A geometric approach to q-Painlevé equations and their hypergeometric solutions

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Topic of this talk:

- ☞ Painlevé and discrete Painlevé equations
- ☞ Affine Weyl group symmetries
- ☞ Formulations of discrete Painlevé equations: Point configuration space and geometry of plane curves on $\mathbb{P}^2$
- ☞ Hypergeometric solutions: beyond the Gauss hypergeometric function?

Painlevé and Discrete Painlevé Equations (1)

List of Painlevé Equations

$$(\mathrm{P_I}) \quad y'' = 6y^2 + t,$$

$$(\mathrm{P_{II}}) \quad y'' = 2y^3 + ty + \alpha,$$

$$(\mathrm{P_{III}}) \quad y'' = \frac{1}{y}(y')^2 - \frac{1}{t}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y},$$

$$(\mathrm{P_{IV}}) \quad y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y},$$

$$(\mathrm{P_V}) \quad y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1},$$

$$(\mathrm{P_{VI}}) \quad y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)(y')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right].$$

Painlevé and Discrete Painlevé Equations (2)

Painlevé Equations:

- ✔ 2nd order nonlinear ODE with the "Painlevé property"
  $=$ No movavle branch points
  (6(8) equations)

- ✔ "Space of initial values" :
  defining manifold (Okamoto, Takano) $\leftarrow$ blow-up of $\mathbb{P}^2$

- ✔ Symmetries:
  Affine Weyl group $=$ reflection $+$ translation

- ✔ Solutions:
  - ☛ Transcendental in general
    (Nishioka, Umemura, Noumi, Okamoto,...)
  - ☛ Particular solutions for special values of parameters
    (1) Hypergeometric solutions
    (classification finished: Okamoto, Umemura, Noumi, Watanabe)
    (2) Algebraic solutions (most cases rational solutions)
    (classification finished except for $P_{\rm VI}$: Murata, Umemura, Kitaev, Watanabe, Dubrovin-Mazocco)

Painlevé and Discrete Painlevé Equations (3)

List of Some Discrete Painlevé Equations:

$$\mathrm{dP_I}: \qquad x_{n+1}+x_n+x_{n-1}=\frac{an+b}{x_n}+c$$

$$x_{n+1}x_n^{\sigma}x_{n-1}=aq^n x_n+b, \quad \sigma=0,1,2$$

$$\mathrm{dP_{II}}: \qquad x_{n+1}+x_{n-1}=\frac{(an+b)x_n+c}{1-x_n^2}$$

$$(x_{n+1}x_n-1)(x_n x_{n-1}-1)=\frac{aq^{2n}x_n}{x_n+q^n}$$

$$\mathrm{dP_{III}}: \qquad x_{n+1}x_{n-1}=\frac{ab(x_n+cq^n)(x_n+dq^n)}{(x_n+a)(x_n+b)}$$

Type $E_7^{(1)}$

$$\begin{cases} \dfrac{(\overline{g}f-t\overline{t})(gf-t^2)}{(\overline{g}f-1)(gf-1)}=\dfrac{(f-b_1t)(f-b_2t)(f-b_3t)(f-b_4t)}{(f-b_5)(f-b_6)(f-b_7)(f-b_8)}, \\ \dfrac{(gf-t^2)(g\underline{f}-\underline{t}t)}{(gf-1)(g\underline{f}-1)}=\dfrac{(g-t/b_1)(g-t/b_2)(g-t/b_3)(g-t/b_4)}{(g-1/b_5)(g-1/b_6)(g-1/b_7)(g-1/b_8)}, \end{cases}$$

$$\overline{t}=qt, \quad b_1b_2b_3b_4=q, \quad b_5b_6b_7b_8=1.$$

Type $E_6^{(1)}$:

$$\begin{cases} (\overline{g}f-1)(gf-1)=t\overline{t}\,\dfrac{(f-b_1)(f-b_2)(f-b_3)(f-b_4)}{(f-b_5t)(f-t/b_5)}, \\ (gf-1)(g\underline{f}-1)=t^2\,\dfrac{(g-1/b_1)(g-1/b_2)(g-1/b_3)(g-1/b_4)}{(g-b_6t)(g-t/b_6)}, \end{cases}$$

$$\overline{t}=qt, \quad b_1b_2b_3b_4=1.$$

Type $(A_2+A_1)^{(1)}$(qP$_{\mathrm{III}}$)

$$\begin{cases} \overline{g}gf=b_0\,\dfrac{1+a_0tf}{a_0t+f}, \\ gf\underline{f}=b_0\,\dfrac{a_1/t+g}{1+ga_1/t}, \end{cases} \qquad \overline{t}=qt.$$

