Structure of totally disconnected groups via compact open subgroups;

an overview of the theory and its applications.

Locally compact groups

Consider connected and totally disconnected case separately.

$$1 \longrightarrow G_0 \longrightarrow G \longrightarrow G/G_0 \longrightarrow 1$$

Methods:

1. G_0 is a projective limit of Lie groups.
   ($\rightsquigarrow$ Approximation by Lie groups.)

2. G/G_0 has a basis at e consisting of compact open subgroups.
   ($\rightsquigarrow$ analysis of action on $\{V : V$ is a compact, open subgroup$\}$.)

Totally Disconnected Case; single automorphism

Analogues of Lie techniques for analyzing $\alpha \in \mathrm{Aut}(G)$:

Definition 1 *The* scale of α *is the positive integer*

$$s(\alpha) := \min\{|\alpha(V)\colon \alpha(V)\cap V| : V \leqslant G \text{ is compact and open}\}.$$

The compact open subgroup O of G is tidy for α *if*

$$|\alpha(O)\colon \alpha(O)\cap O| = s(\alpha),$$

i.e. if the minimum is attained at O.

Identifying subgroups tidy for α corresponds to finding a triangularising basis for a linear transformation.

The scale of α plays a role corresponding to that of the eigenvalues of a linear transformation.

Totally Disconnected Case; groups of automorphisms

$\mathcal{H}$ a finitely generated abelian group of automorphisms of G.

Theorem 1 *There is $O \leqslant G$, tidy for every $\alpha \in \mathcal{H}$ such that*

$$O = O_0 \cdot O_1 \cdots O_q,$$

where each O_j is a closed subgroup of O such that:

1. $\alpha(O_j) \geqslant O_j$ *or* $\alpha(O_j) \leqslant O_j$ *and* $\alpha(O_0) = O_0$; $\quad \alpha \in \mathcal{H}$, $1 \leq j \leq q$

2. $s(\alpha) = \prod_{\alpha(O_j) \geqslant O_j} |\alpha(O_j) : O_j|; \quad \alpha \in \mathcal{H}$

3. $\bigcup_{\alpha \in \mathcal{H}} \alpha(O_j)$ *is a closed $\mathcal{H}$-invariant subgroup of G for each j.*

(1) The numbers

$$s_j(\alpha) := \begin{cases} |\alpha(O_j) \colon O_j| & \text{if } \alpha(O_j) \geq O_j \\ |O_j \colon \alpha(O_j)|^{-1} & \text{if } \alpha(O_j) \leq O_j \end{cases}$$

are analogues of eigenvalues for α and the subgroups

$$\widetilde{O_j} := \bigcup_{\alpha \in \mathcal{H}} \alpha(O_j)$$

are analogues of eigenspaces for the automorphisms in $\mathcal{H}$.

(2) For each j there is $\alpha_j \in \mathcal{H}$ such that $s_j(\alpha_j) =: t_j$ is a minimum and then $s_j(\alpha)$ is a power of t_j for every α.

(3) The subgroups O_j might not commute and they might not be able to be reordered, *e.g.* if $G = SL_n(\mathbb{Q}_p)$ and $\mathcal{H}$ is the group of inner automorhisms induced by diagonal matrices, then O is the Iwahori subgroup and the O_j's are root subgroups.

George A. Willis. The structure of totally disconnected locally compact groups. *Math. Ann.*, **300**:341–363, 1994.

George A. Willis. Further properties of the scale function on a totally disconnected locally compact group. *J. Algebra*, **237**:142–164, 2001.

George A. Willis. Tidy subgroups for commuting automorphisms of totally disconnected locally compact groups: An analogue of simultanous triangularisation of matrices. *New York J. Math.*, **10**:1–35, 2004. `http://nyjm.albany.edu:8000/j/2004/Vol10.htm`.

Application 1: ergodic $\mathbb{Z}^d$-actions by automorphisms

Theorem 2 *G a locally compact group. Suppose $\mathcal{H}$ is a finitely generated abelian group of automorphisms of G with a dense orbit. Then there is $K \trianglelefteq G$, compact and $\mathcal{H}$-invariant such that:*

1. *$G/K \cong V \times F_1 \times \cdots \times F_q$, where V is a finite-dimensional real vector space and $F_1, \ldots, F_q$ are locally compact totally disconnected fields of characteristic 0;*

2. *V and $F_1, \ldots, F_q$ are invariant under the factor action of $\mathcal{H}$ on G/K; and*

3. *the factor action of $\mathcal{H}$ is by linear maps on V and by characters (with values in $(F_i \setminus \{0\}, \times)$) on the F_i.*

History of the Theorem

(1) If $\mathcal{H}$ is generated by a single automorphism then G must be compact. This was conjectured by P. Halmos and proved: for connected G in the 1960's by R. Kaufman and M. Rajagopalan and by T.-S. Wu; and for totally disconnected G in the 1980's by N. Aoki, M. Dateyama and T. Kasuga.

(2) The case of Theorem 1 where G is connected was proved previously by S.G. Dani.

(3) If G is finite-dimensional and 'locally finitely generated', then K belongs to the centre, so that G is nilpotent.

History of the Proofs

(1) Approximation by Lie groups is used to prove the connected case.

(2) Aoki, Dateyama and Kasuga used (long) topological dynamics arguments to prove the totally disconnected case of Halmos' conjecture. Recently W. Previts and T.-S. Wu gave a much shorter proof that uses new structure theorems for totally disconnected groups.

