Aperiodic Order and Dynamical Systems II

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The Plan

Study aperiodic order via dynamical systems:

$$\varLambda < - - - > (\mathbb{X}(\varLambda), \alpha).$$

- Dynamical system arises by gathering together all manifestations of the “same” form of (dis)order.

- Properties of the dynamical system reflect properties of its elements and vice versa.

1. Local topology

- Compactness and finite local complexity.

- Unique ergodicity and uniform patch frequencies.

- A word on symmetry.

- Pure point dynamical spectrum and pure point diffraction.

2. Autocorrelation topology

- Compactness, ε-periods, and pure point diffraction.

3. Where local topology and autocorrelation topology meet: Model sets

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Notation

$\| \cdot \|$ Euclidean norm on $\mathbb{R}^d$.

$\mathcal{D}_r := \{\varLambda \subset \mathbb{R}^d : \|x - y\| \geq r$ for all $x, y \in \varLambda,\ x \neq y\}$.

$B_s := \{x \in \mathbb{R}^d : \|x\| \leq s\}$

$\varLambda$, $\varGamma$ always supposed to be uniformly discrete i.e. to belong to some $\mathcal{D}_r$.

1. Local topology

Idea: Λ, Γ are close if they are **locally close** after a small shift, i.e. if they agree on a large ball after a small shift.

More precisely, the local topology is introduced via the following metric:

$d_{lt}(\Lambda, \Gamma) :=$

$\inf\{\varepsilon > 0 : \exists\, x, y \in B_\varepsilon \text{ s.t. } B_{\frac{1}{\varepsilon}} \cap (x + \Lambda) = B_{\frac{1}{\varepsilon}} \cap (y + \Gamma)\} \wedge 2^{-}$

Theorem 1. $(\mathcal{D}_r, d_{lt})$ *is a complete metric space. The action*

$$\alpha : \mathbb{R}^d \times \mathcal{D}_r \longrightarrow \mathcal{D}_r, \ \ \alpha_x(\Lambda) := x + \Lambda$$

is continuous.

The hull of $\Lambda \in \mathcal{D}_r$ in the local topology is defined by

$$\mathbb{X}(\Lambda) := \overline{\{x + \Lambda : x \in \mathbb{R}^d\}^{lt}}.$$

Then, $\mathbb{X}(\Lambda)$ is invariant under α. Hence, $(\mathbb{X}(\Lambda), \alpha)$ is a topological dynamical system.

Compactness and finite local complexity

Definition 1. *Λ has finite local complexity (FLC) if for every $R > 0$*

$$\sharp\{(-x + \Lambda) \cap B_R : x \in \Lambda\} < \infty.$$

Note: Λ FLC $\iff$ $\Lambda - \Lambda$ locally finite.
Proof. $\cup_{x \in \Lambda}((-x + \Lambda) \cap B_R) = (\Lambda - \Lambda) \cap B_R$

Theorem 2. *$\mathbb{X}(\Lambda)$ compact $\iff$ Λ has FLC.*

See Radin/Wolff '92, Schlottmann '00.

Proof $\Longrightarrow$: Assume the contrary. Then, there exists an $R > 0$ with

$$\sharp\{(-x + \varLambda) \cap B_R : x \in \varLambda\} = \infty.$$

Let $x_1, x_2, x_3, \ldots$ in $\mathbb{R}^d$ be given such that

$$(-x_n + \varLambda) \cap B_R$$

are pairwise disjoint. Consider

$$\varLambda_n := -x_n + \varLambda \in \mathbb{X}(\varLambda).$$

Then, $(\varLambda_n)$ has no converging subsequence.

$\Longleftarrow$: Consider only $\varLambda$ Delone, i.e. there exists $R > 0$ with $(p + B_R) \cap \varLambda \neq \emptyset$ for every $p \in \mathbb{R}^d$.

Let $(\varLambda_n)$ be a sequence in $\{-x + \varLambda : x \in \mathbb{R}^d\}$.

(To show: $(\varLambda_n)$ has converging subsequence).

By assumption, for each $n \in \mathbb{N}$, there exists $x_n \in B_R \cap \varLambda_n$.

W.l.o.g. $x_n \longrightarrow x \in B_R$, $n \to \infty$.

