s_2, \pi$

\[
\begin{align*}
s_0(\alpha_0) &= -\alpha_0 & s_0(\alpha_1) &= \alpha_1 + \alpha_0 & s_0(\alpha_2) &= \alpha_2 + \alpha_0 \\
s_1(\alpha_0) &= \alpha_0 + \alpha_1 & s_1(\alpha_1) &= -\alpha_1 & s_1(\alpha_2) &= \alpha_2 - \alpha_1 \\
s_2(\alpha_0) &= \alpha_0 + \alpha_2 & s_2(\alpha_1) &= \alpha_1 + \alpha_2 & s_2(\alpha_2) &= -\alpha_2
\end{align*}
\]

\[
\begin{align*}
s_0(\varphi_0) &= \varphi_0 & s_0(\varphi_1) &= \varphi_1 - \frac{\alpha_0}{\varphi_0} & s_0(\varphi_2) &= \varphi_2 + \frac{\alpha_0}{\varphi_0} \\
s_1(\varphi_0) &= \varphi_0 + \frac{\alpha_1}{\varphi_1} & s_1(\varphi_1) &= \varphi_1 & s_1(\varphi_2) &= \varphi_2 - \frac{\alpha_0}{\varphi_0} \\
s_2(\varphi_0) &= \varphi_0 - \frac{\alpha_2}{\varphi_2} & s_1(\varphi_1) &= \varphi_1 + \frac{\alpha_2}{\varphi_1} & s_2(\varphi_2) &= \varphi_2
\end{align*}
\]

$$\pi(\varphi_i) = \varphi_{i+1}, \quad \pi(\alpha_i) = \alpha_{i+1}, \quad i \in \mathbb{Z}/3\mathbb{Z}$$
**Affine Weyl group:** \( \langle s_0, s_1, s_2, \pi \rangle = \tilde{W}(A_2^{(1)}) \)

\[
s_i^2 = 1, \quad s_i s_j = s_j s_i, \quad s_i s_j s_i = s_j s_i s_j, \quad \pi^3 = 1, \quad s_{i+1} \pi = \pi s_i
\]

**Translations in the parameter space:**

\[
T_1 = \pi s_1 s_2, \quad T_2 = s_1 \pi s_2, \quad T_0 = s_1 s_2 \pi \quad \rightarrow \quad T_i T_j = T_j T_i, \quad T_0 T_1 T_2 = 1
\]

\[
T_1(\{\alpha_0, \alpha_1, \alpha_2\}) = \{\alpha_0 + 1, \alpha_1 - 1, \alpha_2\}
\]

\[
T_2(\{\alpha_0, \alpha_1, \alpha_2\}) = \{\alpha_0, \alpha_1 + 1, \alpha_2 - 1\}
\]

\[
T_0(\{\alpha_0, \alpha_1, \alpha_2\}) = \{\alpha_0 - 1, \alpha_1, \alpha_2 + 1\}
\]

**A discrete Painlevé II** \( (T_1^\mu(\varphi_i) = \varphi_i(n)) \)

\[
\varphi_1(n + 1) = t - \varphi_0(n) - \varphi_1(n) - \frac{\alpha_0 + n}{\varphi_0(n)}
\]

\[
\varphi_0(n + 1) = t - \varphi_0(n) - \varphi_1(n + 1) + \frac{\alpha_1 - n - 1}{\varphi_1(n + 1)}
\]
**Sakai’s Theory (2001):**
Algebro-geometric theory of Painlevé and discrete Painlevé equations:

☞ Defining manifold of discrete Painlevé equations:
Family of rational surfaces obtained by blow-up of \( \mathbb{P}^2 \) at 9 points.

☞ Action of affine Weyl group:
interchange of points and Cremona transformations

☞ Classification of surfaces:
22 cases obtained by degeneration of points

☞ 8 cases admit continuous flows \( \rightarrow \) Painlevé equations
Continuous flows come from continuous limit of discrete evolutions on “higher” surfaces.