Painlevé and Discrete Painlevé Equations (4)

Discrete Painlevé Equations:

- ✔ 2nd order nonlinear ordinary DIFFERENCE equations with the "singularity confinement property"(MANY equations)

- ✔ Space of initial values" :
  defining manifold (Sakai) $\leftarrow$ blow-up of $\mathbb{P}^2$

- ✔ Symmetries:
  Affine Weyl group $=$ reflection $+$ translation

- ✔ Solutions:
  - ☛ Transcendence?
  - ☛ Particular solutions for special values of parameters

    (1) Hypergeometric solutions

    (2) Algebraic solutions: almost nothing has been done

Particular solutions: hypergeometric solutions

$$\mathrm{P_{II}}: \qquad \frac{d^2y}{dt^2} = 2y^3 + ty + \alpha$$

$$\mathrm{dP_{II}}: \qquad x_{n+1} + x_{n-1} = \frac{(\alpha n + \beta)x_n + \gamma}{1 - x_n^2},$$

✌ **Hypergeometric solution (Riccati solution)**

Riccati equation$\longrightarrow$ linearization

$$y' = a(t)y^2 + b(t)y + c(t)$$

$$\Downarrow$$

$$\frac{dy}{dt} = -y^2 + \frac{t}{2} \longrightarrow \alpha = -\frac{1}{2}$$

$$\Downarrow$$

$$y = \frac{d}{dt}\log f, \qquad \frac{d^2f}{dt^2} = -\frac{t}{2}f$$

$$x_{n+1} = \frac{a_n x_n + b_n}{c_n x_n + d_n}$$

$$\Downarrow$$

$$x_{n+1} = \frac{x_n - (pn+q)}{1 + x_n} \quad \longrightarrow \alpha = 2p, \quad \beta = -p + 2q + 2, \quad \gamma = -p$$

$$\Downarrow$$

$$x_n = \frac{g_{n+1} - g_n}{g_n}, \quad g_{n+2} - 2g_{n+1} + g_n = -(pn+q)g_n$$

Affine Weyl group symmetry: $\mathrm{P_{IV}}$

$$\mathrm{P_{IV}}: \quad y'' = \frac{y'^2}{2y} + \frac{3}{2}y^3 - 2ty + \left(\frac{t^2}{2} - \alpha_0 + \alpha_2\right) y - \frac{\alpha_1^2}{2y}$$

$$\Updownarrow$$

“symmetric form” of $\mathbf{P}_{\mathrm{IV}}$ (Noumi-Yamada, 1998)

$$\begin{aligned} \varphi_0' &= \varphi_0(\varphi_1 - \varphi_2) + \alpha_0 \\ \varphi_1' &= \varphi_1(\varphi_2 - \varphi_0) + \alpha_1 \\ \varphi_2' &= \varphi_2(\varphi_0 - \varphi_1) + \alpha_2 \end{aligned}$$

$y = \varphi_1$, normalization: $\alpha_0 + \alpha_1 + \alpha_2 = 1$, $\varphi_0 + \varphi_1 + \varphi_2 = t$

Bäcklund transformations s_0, s_1, s_2, π

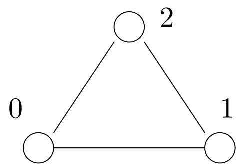
$$\begin{array}{lll} s_0(\alpha_0) = -\alpha_0 & s_0(\alpha_1) = \alpha_1 + \alpha_0 & s_0(\alpha_2) = \alpha_2 + \alpha_0 \\ s_1(\alpha_0) = \alpha_0 + \alpha_1 & s_1(\alpha_1) = -\alpha_1 & s_1(\alpha_2) = \alpha_2 - \alpha_1 \\ s_2(\alpha_0) = \alpha_0 + \alpha_2 & s_2(\alpha_1) = \alpha_1 + \alpha_2 & s_2(\alpha_2) = -\alpha_2 \end{array}$$

$$\begin{array}{lll} s_0(\varphi_0) = \varphi_0 & s_0(\varphi_1) = \varphi_1 - \dfrac{\alpha_0}{\varphi_0} & s_0(\varphi_2) = \varphi_2 + \dfrac{\alpha_0}{\varphi_0} \\ s_1(\varphi_0) = \varphi_0 + \dfrac{\alpha_1}{\varphi_1} & s_1(\varphi_1) = \varphi_1 & s_1(\varphi_2) = \varphi_2 - \dfrac{\alpha_0}{\varphi_0} \\ s_2(\varphi_0) = \varphi_0 - \dfrac{\alpha_2}{\varphi_2} & s_1(\varphi_1) = \varphi_1 + \dfrac{\alpha_2}{\varphi_1} & s_2(\varphi_2) = \varphi_2 \end{array}$$

$$\pi(\varphi_i) = \varphi_{i+1}, \quad \pi(\alpha_i) = \alpha_{i+1}, \quad i \in \mathbb{Z}/3\mathbb{Z}$$

✎ Affine Weyl group: $\langle s_0, s_1, s_2, \pi \rangle = \widetilde{W}(A_2^{(1)})$

$$s_i^2 = 1, \qquad \overset{i\circ \quad \circ j}{s_i s_j = s_j s_i}, \qquad \overset{i\circ\!\!-\!\!\!-\!\!\circ j}{s_i s_j s_i = s_j s_i s_j}, \qquad \pi^3 = 1, \quad s_{i+1}\pi = \pi s_i$$



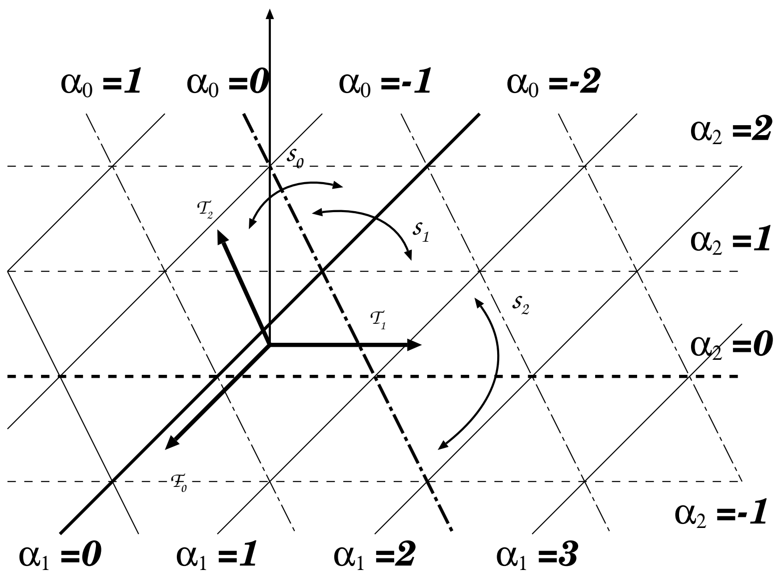
✎ Translations in the parameter space:

$$T_1 = \pi s_1 s_2, \quad T_2 = s_1 \pi s_2, \quad T_0 = s_1 s_2 \pi \ \rightarrow \ T_i T_j = T_j T_i, \ T_0 T_1 T_2 = 1$$

$$T_1(\{\alpha_0, \alpha_1, \alpha_2\}) = \{\alpha_0 + 1, \alpha_1 - 1, \alpha_2\}$$
$$T_2(\{\alpha_0, \alpha_1, \alpha_2\}) = \{\alpha_0, \alpha_1 + 1, \alpha_2 - 1\}$$
$$T_0(\{\alpha_0, \alpha_1, \alpha_2\}) = \{\alpha_0 - 1, \alpha_1, \alpha_2 + 1\}$$

A discrete Painlevé II $(T_1^n(\varphi_i) = \varphi_i(n))$

$$\varphi_1(n+1) = t - \varphi_0(n) - \varphi_1(n) - \frac{\alpha_0 + n}{\varphi_0(n)}$$
$$\varphi_0(n+1) = t - \varphi_0(n) - \varphi_1(n+1) + \frac{\alpha_1 - n - 1}{\varphi_1(n+1)}$$



Sakai's Theory (2001):

Algebro-geometric theory of Painlevé and discrete Painlevé equations:

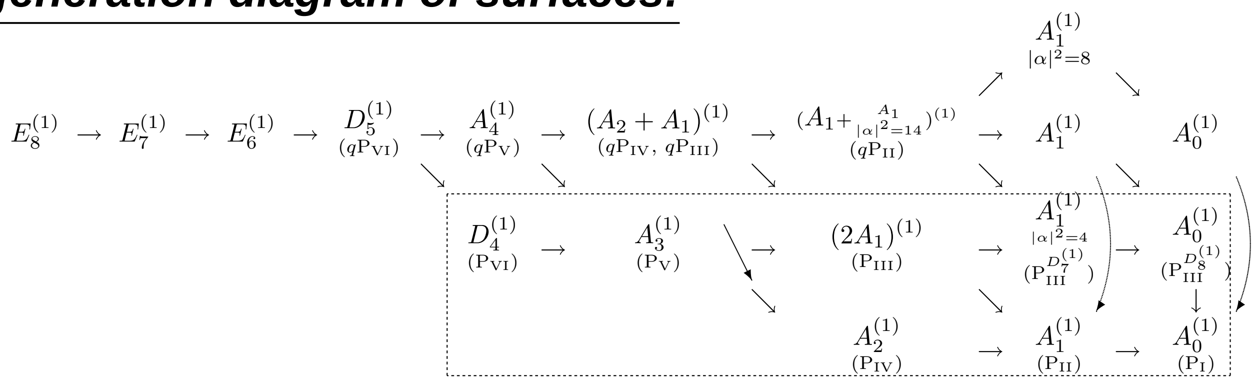
- Defining manifold of discrete Painlevé equations:
  Family of rational surfaces obtained by blow-up of $\mathbb{P}^2$ at 9 points.

- Action of affine Weyl group:
  interchange of points and Cremona transformations

- Classification of surfaces:
  22 cases obtained by degeneration of points

- 8 cases admit continuous flows $\rightarrow$ Painlevé equations
  Continuous flows come from continous limit of discrete evolutions on "higher" surfaces.

Degeneration diagram of surfaces:



Type of time evolution (action of Weyl group):

- Elliptic: $E_8^{(1)}$ (1) $\rightarrow$ Elliptic Painlevé equation
- Multiplicative: $E_8^{(1)} \cdots A_0^{(1)}$ (10) $\rightarrow$ q-Painlevé equations
- Additive: $E_8^{(1)}$, $E_7^{(1)}$, $E_6^{(1)}$ $+$ inside the box (11)

Hypergeometric Solutions:

Coalescence cascade

$$\begin{array}{ccccccccc}
\underset{\text{Gauss}}{\mathrm{P_{VI}}} & \to & \underset{\text{Kummer}}{\mathrm{P_{V}}} & \to & \underset{\text{Bessel}}{\mathrm{P_{III}}} & \to & \underset{\times}{\mathrm{P}_{\mathrm{III}}^{D_7^{(1)}}} & \to & \underset{\times}{\mathrm{P}_{\mathrm{III}}^{D_8^{(1)}}} \\
 & & & \searrow & & \searrow & & & \\
 & & & & \underset{\text{Parabolic Cylinder}}{\mathrm{P_{IV}}} & \to & \underset{\text{Airy}}{\mathrm{P_{II}}} & \to & \underset{\times}{\mathrm{P_{I}}}
\end{array}$$

Series of q-Painlevé equations:

$$E_8^{(1)} \to E_7^{(1)} \to E_6^{(1)} \to D_5^{(1)} \to A_4^{(1)} \to (A_2+A_1)^{(1)} \to (A_1 + \underset{|\alpha|^2=14}{A_1})^{(1)}$$

- ✔ What kind of hypergeometric functions appear for q-Painlevé equations, in particular for $E_6^{(1)}$, $E_7^{(8)}$, $E_8^{(1)}$?
  (beyond the Gauss hypergeometric function!)