W.l.o.g. $x = x_n = 0$ for all $n \in \mathbb{N}$. $(\varLambda_n - -- > \varLambda_n - x_n)$.

Consider for each $k \in \mathbb{N}$ $(\varLambda_n \cap B_k)_n$.

FLC $\Longrightarrow$ exists subsequence $(n_j^{(1)})_j$ of (n) s.t. $\varLambda_{n_j^{(1)}} \cap B_1$ the same for all j.

FLC $\Longrightarrow$ exists subsequence $(n_j^{(2)})_j$ of $(n_j^{(1)})$ s.t. $\varLambda_{n_j^{(2)}} \cap B_2$ the same for all j.

FLC $\Longrightarrow$ exists subsequence $(n_j^{(3)})_j$ of $(n_j^{(2)})$ s.t. $\varLambda_{n_j^{(3)}} \cap B_3$ the same for all j.

Then $(\varLambda_{n_k^{(k)}})_k$ converges. $\blacksquare$

Unique ergodicity and uniform patch frequencies

Definition 2. *For $\Lambda \in \mathcal{D}_r$ and $P \subset B_s$ with 0 $\in P$ define the locator set of P in Λ by*

$$L(\Lambda, P) := \{x \in \Lambda : (-x + \Lambda) \cap B_s = P\}.$$

Note: $L(\Lambda, P) = \emptyset$ is possible.

Definition 3. *Λ has uniform patch frequencies (UPF) if for every P*

$$\lim_{n\to\infty} \frac{\sharp L(\Lambda, P) \cap (q + B_n)}{|B_n|}$$

exists uniformly in $q \in \mathbb{R}^d$.

Theorem 3. *Let Λ have (FLC). Then:*
Λ has UPF $\Longleftrightarrow$ $(\mathbb{X}(\Lambda), \alpha)$ is uniquely ergodic.

For a proof see e.g. Solomyak '96, Schlottmann '00, Lee/Moody/Soloymak '02.

Proof. For $\varphi \in C_c(\mathbb{R}^d)$ and P patch define

$$f_{\varphi,P} : \mathcal{D}_r \longrightarrow \mathbb{C}, \;\; f_{\varphi,P}(\mathit{\Gamma}) := \sum_{x \in L(\mathit{\Gamma},P)} \varphi(-x).$$

Then,

$$\int_{B_n} f_{\varphi,P}(t+\mathit{\Gamma})dt = \int \varphi(t)dt \cdot \sharp L(\mathit{\Gamma},P) \cap B_n + \mathsf{BT}.$$

Therfore,

$$\begin{aligned}
(UPF) \;\; &\Longleftrightarrow \;\; \lim_{n\to\infty} \frac{1}{B_n} \int_{B_n} f_{\varphi,P}(t+\mathit{\Gamma})\, dt \text{ ex. all } f_{\varphi,P} \\
&\Longleftrightarrow \;\; \lim_{n\to\infty} \frac{1}{B_n} \int_{B_n} f(t+\mathit{\Gamma})\, dt \text{ ex. all cont. } f \\
&\Longleftrightarrow \;\; \text{Unique ergodicity.}
\end{aligned}$$

$\blacksquare$

A word on symmetry

$\mathbb{X}(\Lambda)$ may have "more" symmetry than Λ.

More precisely, consider a rotation

$$S : \mathcal{D}_r \longrightarrow \mathcal{D}_r,\ \ \Gamma \mapsto S(\Gamma).$$

Then, S may leave $\mathbb{X}(\Lambda)$ invariant and then act on $\mathbb{X}(\Lambda)$ via

$$S : \mathbb{X}(\Lambda) \longrightarrow \mathbb{X}(\Lambda),\ \ \Gamma \mapsto S(\Gamma)$$

without Λ being fixed by S.

In this case, if $\mathbb{X}(\Lambda)$ admits a unique α invariant probability measure m, then m is invariant under S as well (as $S(m)$ is another α-invariant probability measure).

Pure point dynamical and pure point diffraction spectrum

Let $\varLambda$ with FLC and UPF be fixed.

Thus, $(\mathbb{X}(\varLambda), \alpha)$ is compact and uniquely ergodic.