(3) The proof of Theorem 1 also uses these structure theorems, which contain analogues of Lie techniques.

N. Aoki, Dense orbits of automorphisms and compactness of groups, *Topology Appl.*, **20**: 1–15, 1985.

S.G. Dani, On ergodic $\mathbb{Z}^d$-actions on Lie groups by automorphisms, *Israel J. Math.*, **126**: 327–344, 2001.

W.H. Previts and T.-S. Wu, Dense orbits and compactness of groups, *Bull. Aust. Math, Soc.*, **68**: 155–159, 2003.

S. G. Dani, Nimish A. Shah and George A. Willis, Locally compact groups with dense orbits under $\mathbb{Z}^d$-actions by automorphisms, *preprint, September 2004*.

Application 2: Dissipation of random walks

Definition 2 *Let μ be a regular Borel probability measure on a locally compact group G. The sequence of functions*

$$f_n\colon \{K\colon K \text{ is a compact subset of } G\} \to [0;1]$$
$$f_n(K) := \sup\{\mu^n(Kg)\colon g \in G\}$$

are called concentration functions *of μ.*

The pointwise behaviour of the concentration functions gives a measure of the dissipation of the random walk of law μ on G.

Theorem 3 *Suppose that G is not compact and supp(μ) generates G as a topological group.*
Then $f_n(K) \xrightarrow[n\to\infty]{} 0$ for all compact subsets of G.

Karl H. Hofmann and Arun Mukherjea, Concentration functions and a class of locally compact groups, *Math. Ann.* **256**: 535–548, 1981.

Wojciech Jaworski, Joseph Rosenblatt and George A. Willis, Concentration functions in locally compact groups, *Math. Ann.* **305**: 673–691, 1996.

Geometric interpretation of Willis' theory

G a totally disconnected locally compact group.
$\mathrm{Aut}(G)$ acts on the non-empty set

$$\mathcal{B}(G) := \{V : V \text{ is a compact, open subgroup of } G\}\,.$$

Define, for V and W in $\mathcal{B}(G)$

$$d(V, W) := \log(|V : V \cap W| \cdot |W : W \cap V|)\,.$$

Lemma 4 *The function d is a metric on $\mathcal{B}(G)$, and* $\mathrm{Aut}(G)$ *acts on the discrete metric space $(\mathcal{B}(G), d)$ by isometries.*

A compact, open subgroup O is tidy for α if and only if the displacement $d(\alpha(O), O)$ is minimal.

Definition 3 *A group $\mathcal{H}$ of automorphisms of G is called* flat *if there is an O tidy for every element of $\mathcal{H}$.*

1. A finitely generated abelian $\mathcal{H}$ is flat by Theorem 1.

2. Theorem 1 can be generalized to flat groups.

3. Let $\mathcal{H}$ be flat. Put $\mathcal{H}(1) := \{\eta \in \mathcal{H} \colon s(\eta) = 1 = s(\eta^{-1})\}$.
   $N_{\mathcal{H}}(O) := \{\eta \in \mathcal{H} \colon \eta(O) = O\} = \mathcal{H}(1)$ if O is tidy for $\mathcal{H}$.
   The group $\mathcal{H}/\mathcal{H}(1)$ is free abelian.

Lemma 5 *Let $\mathcal{H}$ be a flat group of automorphisms and let O be tidy for $\mathcal{H}$. Then $p(\gamma\mathcal{H}(1)) := d(\gamma(O), O)$ is a norm on $\mathcal{H}/\mathcal{H}(1)$.*

Application 3: Isometry groups of CAT(0) cell complexes

Definition 4 *The* flat rank *of a flat group $\mathcal{H}$ is the $\mathbb{Z}$-rank of the group $\mathcal{H}/\mathcal{H}(1)$.*
The flat rank of a group of automorphisms $\mathcal{A}$ is the supremum of the flat ranks of all flat subgroups of $\mathcal{A}$.

Lemma 6 *Suppose that G acts on a metric space X such that the map $X \to \mathcal{B}(G)$, $x \mapsto G_x$ is a quasi-isometric embedding.*
Let $H \leqslant G$ be a flat of finite flat rank n.
Then, for $x \in X$ the map $H.x \hookrightarrow X$ defines a n-quasi-flat in X.

Corollary 7 *Suppose that G acts on a complete cocompact* CAT(0)*-space X such that $x \mapsto G_x$ is a quasi-isometric embedding. Then* flat-rk$(G) \leq$ rk(X)*; in particular* flat-rk$(G) < \infty$.

The hypotheses of Corollary 7 apply to many isometry groups of Davis-realizations of buildings.

Relative position of flat groups; space of directions

Using the action of a group of automorphisms $\mathcal{A}$ on the set of compact open subgroups, one can give sense to the *direction* of automorphisms in $\mathcal{A}$.

For a flat group $\mathcal{H}$, the set of directions of elements in $\mathcal{H}$ correspond to a cone of rays in the lattice $\mathcal{H}/\mathcal{H}(1)$.

The set of directions $\partial\mathcal{A}$, of a group of automorphisms $\mathcal{A}$ carries a natural pseudo-metric. $\mathcal{A}$ acts by isometries on the completion of this space.

Udo Baumgartner, Rögnvaldur G. Möller and George A. Willis, Groups of flat rank at most 1, *in preparation*.

Udo Baumgartner, Bertrand Rémy and George A. Willis, Flat rank of topological Kac-Moody groups, *in preparation*.

Udo Baumgartner and George A. Willis, The direction of an automorphism of a totally disconnected locally compact group, *preprint July 2004*.