Basic hypergeometric series:

$${}_r\varphi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s\end{matrix};q,z\right) = \sum_{n=0}^{\infty} \frac{(a_1,\cdots a_r;q)_n}{(b_1,\cdots,b_s;q)_n(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n,$$

$$(a_1,\ldots,a_r;q)_n = (a_1;q)_n\cdots(a_r;q)_n,$$

$$(a;q)_n = \underbrace{(1-a)(1-qa)\cdots(1-q^{n-1}a)}_{n}$$

- ✔ balanced: $qa_1a_2\cdots a_{r+1} = b_1b_2\cdots b_r, \quad z=q$
- ✔ well-poised: $qa_1 = a_2b_1 = \cdots = a_{r+1}b_r$
- ✔ very-well-poised: well-poised $+\ a_2 = qa_1^{\frac{1}{2}}, \quad a_3 = -qa_1^{\frac{1}{2}}$

Very-well-poised basic hypergeometric series ${}_{r+1}W_r$:

$${}_{r+1}W_r(a_1;a_4,\ldots,a_{r+1};q,z) = {}_{r+1}\varphi_r\left(\begin{matrix} a_1, qa_1^{\frac{1}{2}}, -qa_1^{\frac{1}{2}}, a_4\ldots,a_{r+1} \\ a_1^{\frac{1}{2}}, -a_1^{\frac{1}{2}}, qa_1/a_4,\ldots,qa_1/a_{r+1}\end{matrix};q,z\right)$$

Formulation of discrete Painlevé equations (1)

"Configuration space of points" on $\mathbb{P}^2$:

$(x:y:z)$: homogeneous coordinate of $\mathbb{P}^2$

$$\mathcal{M}_{3,n} = GL(3)\backslash \left\{ \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ y_1 & y_2 & y_3 & \cdots & y_n \\ z_1 & z_2 & z_3 & \cdots & z_n \end{bmatrix} \right\} /(\mathbb{C}^\times)^n.$$

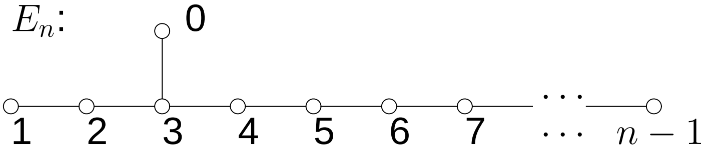
Birational transformations on $\mathcal{M}_{3,n}$:

1. s_i ($i=1,\ldots,n-1$): interchanging P_i and P_{i+1}
2. s_0: standard Cremona transformation with base points $P_1(1:0:0)$, $P_2(0:1:0)$, $P_3(0:0:1)$

$$s_0:\ (x:y:z)\ \mapsto (\frac{1}{x},\frac{1}{y},\frac{1}{z}).$$

$\langle s_0, s_1, \ldots, s_n\rangle = W(E_n)$: (A. Coble, 1922)

E_n:



$i\circ \quad \circ j$ $\qquad$ $i\circ\!\!-\!\!\circ j$

$$s_i^2 = 1, \qquad s_i s_j = s_j s_i, \qquad s_i s_j s_i = s_j s_i s_j,$$

$n=10$ $\quad \mathbb{Z}^8 \subset \langle s_0, s_1, \ldots, s_9\rangle = W(E_9) = W(E_8^{(1)}) \subset W(E_{10})$

action of translation subgroup $\mathbb{Z}^8$	=	Elliptic Painlevé equation
9 points $P_1, \ldots, P_9$	=	parameters
10th pointP_{10}	=	dependent variable

Formulation with the pencil of cubic curves (1)

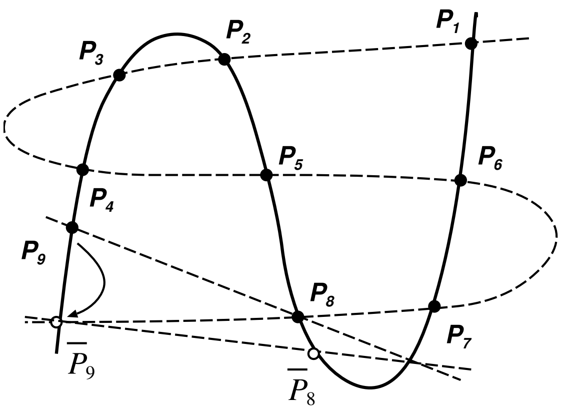
Translation = Addtion on *moving* cubic curve

Example: T_{89}, C_0: Cubic curve passing $P_1, \ldots P_9$

- ☞ Determine new points $\overline{P}_8$, $\overline{P}_9$ by using the addition on C_0:

  $$\overline{P}_i = P_i,\ (i \neq 8, 9), \quad P_1 + \cdots + P_8 + \overline{P}_9 = O, \quad P_8 + P_9 = \overline{P}_8 + \overline{P}_9$$

- ☞ Let $C_0\ :\ F = 0$ and let the cubic pencil be $\lambda F + \mu G = 0$. Choose λ, μ so that the pencil passes P_{10}. Denote this new cubic curve passing through $P_1, \ldots, P_8$, $\overline{P}_9$ and P_{10} as C.



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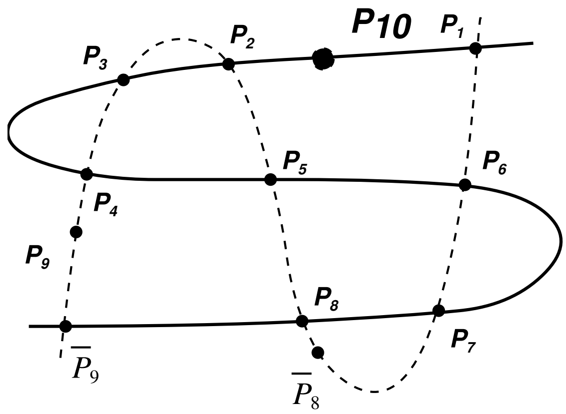
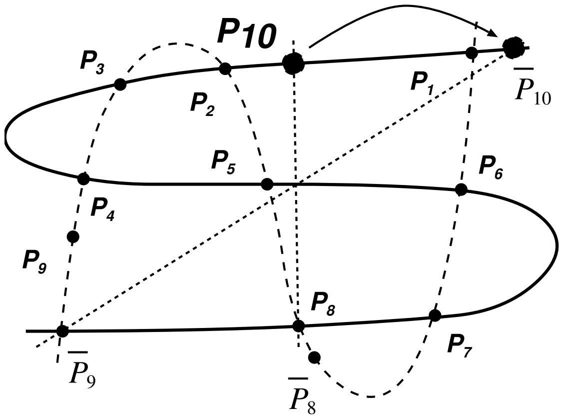


- ☞ Determine new point $\overline{P}_{10}$ by using the addition on C as

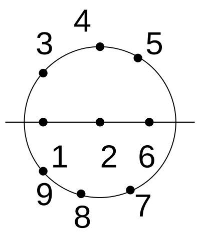
  $$P_{10} + P_8 = \overline{P}_{10} + \overline{P}_9$$



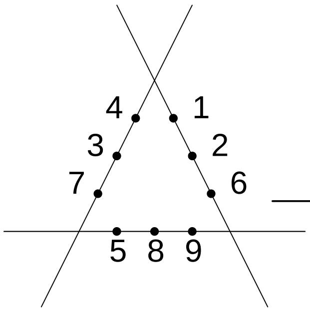
Comments:

- Generic nine points : Elliptic Painlevé equation(KMNOY, 2003). Varrious degenerate configurations: other discrete Painlevé equations

  ☞ 9 points on nodal cubic curve: q-Painlevé of type $E_8^{(1)}$



$\longrightarrow E_7^{(1)}$



$\longrightarrow E_6^{(1)}$

- The same procedure for the stationary cubic gives QRT system (Tsuda)

- The procedure for statinary cubic was first discussed by Manin and shown that it is equivalent to translation.

Description of hypergeometric solution (1)

✐ *Reduction to Riccati Equation:*

✔ Three points among the nine points in the cubic are colinear.

✔ A point is infinitesimally near to another point

(The second case is essential only for much degenerate cases)

Example: P_5, P_6, P_7 are colinear

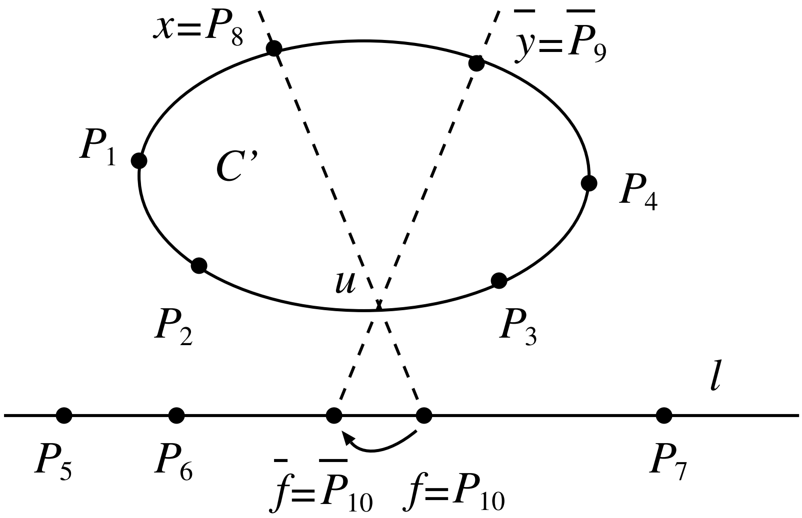
Cubic C is decomposed into a line $\ell(P_5, P_6, P_7)$ and a conic $C'(P_2, \ldots, P_4, P_8, \overline{P}_9)$.

$$\because\ P_1 + \cdots + P_8 + \overline{P}_9 = O, \quad P_5 + P_6 + P_7 = O \ \to \ P_1 + \cdots + P_4 + P_8 + \overline{P}_9 = O.$$

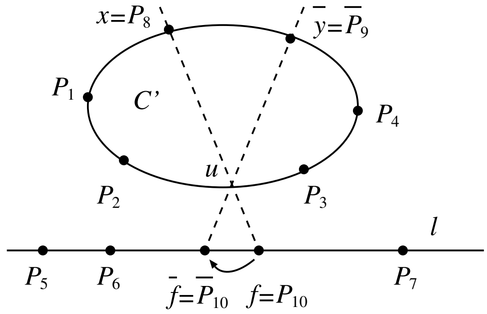
$$\Downarrow$$

$P_{10} \in \ell \ \to \overline{P}_{10} \in \ell$ $\quad \ell$ is an "invariant divisor"

✔ In this case, discrete time evolution of f can be described in terms of linear equation.



Description of hypergeometric solution (2)



Choose coordinate such that
$P_1(1:0:0)$, $P_2(0:1:0)$, $P_3(0:0:1)$, and denote:

$$P_{10} = f = (f_1 : f_2 : f_3),\ \ \overline{P}_{10} = \overline{f} = (\overline{f}_1 : \overline{f}_2 : \overline{f}_3),$$
$$P_8 = x = (x_1 : x_2 : x_3),\ \ \ P_9 = \overline{y} = (\overline{y}_1 : \overline{y}_2 : \overline{y}_3),$$

$$\ell : \quad (a,f) = a_1f_1 + a_2f_2 + a_3f_3 = 0, \quad (a,\overline{f}) = a_1\overline{f}_1 + a_2\overline{f}_2 + a_3\overline{f}_3 = 0$$

Linear equation for f:

$$\lambda\overline{f} = (a,\overline{y})Df - (a,Df)\overline{y},$$
$$\mu f = (a,x)D^{-1}\overline{f} - (a,D^{-1}\overline{f})x,$$
$$D = \mathrm{diag}\left(\frac{x_2x_3}{\overline{y}_2\overline{y}_3}, \frac{x_3x_1}{\overline{y}_3\overline{y}_1}, \frac{x_1x_2}{\overline{y}_1\overline{y}_2}\right), \quad \lambda = (a,x), \quad \mu = (a,\overline{y}).$$

$GL(3)$*-invariant linear equation:* $d_{ijk} = \det[P_i, P_j, P_k]$

$$\frac{d_{239}d_{12\overline{9}}d_{568}d_{318}}{d_{18\overline{9}}}\left(\frac{d_{31\overline{9}}}{d_{318}}\boxed{d_{23\,\overline{10}}} - \boxed{d_{23\,10}}\right)$$
$$+\ \frac{d_{238}d_{12\underline{8}}d_{569}d_{319}}{d_{1\underline{8}9}}\left(\frac{d_{31\underline{8}}}{d_{319}}\boxed{d_{23\,\underline{10}}} - \boxed{d_{23\,10}}\right) = d_{562}d_{389}d_{123}\boxed{d_{23\,10}}$$

Description of hypergeometric solution (3): Case of $E_7^{(1)}$

$$\begin{cases} \dfrac{(\overline{g}f - t\overline{t})(gf - t^2)}{(\overline{g}f - 1)(gf - 1)} = \dfrac{(f - b_1 t)(f - b_2 t)(f - b_3 t)(f - b_4 t)}{(f - b_5)(f - b_6)(f - b_7)(f - b_8)}, \\ \dfrac{(gf - t^2)(g\underline{f} - \underline{t}t)}{(gf - 1)(g\underline{f} - 1)} = \dfrac{(g - t/b_1)(g - t/b_2)(g - t/b_3)(g - t/b_4)}{(g - 1/b_5)(g - 1/b_6)(g - 1/b_7)(g - 1/b_8)}, \end{cases}$$

$$\overline{t} = qt, \quad b_1 b_2 b_3 b_4 = q, \quad b_5 b_6 b_7 b_8 = 1.$$

$$z = \frac{1 - b_3/b_1}{1 - b_3/b_5 t}\, \frac{{}_8W_7(a; qb, c, d, e, f; q, qa^2/bcdef)}{{}_8W_7(a; b, c, d, e, f; q, q^2a^2/bcdef)}, \quad z = \frac{g - t/b_1}{g - 1/b_5}$$

where

$$\begin{aligned} &a = b_1 b_8 / b_3 b_5, \quad b = b_8/b_5, \quad c = b_2/b_3, \\ &d = b_1 t / b_5, \qquad e = b_1 / b_5 t, \quad f = b_4/b_3, \end{aligned}$$

gives a solution of the q-Painlevé equation of type $E_7^{(1)}$ with

$$b_1 b_3 = b_5 b_7 \quad (b_2 b_4 = q b_6 b_8).$$

Comments:

- ✔ In the terminating case, i.e. $f = b_4/b_3 = q^{-n}$, $n \in \mathbb{Z}_{>0}$, the solution is expressed by *terminating balanced* ${}_4\varphi_3$ (*Askey-Wilson Polynomials*)
- ✔ In the elliptic case, the solution is expressible in terms of the *terminating balanced very-well-poised elliptic hypergeometric series*

$$ {}_{10}E_9(u_0, u_1, \ldots, u_7) = \sum_{n=0}^{\infty} \frac{[u_0 + 2n\delta]}{[u_0]} \prod_{r=0}^{7} \frac{[u_r]_n}{[u_0 - u_r + \delta]_n}, $$
$$ [z]_n = [z][z + \delta] \cdots [z + (n-1)\delta], $$

$$2\delta + 3u_0 - \sum_{i=1}^{7} u_i = 0.$$

Summary

- ☛ Addition on moving cubic curve: discrete Painlevé equations
- ☛ Diagram of hypergeometric functions:

$$
\begin{array}{l}
E_8^{(1)}(\text{e.}) \\
\quad\downarrow \\
E_8^{(1)}(q) \;\to\; E_7^{(1)} \;\to\; E_6^{(1)} \;\to\; D_5^{(1)} \;\to\; A_4^{(1)} \;\to\; (A_2+A_1)^{(1)} \;\to\; (A_1 + \underset{|\alpha|^2=14}{A_1})^{(1)}
\end{array}
$$

$$
\begin{array}{l}
\begin{array}{c}\text{balanced}\\ {}_{10}E_9\end{array} \\
\quad\downarrow \\
\begin{array}{c}\text{balanced}\\ {}_{10}W_9\end{array} \;\to\; {}_8W_7 \;\to\; \begin{array}{c}\text{balanced}\\ {}_3\phi_2\end{array} \;\to\; {}_2\phi_1 \;\to\; {}_1\phi_1 \;\to\; \begin{array}{c} {}_1\phi_1\left(\begin{array}{c}a\\0\end{array};q,z\right) \\ {}_1\phi_1\left(\begin{array}{c}0\\b\end{array};q,z\right)\end{array} \;\to\; {}_1\phi_1\left(\begin{array}{c}0\\-q\end{array};q,z\right)
\end{array}
$$

- ☛ Things to be done: Many things! Only the formulations and simplest solutions have been presented.
  - ✐ Lax pair
  - ✐ Particular solutions
  - ✐ Asymptotics
  - ✐ τ functions
  - ✐ Relation to other fields (random matrices, geometry, ...)

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