Denote unique α-invariant probability measure on $\mathbb{X}(\varLambda)$ by m.

$$L^2(\mathbb{X}(\varLambda), m) := \{f : \mathbb{X}(\varLambda) \longrightarrow \mathbb{C} : \int |f|^2 dm < \infty\}$$

Hilbert space with inner product

$$\langle f, g \rangle := \int \overline{f} g \, dm.$$

Unitary representation T of $\mathbb{R}^d$ on $L^2(\mathbb{X}(\varLambda), m)$:

For each $x \in \mathbb{R}^d$

$$T_x : L^2(\mathbb{X}(\varLambda), m) \longrightarrow L^2(\mathbb{X}(\varLambda), m)$$

$$(T_x f)(\varGamma) := f(\alpha_{-x}\varGamma)$$

is unitary (i.e. isometric and onto).

An $f \in L^2(\mathbb{X}(\varLambda), m)$ is called *eigenfunction* of T to the eigenvalue $y \in \mathbb{R}^d$ if

$$T_x f = e^{ixy} f \text{ for all } x \in \mathbb{R}^d.$$

$\mathcal{H}_{pp}(T) := \overline{Lin\{\text{eigenfunctions of } T\ \}} \subset L^2(\mathbb{X}(\varLambda), m)$.

T is said to have *pure point spectrum* if

$$\mathcal{H}_{pp}(T) = L^2(\mathbb{X}(\varLambda), m).$$

Then, $(\mathbb{X}(\varLambda), \alpha, m)$ is said to *have pure point dynamical spectrum*.

We now come to a circle of ideas going back to Dworkin '93 (see Enter/Miękisz '92, Hof '98, Schlottmann '00, Lee/Moody/Solomyak '02... as well).

For $\varphi \in C_c(\mathbb{R}^d)$ define

$$f_\varphi : \mathcal{D}_r \longrightarrow \mathbb{C}, \;\; f_\varphi(\Gamma) := \sum_{x \in \Gamma} \varphi(-x) = \varphi * \delta_{\Gamma}(0).$$

Proposition 1.

$$\lim_{n\to\infty} \varphi * \widetilde{\varphi} * \frac{1}{|B_n|}(\delta_{\Gamma\cap B_n} * \delta_{-(\Gamma\cap B_n)})(t) = \langle f_\varphi, T_t f_\varphi \rangle$$

for every $\Gamma \in \mathbb{X}(\Lambda)$ and $t \in \mathbb{R}^d$.

Proof. By unique ergodicity we have

$$\begin{aligned}
&\langle f_\varphi, T_t f_\varphi \rangle \\
&= \lim_{n\to\infty} \frac{1}{|B_n|} \int_{B_n} \overline{f_\varphi(\alpha_s \Gamma)} f_\varphi(\alpha_{s-t}\Gamma) ds \\
&= \lim_{n\to\infty} \frac{1}{|B_n|} \int_{B_n} \sum_{x\in\Gamma} \overline{\varphi(-s-x)} \sum_{y\in\Gamma} \varphi(-s-t-y) ds \\
&= \lim_{n\to\infty} \frac{1}{|B_n|} \int \sum_{x\in\Gamma\cap B_n} \overline{\varphi(-s-x)} \sum_{y\in\Gamma\cap B_n} \varphi(-s-t-y)d \\
&= \lim_{n\to\infty} \frac{1}{|B_n|} (\widetilde{\varphi} * \delta_{-(\Gamma\cap B_n)}) * (\varphi * \delta_{\Gamma\cap B_n})(t).
\end{aligned}$$

From this result (or by other means) we may infer that

$$\gamma := \lim_{n\to\infty} \frac{1}{|B_n|} \delta_{\Gamma\cap B_n} * \delta_{-(\Gamma\cap B_n)}$$

exists for every $\Gamma \in \mathbb{X}(\Lambda)$ and does not depend on Γ.

Then, the proposition may be reformulated as saying that

$$\langle f_\varphi, T_t f_\varphi \rangle = \varphi * \widetilde{\varphi} * \gamma(t)$$

for every $t \in \mathbb{R}^d$.