**Degeneration diagram of surfaces:**

![Degeneration Diagram]

Type of time evolution (action of Weyl group):

☞ Elliptic: \( E_8^{(1)} (1) \rightarrow \) Elliptic Painlevé equation

☞ Multiplicative: \( E_8^{(1)} \cdots A_0^{(1)} (10) \rightarrow \) \(q\)-Painlevé equations

☞ Additive: \( E_8^{(1)}, E_7^{(1)}, E_6^{(1)} \) + inside the box (11)
Hypergeometric Solutions:

Coalescence cascade

\[
P_{VI} \rightarrow P_V \rightarrow P_{III} \rightarrow P_{III}^{D(1)} \rightarrow P_{III}^{D(1)} \rightarrow P_{IV} \rightarrow P_{II} \rightarrow P_I
\]

Parabolic Cylinder

Series of \(q\)-Painlevé equations:

\[E_{8}^{(1)} \rightarrow E_{7}^{(1)} \rightarrow E_{6}^{(1)} \rightarrow D_{5}^{(1)} \rightarrow A_{4}^{(1)} \rightarrow (A_2 + A_1)^{(1)} \rightarrow (A_1 + A_{1/2})^{(1)}\]

✔ What kind of hypergeometric functions appear for \(q\)-Painlevé equations, in particular for \(E_{6}^{(1)}, E_{7}^{(8)}, E_{8}^{(1)}\)?
(beyond the Gauss hypergeometric function!)

Basic hypergeometric series:

\[
_{r+s} \varphi_{s} \left( \begin{array}{c}
a_1, \ldots, a_r \\
b_1, \ldots, b_s \\
\end{array} ; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n(q; q)_n} \left[ (-1)^n q \left( \frac{n}{2} \right) \right]^{1+s-r} z^n,
\]

\[
(a_1, \ldots, a_r; q)_n = (a_1; q)_n \cdots (a_r; q)_n,
\]

\[
(a; q)_n = (1 - a)(1 - qa) \cdots (1 - q^{n-1}a)
\]

✔ balanced: \(qa_1a_2 \cdots a_{r+1} = b_1b_2 \cdots b_r, \quad z = q\)

✔ well-poised: \(qa_1 = a_2b_1 = \cdots = a_{r+1}b_r\)

✔ very-well-poised: well-poised + \(a_2 = qa_1^{1/2}, \quad a_3 = -qa_1^{1/2}\)

Very-well-poised basic hypergeometric series \(r+1 W_r\):

\[
r+1 W_r(a_1; a_4, \ldots, a_{r+1}; q, z) = r+1 \varphi_r \left( \begin{array}{c}
a_1, qa_1^{1/2}, -qa_1^{1/2}, a_4 \ldots, a_{r+1} \\
a_1^{1/2}, -a_1^{1/2}, qa_1/a_4, \ldots, qa_1/a_{r+1} \\
\end{array} ; q, z \right)
\]
Formulation of discrete Painlevé equations (1)

“Configuration space of points” on \( \mathbb{P}^2 \):

\((x : y : z)\): homogeneous coordinate of \( \mathbb{P}^2 \)

\[
\mathcal{M}_{3,n} = GL(3) \backslash \left\{ \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ y_1 & y_2 & y_3 & \cdots & y_n \\ z_1 & z_2 & z_3 & \cdots & z_n \end{bmatrix} \right\} / (\mathbb{C}^*)^n.
\]

Birational transformations on \( \mathcal{M}_{3,n} \):

1. \( s_i \) \((i = 1, \ldots, n - 1)\): interchanging \( P_i \) and \( P_{i+1} \)

2. \( s_0 \): standard Cremona transformation with base points \( P_1(1 : 0 : 0), P_2(0 : 1 : 0), P_3(0 : 0 : 1) \)

\[
s_0 : (x : y : z) \mapsto \left( \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right).
\]

\( \langle s_0, s_1, \ldots, s_n \rangle = W(E_n) \): (A. Coble, 1922)

\[
E_n: E_0 \quad \begin{array}{ccccccccccc} & & & & & & & & & & \end{array}
\]

\[
\begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots & n & 1
\end{array}
\]

\[
s_i^2 = 1, \quad s_is_j = s_js_i, \quad s_is_js_i = s_js_is_j,
\]

\[
n = 10 \quad \mathbb{Z}^8 \subset \langle s_0, s_1, \ldots, s_9 \rangle = W(E_9) = W(E_8^{(1)}) \subset W(E_{10})
\]

action of translation subgroup \( \mathbb{Z}^8 \) = Elliptic Painlevé equation

9 points \( P_1, \ldots, P_9 \) = parameters

10th point \( P_{10} \) = dependent variable
Formulation with the pencil of cubic curves (1)