On the other hand, by spectral theory, for every $f \in L^2(\mathbb{X}(\Lambda), m)$ there exists a finite measure ρ_f on $\mathbb{R}^d$ with

$$\langle f, T_t f \rangle = \int e^{ity} \, d\rho_f(y)$$

for every $t \in \mathbb{R}^d$.

Putting this together we infer

$$\varphi * \widetilde{\varphi} * \gamma(t) = \int e^{ity} \, d\rho_{f_\varphi}(y)$$

for every $\varphi \in C_c(\mathbb{R}^d)$ and $t \in \mathbb{R}^d$.

Theorem 4. *Let $\varLambda$ with (FLC) and (UPF) be given. Then, $\widehat{\gamma}$ is pure point if and only if $(\mathbb{X}(\varLambda),\alpha)$ has pure point dynamical spectrum.*

Remark. In this form due to Lee/Moody/Solomyak '02; later generalised see Gouéré '02, '03, Baake/L. '03; for an earlier result in symbolic dynamics see Queffélec '87.

Proof. As shown above

$$\varphi * \widetilde{\varphi} * \gamma(t) = \int e^{ity}\, d\rho_{f_\varphi}(y)$$

for all $\varphi \in C_c(\mathbb{R}^d)$ and $t \in \mathbb{R}^d$. Fourier transform yields

$$|\widehat{\varphi}|^2 \widehat{\gamma} = \rho_{f_\varphi}$$

for all $\varphi \in C_c(\mathbb{R}^d)$. This gives

$$\begin{aligned}
\widehat{\gamma}\, pp &\iff \rho_{f_\varphi} \text{ pure point for all } \varphi \in C_c(\mathbb{R}^d)\\
&\iff \rho_f \text{ pure point for all } f \in C(\mathbb{X}(\varLambda))\\
&\iff f \in \mathcal{H}_{pp}(T) \text{ for all } f \in C(\mathbb{X}(\varLambda))\\
&\iff \mathcal{H}_{pp}(T) = L^2(\mathbb{X}(\varLambda), m).
\end{aligned}$$

2. Autocorrelation topology

(Introduced in Baake/Moody '02; further studied in Moody/Strungaru '03.)

Idea: $\mathit{\Lambda}$, $\mathit{\Gamma}$ are close if they are **statistically close** after a small shift.

Statistical closeness captured by

$$\rho(\mathit{\Lambda},\mathit{\Gamma}) := \limsup_{n\to\infty} \frac{\sharp(\mathit{\Lambda}\setminus\mathit{\Gamma}\cup\mathit{\Gamma}\setminus\mathit{\Lambda})\cap B_n}{|B_n|}.$$

Note: $\rho(x+\mathit{\Lambda},x+\mathit{\Gamma}) = \rho(\mathit{\Lambda},\mathit{\Gamma}))$ for all $x\in\mathbb{R}^d$.

Define pseudo-metric on $\mathcal{D}_r$ by

$d_{at}(\mathit{\Lambda},\mathit{\Gamma}) := \inf\{\varepsilon > 0 : \exists x, y \in B_{\frac{\varepsilon}{2}}\ \ \rho(x+\mathit{\Lambda}, y+\mathit{\Gamma}) \le \varepsilon\}$.

Define $\mathit{\Lambda}\equiv\mathit{\Gamma}$ if and only if $d_{at}(\mathit{\Lambda},\mathit{\Gamma}) = 0$ and

$$\mathcal{D}_r^{\equiv} := \mathcal{D}_r/\equiv .$$

Theorem 5. *(Moody/Strungaru)* $(\mathcal{D}_r^{\equiv}, d_{at})$ *is a complete metric space. The action*

$$\alpha : \mathbb{R}^d \times \mathcal{D}_r^{\equiv} \longrightarrow \mathcal{D}_r^{\equiv},\ \alpha_x([\Lambda]) := [x + \Lambda],$$

is continuous.

This leads to a new notion of hull

$$\mathbb{A}(\Lambda) := \overline{\{\alpha_x([\Lambda]) : x \in \mathbb{R}^d\}^{at}}$$

and a new dynamical system

$$(\mathbb{A}(\Lambda), \alpha).$$

Important: Due to translation invariance of ρ the hull $\mathbb{A}(\Lambda)$ is actually a group. It can be considered to be the completion of $\mathbb{R}^d$ under the translation invariant pseudo-metric

$$\rho_\Lambda(t, s) = \rho_\Lambda(t - s, 0) = \rho(t + \Lambda, s + \Lambda).$$

Assume:

- $\varLambda$ Meyer (i.e. $\varLambda - \varLambda$ uniformly discrete).

- $\gamma = \lim_{n\to\infty} \frac{1}{|B_n|}\delta_{\varLambda\cap B_n} * \delta_{-(\varLambda\cap B_n)} = \sum_{t\in\varLambda-\varLambda} \eta(t)\delta_t$ exists, where

$$\eta(t) = \lim_{n\to\infty} \frac{\sharp(\varLambda \cap (t+\varLambda)\cap B_n)}{|B_n|}.$$

Recall $\rho_\varLambda(t,s) = \rho_\varLambda(t-s,0) = \rho(t+\varLambda, s+\varLambda)$.

Theorem 6. *TFAE:*

(i) $\varLambda$ is pure point diffractive (i.e. $\widehat{\gamma}$ is pure point).

(ii) For every $\varepsilon > 0$ the set of ε-periods $P_\epsilon := \{t \in \mathbb{R}^d : \rho_\varLambda(t,0) \le \varepsilon\}$ is relatively dense in $\mathbb{R}^d$.

(iii) $\mathbb{A}(\varLambda)$ is compact.

Equivalence of (i) and (ii) shown in Baake/Moody '02, equivalence of (ii) and (iii) shown in Moody/Strungaru '03. Almost periodicity enters (see Gouéré '02, '03 as well).

Crucial link $\rho_\Lambda(t,0) = 2(\eta(0) - \eta(t))$.

Proof.

$$\begin{aligned}
\rho_\Lambda(t,0) &= \limsup_{n\to\infty} \frac{\sharp(((t+\Lambda)\setminus\Lambda \cup \Lambda\setminus(t+\Lambda))\cap B_n}{|B_n|} \\
&= \limsup_{n\to\infty}\Big(\frac{\sharp(t+\Lambda)\cap B_n - \sharp(t+\Lambda)\cap\Lambda\cap B_n}{|B_n|} \\
&\quad + \frac{\sharp(\Lambda\cap B_n) - \sharp\Lambda\cap(t+\Lambda)\cap B_n}{|B_n|}\Big) \\
&= \eta(0) - \eta(t) + \eta(0) - \eta(t).
\end{aligned}$$

3. Where local topology and autocorrelation topology meet: Model sets

We have provided two frameworks to study aperiodic order:

$$(\mathcal{D}_r, d_{lt}) \text{ and } (\mathcal{D}_r^{\equiv}, d_{at}).$$

Here, d_{lt} measures local complete coincidence and d_{at} measures long range statistical coincidence.

Accordingly, a Meyer set $\varLambda$ gives rise to two dynamical systems

$$(\mathbb{X}(\varLambda), \alpha) \text{ and } (\mathbb{A}(\varLambda), \alpha).$$

Apriori the two frameworks (and then these two dynamical systems) are unrelated, even though there is a natural map

$$\beta : \mathcal{D}_r \longrightarrow \mathcal{D}_r^{\equiv},\ \varLambda \mapsto [\varLambda].$$

For model sets these two frameworks meet:

Theorem 7. *(Baake/L./Moody) Λ Meyer. TFAE:*

(i) $(\mathbb{X}(\Lambda), \alpha)$ comes from a regular model set.

(ii) $\beta : \mathbb{X}(\Lambda) \longrightarrow \mathbb{A}(\Lambda)$ is continuous and almost everywhere $1:1$.

(iii) $(\mathbb{X}(\Lambda), \alpha)$ has pure point dynamical spectrum with continuous eigenfunctions, which separate almost all points.

In some sense aperiodic model sets mark the border between periodicity and aperiodicity:

Theorem 8. *(Baake/L./Moody) Λ Meyer. TFAE:*

(i) Λ is crystallographic.

(ii) $\beta : \mathbb{X}(\Lambda) \longrightarrow \mathbb{A}(\Lambda)$ is continuous and injective.

(iii) $(\mathbb{X}(\Lambda), \alpha)$ has pure point dynamical spectrum with continuous eigenfunctions, which separate all points.