Translation = Addition on moving cubic curve

Example: $T_{89}, C_0$: Cubic curve passing $P_1, \ldots, P_9$

- Determine new points $\overline{P}_8, \overline{P}_9$ by using the addition on $C_0$: $\overline{P}_i = P_i, (i \neq 8, 9), \ P_1 + \cdots + P_8 + \overline{P}_9 = O, \ P_8 + P_9 = \overline{P}_8 + \overline{P}_9$

- Let $C_0: F = 0$ and let the cubic pencil be $\lambda F + \mu G = 0$. Choose $\lambda, \mu$ so that the pencil passes $P_{10}$. Denote this new cubic curve passing through $P_1, \ldots, P_8, \overline{P}_9$ and $P_{10}$ as $C$.

- Determine new point $\overline{P}_{10}$ by using the addition on $C$ as $P_{10} + P_8 = \overline{P}_{10} + \overline{P}_9$
**Comments:**


- 9 points on nodal cubic curve: $q$-Painlevé of type $E_8^{(1)}$

- The same procedure for the stationary cubic gives QRT system (Tsuda)

- The procedure for stationary cubic was first discussed by Manin and shown that it is equivalent to translation.
Description of hypergeometric solution (1)

\(\Leftrightarrow\) Reduction to Riccati Equation:

- Three points among the nine points in the cubic are colinear.
- A point is infinitesimally near to another point
  (The second case is essential only for much degenerate cases)

**Example:** \(P_5, P_6, P_7\) are colinear

Cubic \(C\) is decomposed into a line \(\ell(P_5, P_6, P_7)\) and a conic \(C'(P_2, \ldots, P_4, P_8, \overline{P_9})\).

\[ P_1 + \cdots + P_8 + \overline{P_9} = O, \quad P_5 + P_6 + P_7 = O \rightarrow P_1 + \cdots + P_4 + P_8 + \overline{P_9} = O. \]

\[ \therefore \quad P_{10} \in \ell \rightarrow \overline{P}_{10} \in \ell \quad \ell \text{ is an "invariant divisor"} \]

- In this case, discrete time evolution of \(f\) can be described in terms of linear equation.

![Diagram showing cubic and line decomposition](image-url)
Choose coordinate such that 
\( P_1(1 : 0 : 0), P_2(0 : 1 : 0), P_3(0 : 0 : 1) \), and denote:

\[
P_{10} = f = (f_1 : f_2 : f_3), \quad \overline{P}_{10} = \overline{f} = (\overline{f}_1 : \overline{f}_2 : \overline{f}_3),
\]
\[
P_8 = x = (x_1 : x_2 : x_3), \quad P_9 = \overline{y} = (\overline{y}_1 : \overline{y}_2 : \overline{y}_3),
\]

\[\ell : \quad (a, f) = a_1 f_1 + a_2 f_2 + a_3 f_3 = 0, \quad (a, \overline{f}) = a_1 \overline{f}_1 + a_2 \overline{f}_2 + a_3 \overline{f}_3 = 0\]

Linear equation for \( f \):

\[
\lambda \overline{f} = (a, \overline{y}) D f - (a, D f) \overline{y},
\]
\[
\mu f = (a, x) D^{-1} f - (a, D^{-1} f) x,
\]

\[
D = \text{diag} \left( \frac{x_2 x_3}{\overline{y}_2 \overline{y}_3}, \frac{x_3 x_1}{\overline{y}_3 \overline{y}_1}, \frac{x_1 x_2}{\overline{y}_1 \overline{y}_2} \right), \quad \lambda = (a, x), \quad \mu = (a, \overline{y}).
\]

**GL(3)-invariant linear equation:** \( d_{ijk} = \det[P_i, P_j, P_k] \)

\[
\begin{align*}
\frac{d_{239}d_{125}d_{568}d_{318}}{d_{185}} & \begin{pmatrix} d_{315} & \cdots & d_{23 10} \\ \cdots & \cdots & \cdots \\ d_{318} & \cdots & \cdots \end{pmatrix} - \begin{pmatrix} d_{23 10} \\ \cdots \\ d_{23 10} \end{pmatrix} \\
+ \frac{d_{238}d_{128}d_{569}d_{319}}{d_{189}} & \begin{pmatrix} d_{318} & \cdots & d_{23 10} \\ \cdots & \cdots & \cdots \\ d_{319} & \cdots & \cdots \end{pmatrix} - \begin{pmatrix} d_{23 10} \\ \cdots \\ d_{23 10} \end{pmatrix} = d_{562}d_{389}d_{123}d_{23 10}
\end{align*}
\]
Description of hypergeometric solution (3): Case of $E_7^{(1)}$

\[
\begin{aligned}
\frac{(gf - t\overline{t})(gf - t^2)}{(gf - 1)(gf - 1)} &= \frac{(f - b_1t)(f - b_2t)(f - b_3t)(f - b_4t)}{(f - b_5)(f - b_6)(f - b_7)(f - b_8)}, \\
\frac{(gf - t^2)(gf - t\overline{t})}{(gf - 1)(gf - 1)} &= \frac{(g - t/b_1)(g - t/b_2)(g - t/b_3)(g - t/b_4)}{(g - 1/b_5)(g - 1/b_6)(g - 1/b_7)(g - 1/b_8)},
\end{aligned}
\]

\[
\overline{t} = qt, \quad b_1b_2b_3b_4 = q, \quad b_5b_6b_7b_8 = 1.
\]

\[
z = \frac{1 - b_3/b_1}{1 - b_3/b_5} \frac{8W_7(a; q, b, c, d, e, f; q, qa^2/bcdef)}{8W_7(a; b, c, d, e, f; q, q^2a^2/bcdef)}, \quad z = \frac{g - t/b_1}{g - 1/b_5}
\]

where

\[
\begin{aligned}
a &= b_1b_8/b_3b_5, \quad b = b_8/b_5, \quad c = b_2/b_3, \\
d &= b_1t/b_5, \quad e = b_1/b_5, \quad f = b_4/b_3,
\end{aligned}
\]

gives a solution of the $q$-Painlevé equation of type $E_7^{(1)}$ with

\[
b_1b_3 = b_5b_7 \quad (b_2b_4 = qb_6b_8).
\]

Comments:

- ✔ In the terminating case, i.e. $f = b_4/b_3 = q^{-n}$, $n \in \mathbb{Z}_{>0}$, the solution is expressed by terminating balanced $_4\varphi_3$ (Askey-Wilson Polynomials)

- ✔ In the elliptic case, the solution is expressible in terms of the terminating balanced very-well-poised elliptic hypergeometric series

\[
_{10}E_9(u_0, u_1, \ldots, u_7) = \sum_{n=0}^{\infty} \frac{[u_0 + 2n\delta]}{[u_0]} \prod_{r=0}^{7} \frac{[u_r]_n}{[u_0 - u_r + \delta]_n},
\]

\[
[z]_n = [z][z + \delta] \cdots [z + (n - 1)\delta],
\]

\[
2\delta + 3u_0 - \sum_{i=1}^{7} u_i = 0.
\]
**Summary**

☛ Addition on moving cubic curve: discrete Painlevé equations

☛ Diagram of hypergeometric functions:

\[ E_8^{(1)}(e.) \]
\[ \downarrow \]
\[ E_8^{(1)}(q) \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow D_5^{(1)} \rightarrow A_4^{(1)} \rightarrow (A_2 + A_1)^{(1)} \rightarrow (A_1 + A_1 |_{\alpha^2 = 14})^{(1)} \]

balanced
\[ 10 E_9 \]
\[ \downarrow \]

balanced
\[ 10 W_9 \rightarrow 8 W_7 \rightarrow \text{balanced} \rightarrow 2 \phi_1 \rightarrow 1 \phi_1 \rightarrow 1 \phi_1 \left( \begin{array}{c} a \ 0 \ q, z \end{array} \right) \rightarrow 1 \phi_1 \left( \begin{array}{c} 0 \ -q \ q, z \end{array} \right) \]

☛ Things to be done: Many things! Only the formulations and simplest solutions have been presented.

★ Lax pair

★ Particular solutions

★ Asymptotics

★ \( \tau \) functions

★ Relation to other fields (random matrices, geometry, .....)

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References

This talk is based on:


Painlevé equations and affine Weyl group symmetry:


Formulation of discrete Painlevé equations:

