Abstract

Much work has been done studying amenable group actions, but until recently, it has been difficult to handle non-amenable actions. A break-through was made with work of Levitt, Kechris, Gaboriau, which defines a new invariant, the cost of a group action (or equivalence relation). Gaboriau showed how to use this invariant to distinguish between group actions of, for example, the free group on two generators and the free group on three generators.

In joint work with Golodets, we used the theory of index cocycles of Feldman, Sutherland and Zimmer to calculate the cost of equivalence relations which are finite extensions. This enables us to resolve some conjectures of Gaboriau, and also to show that many group actions cannot be isomorphic.

I will give an introduction to the theory of costs and an outline of our main results.

Costs of equivalence relations and group actions

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1 History

J. Feldman and C. Moore (1977) introduced the notion of a standard countable measure-preserving equivalence relation to investigate the orbit properties of dynamical systems. This part of ergodic theory has deep relations with von Neumann algebras.

Connes-Feldman-Weiss and Ornstein-Weiss(1980, 1981) Investigated amenable countable equivalence relations. They proved that any ergodic measure-preserving free action of a countable amenable group is orbit equivalent to a free action of $\mathbb{Z}$.

Non-amenable measure-preserving equivalence relations are more complicated. For example, there are countable groups which have uncountably many actions which are pairwise non-orbit equivalent. Further, there exist measure-preserving countable equivalence relations which have countable fundamental groups (Golodets-Gefter), analogue of phenomena for von Neumann algebras discovered by Connes, Popa.

Zimmer's(1977 – 1980) work on strong rigidity for ergodic actions of semi-simple Lie groups and their lattices made a start in the study of non-amenable actions. But it was still not possible to distinguish between actions of the free group on 2 generators and the free group on 3 generators!

Recent results of **Adams** (1990), **Furman**(1999), **Gaboriau** (2000–2004), **Levitt** (1995) and other authors develop the notion of costs ℓ^2-Betti numbers for measured equivalence relations. Kechris and Miller 's lecture notes on costs of equivalence relations and groups are a good source for this theory.

In particular, **Gaboriau** showed that free actions of groups F_n and F_m, are not orbit equivalent if $n \neq m$. He did this by calculating the cost of $F_n = n$.

2 Our results

Let $E \subseteq F$ be aperiodic countable Borel equivalence relations on a Lebesgue space (X, μ), where μ is a finite F-invariant measure. Let $[F : E] = n < \infty$. Then

$$C_\mu(E) - \mu(X) = n(C_\mu(F) - \mu(X)). \tag{1}$$

Here $C_\mu(E)$ and $C_\mu(F)$ are the costs of E and F respectively.

In particular, this answers a question of Kechris and Miller.

We also prove if $E \subseteq F$ are as above then F is treeable if and only if E is treeable.

3 Measured equivalence relations

A **relation** R on a set X is a set of ordered pairs from X, $R \subset X^2$. If R is a relation we write

$$xRy \Leftrightarrow (x, y) \in R.$$

A **graph** $\mathcal{G}$ with vertex set X is a non-reflexive (i.e., $(x,x) \notin \mathcal{G}\ \forall x \in X$), symmetric (i.e., $\mathcal{G} = \mathcal{G}^{-1}$) relation on X. The **neighbours** of $x \in X$ in the graph $\mathcal{G}$ are $\{y \in X : (x,y) \in \mathcal{G}\}$.

A $\mathcal{G}$**-path** from x to y is a finite sequence of vertices $x = x_0, x_1, \cdots, x_n = y$ such that $(x_i, x_{i+1}) \in \mathcal{G},\ \forall i < n$, and $x_i \neq x_j$ if $i \neq j$. Define an equivalence relation on X given by

$$xEy \Leftrightarrow \exists\ a\ \mathcal{G}\text{-path from } x \text{ to } y.$$

Its equivalence classes are the **connected components** of $\mathcal{G}$. A **cycle** is a $\mathcal{G}$-path $x_0, x_1, \cdots, x_n = x_0$, starting and ending at the same point. A graph $\mathcal{G}$ is **acyclic** if it contains no $\mathcal{G}$-cycles. An acyclic graph containing only one connected component is called a **tree.**

3.1 Countable Borel equivalence relations.

Let X be a standard Borel space. An equivalence relation E on X is called **Borel** if it is a Borel subset of the product space X^2. A Borel equivalence relation E is **countable** if every equivalence class $[x]_E,\ x \in X$, is countable.

If Γ is a countable group Γ and $(g,x) \mapsto g \cdot x$ is a Borel action of Γ on X, then the orbit equivalence relation

$$xE_\Gamma^X y \Leftrightarrow \exists g \in \Gamma \text{ such that } g \cdot x = y$$

is countable. The converse assertion is also true. Feldman and Moore showed that a countable Borel equivalence relation always arises from a countable group Borel action.

A countable equivalence relation E in X is called **aperiodic** if every equivalence class $[x]_E$ is infinite. A Borel subset S of X is called a **complete section** if it meets every equivalence class.

We denote by $[E]$ the set of all Borel automorphisms f of X with $f(x)Ex$ for all $x \in E$, and by $[[E]]$ the set of all partial Borel automorphisms $f : A \to B$, where A, B are Borel subsets of X, with $f(x)Ex$, $\forall x \in A$.

We further denote by $\mathrm{Aut}(E)$ the group of all Borel automorphisms of E, that is to say, $f \in \mathrm{Aut}(E)$ if f is a Borel automorphism of X and $xEy \Leftrightarrow f(x)Ef(y)$. Observe that $[E] \subset \mathrm{Aut}(E)$.

We call elements of $[E]$ **inner automorphisms** of E and we call $f \in \mathrm{Aut}(E) \setminus [E]$ an **outer automorphisms** if x is not E-equivalent to $f(x)$ for all $x \in X$.

For $f, g \in \mathrm{Aut}(E)$ we write $f = g(\mathrm{mod}[E])$ if there exists $h \in [E]$ with $f = g \circ h$.

3.2 Invariant measures.

Let μ be a measure on a standard Borel space X and E a countable Borel equivalence relation on X.

We say that μ is E-**invariant** if there is a countable group Γ and a Borel action of Γ on X with $E^X_\Gamma = E$, such that μ is Γ-invariant. We call E a **measured equivalence relation** if there exists an E-invariant measure μ on X. An E-invariant measure μ is **ergodic** if every E-invariant Borel subset of X is either null or conull.

If μ is an E-invariant measure we can define a measure M_μ on E as follows

$$M_\mu(A) = \int |A_x| d\mu(x),$$

where A is a Borel subset of E, $|S| = \text{card } S$, $A_x = \{y : (x; y) \in$

$A\}$. It turns out that

$$\int |A_x| d\mu(x) = \int |A^y| d\mu(y)$$

where $A^y = \{x : (x, y) \in A\}$.

3.3 Graphings.

A graph on a standard Borel space $(X, \mathcal{B})$ is a graph $\mathcal{G}$ on the set X, such that $\mathcal{G} \subseteq X^2$ is Borel, and every $x \in X$ has at most countably many neighbours. Let E be a countable Borel equivalence relation. A **Borel graphing** of E is a graph $\mathcal{G}$ such that the connected components of $\mathcal{G}$ are exactly the E-equivalence classes. If $\mathcal{G}$ is a tree, this is called a **treeing**.

There is another concept of graph which is called an L-graph (L stands for Levitt). This is a countable family $\Phi = \{\varphi_i\}, i \in I$, of partial Borel isomorphisms $\varphi_i : A_i \to B_i$ where A_i, B_i are Borel subsets of $X, \varphi_i \in [[E]]$. We say that Φ is an L-**graphing** of E if Φ generates E, i.e. xEy means that $x = y$ or there is a sequence $i_1, \cdots, i_k \in I$ and $\varepsilon_1, \cdots, \varepsilon_k \in \{\pm 1\}$ such that $x = \varphi_{i_1}^{\varepsilon_1} \cdots \varphi_{i_k}^{\varepsilon_k}(y)$. Similarly one defines an **L-treeing**.[1]

For every L-graph $\Phi = \{\varphi_i\}$ we can define an associated graph $\mathcal{G}_\Phi$ which generates the same equivalence relation as Φ. Conversely, for every graph $\mathcal{G}$ one can find an L-graph $\Phi_\mathcal{G}$ such that $\mathcal{G} = \mathcal{G}_{\Phi_\mathcal{G}}$.

3.4 Cost of an equivalence relation.

Let E be a countable Borel equivalence relation on X and μ an E-invariant measure. Now we define the (μ) **cost** of E, which will be denoted by $C_\mu(E)$.

[1] $\mu\{x \in \mathrm{dom} w : wx = x\} = 0$ for all non-empty reduced words w.

If $\mathcal{G}$ is a graphing of E we define its **cost** by $C_\mu(\mathcal{G}) = \frac{1}{2}M(\mathcal{G})$. If $\Phi = \{\varphi_i\}_{i\in I},\ \varphi_i \subseteq [[E]]$ is an L-graph, define its cost by

$$\begin{aligned} C_\mu(\Phi) &= \sum_{i\in I} \mu(\mathrm{dom}(\varphi_i)) \\ &= \sum_{i\in I} \mu(\mathrm{rng}(\varphi_i)) \end{aligned}$$

Then

$$C_\mu(\Phi) = \frac{1}{2}\int \sum_{i\in I}(\chi_{A_i}(x) + \chi_{B_i}(x))d\mu(x)$$

where $A_i = \mathrm{dom}(\varphi_i),\ B_i = \mathrm{rng}(\varphi_i)$. Hence if $\mathcal{G}_\Phi$ is the graph associated to Φ, then $|(\mathcal{G}_\Phi)_x| \leq \sum_{i\in I}(\chi_{A_i}(x) + \chi_{B_i}(x))$, and $C_\mu(\mathcal{G}_\Phi) = \frac{1}{2}M(\mathcal{G}_\Phi) \leq C_\mu(\Phi)$.

Conversely, let $\mathcal{G}$ be a graphing of E and $\Phi_\mathcal{G}$ be the associated L-graph. Then $C_\mu(\mathcal{G}) = C_\mu(\Phi_\mathcal{G})$. Thus we can define the μ **cost** of E as

$$\begin{aligned} C_\mu(E) &= \inf\{C_\mu(\mathcal{G}) : \mathcal{G} \text{ is a graphing of } E \text{ a.e.}\} \\ &= \inf\{C_\mu(\Phi) : \Phi \text{ is an } L-\text{graphing of } E \text{ a.e.}\} \end{aligned}$$

This notion was introduced by G. Levitt. It is clear that $0 \leq C_\mu(E) \leq \infty$.

Levitt and Gaboriau showed that $C_\mu(E) = C_\mu(\mathcal{G})$, if and only if $\mathcal{G}$ is a treeing. The same result holds for L-graphings and L-treeings.

Gaboriau showed:

- Let $S \subseteq X$ a Borel complete section for E and μ an E-invariant measure. Then

$$C_\mu(E) = C_{\mu|_S}(E|_S) + \mu(X\backslash S)$$

- $C_\mu(E_1 *_{E_3} E_2) = C_\mu(E_1) + C_\mu(E_2) - C_\mu(E_3)$ where $*$ is the amalgamated join of equivalence relations.

If Γ is a group, one defines $C(\Gamma) = \inf\{C_\mu(E) : E$ is induced by a free action of $\Gamma\}$.

Gaboriau showed further that:

- $C(F_n) = n$
- $SL(2, \mathbb{Z})$ is treeable with price $1 + \frac{1}{12}$
- The fundamental group $\pi_1(\Sigma_g)$ of a surface of genus g is fixed price, of cost $2g - 1$
- If Δ is a closed normal subgroup of Γ then $C(\Gamma) - 1 = [\Gamma : \Delta](C(\Delta) - 1)$

A group is said to have **fixed price** if $C(\Gamma) = C_\mu(E)$ for all E induced by a free action of Γ.

A group is **cheap** if $C(\Gamma) = 1$

A re-phrasing of some theorems of Ornstein and Weiss says that every infinite amenable group is cheap, treeable and of fixed price!

If $E \subseteq F$ are equivalence relations and every F-class contains exactly n E-classes then we say that **index** of E in F is n, or

$$[F : E] = n.$$

We say that E **has finite index** in F, in symbols

$$[F : E] < \infty,$$

if every F-class contains only finitely many E-classes.

Gaboriau's theorem above will follow from our result, which had been conjectured by Kechris and Miller.

Theorem 3.1 *Let F be an aperiodic countable Borel equivalence relation on a standard Borel measure space (X, μ) and μ an F-invariant finite measure, E be a Borel subrelation of F with finite index: $[F : E] = n$ then*

$$C_\mu(E) - \mu(X) = n(C_\mu(F) - \mu(X)). \tag{2}$$

There are 3 main steps in the proof of the theorem:

1. ***Step 1*** Suppose that E is generated by F together with a finite group K.
2. ***Step 2*** Suppose F is ergodic and use index cocycles to reduce to Step 1.
3. ***Step 3*** Use the ergodic decomposition of F to reduce to Step 2.

4 Step 1: Equivalence relations of finite index

Let $E \subseteq F$ be two countable Borel equivalence relations on the standard Borel space X. E is said to be **normal** in F, written $E \lhd F$, if there is a countable group of Borel automorphisms $\{g_i\}_{i\in\mathbb{N}}$, of X which generates F, i.e.

$$xFy \text{ if and only if there exists } i \in \mathbb{N} \text{ such that } g_i x = y \tag{3}$$

and such that each g_i preserves E, i.e.

$$xEy \,\text{if and only if } g_i(x)Eg_i(y). \tag{4}$$

For example, if G is a countable group with a Borel action on X, and N is a normal subgroup of G then $E_N^X \lhd E_G^X$.

A slightly stronger notion is:

Definition 4.1 *We say that E is a* ***strongly normal subrelation*** *of F, denoted $E \overset{s}{\lhd} F$, if there is a family $\{f_i\}_{i\in I\subseteq \mathbb{N}}$, of Borel automorphisms of X, such that the following conditions hold*

(i) $f_i \in \mathrm{Aut}(E)$, *i.e.* $xEy \Rightarrow f_i(x)Ef_i(y)$,

(ii) $f_1 = \mathrm{id} \mod ([E])$ *and for* $i > 1$, f_i *is an outer automorphism of E, i.e. for all $x \in X$, x is not E-equivalent to* $f_i(x)$.

(iii) The functions $\{f_i\}_{i\in I}$ form a group mod $[E]$, that is, for all i, j there exists $k(i,j) \in I$ such that $f_i f_j = f_{k(i,j)} c(i,j)$, for some $c(i,j) \in [E]$.

(iv) E and $\{f_i\}$ generate F, in the following sense: $xFy \Leftrightarrow f_i(x)Ey$ for some $i \in I$.

It is obvious how to adapt this definition to the case where E is a measured equivalence relation.

Below, we shall consider the case where $|I| < \infty$ so that $\{f_i\}_{i\in I}$ is a finite group (mod$[E]$).

Lemma 4.2 *Let F be a measured equivalence relation on (X, μ) and suppose that $E \overset{s}{\lhd} F$. Then $E \lhd F$.*

If F is not ergodic, then it is not hard to give an example where $E \lhd F$ but $E \overset{s}{\ntriangleleft} F$.

The situation changes if E is ergodic. In this case, F is also ergodic, and if $E \subseteq F$ with $E \lhd F$, then $E \overset{s}{\lhd} F$.

In the case of a normal subrelation of finite index, we can weaken the hypotheses.

Lemma 4.3 *Let E, F be as above, with F ergodic, $E \lhd F$ and $[F : E] < \infty$. Then there exists $n \in \mathbb{N}$ such that $[F : E] = n$, and $E \overset{s}{\lhd} F$.*

Theorem 4.4 *Let $E \subseteq F$ be aperiodic countable Borel equivalence relations on a standard Borel space $(X, \mathcal{B})$ equipped with a finite F-invariant measure μ. Suppose that $[F : E] = n < \infty$ and that there is a finite group K, of order n, which consists of outer automorphisms of E and is such that F is generated by E and K. Then*

$$C_\mu(E) - \mu(X) = n(C_\mu(F) - \mu(X)). \tag{5}$$

The condition $[F : E] = n$ in the statement of Theorem 4.4 is equivalent to the following conditions:

- if $(x, y) \in E$, then $(kx, ky) \in E$ for a.e. $(x, y) \in X \times X$,
- for all $k \in K \setminus \{e\}$, we have $(x, kx) \notin E$ for a.e. $x \in X$.

In particular, it follows that every nontrivial element of K acts freely on (X, μ). Note also that every element $k \in K$ either acts inside each ergodic component of E or transposes ergodic components.

The main technique is the following idea:

4.1 Invariant graphings.

Definition 4.5 *Let $(X, \mu), E$ and K be as in the statement of Theorem 4.4. We will say that a graphing $\mathcal{G}$ of E is K**-invariant** if $k\mathcal{G}k^{-1} = \mathcal{G}$ for all $k \in K$. An L-graphing $\Phi = \{\varphi_i\}_{i\in\mathbb{N}}$, of E is K-invariant if $k\Phi k^{-1} = \{k\varphi_i k^{-1}\} = \Phi$ for any $k \in K$.*

Note that $\mathcal{G}$ is K-invariant if and only if $(x,y) \in \mathcal{G}$ implies $(kx,ky) \in \mathcal{G}$ for all $k \in K$.

It is clear that a K-invariant L-graphing Φ of E defines a K-invariant graphing $\mathcal{G}$ of E. The converse statement is also true.

Proposition 4.6 *Suppose that $C_\mu(E) < \infty$. If for any $\varepsilon > 0$ there is a K-invariant L-graphing $\Phi = \{\varphi_i\}_{i\in\mathbb{N}}$, of E such that $C_\mu(\Phi) \leq C_\mu(E) + \varepsilon$, then (5) holds.*

Proposition 4.7 *Let $C_\mu(E) < \infty$. Then for any given $\varepsilon > 0$ there is a K-invariant L-graphing Φ_ε of E such that*

$$C_\mu(E) \leq C_\mu(\Phi_\varepsilon) < C_\mu(E) + \varepsilon \tag{6}$$

Note that Theorem 4.4 is a consequence of Propositions 4.6 and 4.7.

Let $\Phi^K = \{\varphi_i \in \Phi : \forall k \in K,\ k^{-1}\varphi_i k \in \Phi\}$, and $\Phi^r = \Phi\backslash\Phi^K$. Notice that if $\varphi \in \Phi^r$ and $A \subseteq \mathrm{dom}\varphi$, with $\mu(A) > 0$, then $\varphi|_A$ does not belong to Φ^K.

The key idea in the proof is:

Lemma 4.8 *Let E, K and Φ be as above and suppose that $\varphi_1 \in \Phi^r$. Then for any $Y \subseteq \mathrm{dom}\varphi_1$ of positive measure, there exists $Y_1 \subset Y$, also of positive measure, such that if we define the family $\{f_k\}$ of Borel functions by*

$$f_k(x) = k\varphi_1 k^{-1}(x),\ x \in kY_1,\ k \in K,$$

and set $\Phi_1^K = \Phi^K \cup \{f_k, k \in K\}$, then there exists an L-graphing Φ_1 of E for which

$$C_\mu(\Phi) = C_\mu(\Phi_1),\ (\Phi_1)^K = \Phi_1^K \text{ and } C_\mu(\Phi_1^K) > C_\mu(\Phi^K).$$

Proof. We may suppose that there is a subset $Y_1' \subset Y$, with $\mu(Y_1') > 0$, and $k_1 \in K$ such that the function $k_1\varphi_1 k_1^{-1}(x)$ does not belong to Φ for $x \in k_1 Y_1'$. It follows from the definition of an L-graphing that there are Borel functions $\psi_1, \cdots \psi_t,\ t < \infty$, belonging to Φ and $\varepsilon_1, \cdots, \varepsilon_t$, where $\varepsilon_i = \pm 1, 1 \leq i \leq t$, such that:

$$k_1\varphi_1 k_1^{-1}(x) = \psi_1^{\varepsilon_1} \ldots \psi_t^{\varepsilon_t}(x) \text{ for } x \in k_1 Y_1'. \tag{7}$$

Furthermore, there is a function ψ_p amongst the functions $\{\psi_i\}, 1 \leq i \leq t$, such that $\psi_p \notin \Phi^K$. Let $f_1' = \varphi_1|_{Y_1'}$. As $(x, k_1 x) \notin E$, we can choose Y_1' such that f_1' is not the restriction of one of the functions $\{\psi_i\}, 1 \leq i \leq t$, from (7).

We shall define a new L-graphing Φ_1'. To simplify the discussion we assume that $\varepsilon_t = 1$ and $\psi_t \notin \Phi^K$. (The other cases are similar). Then $\psi_t = \varphi_{m_2} \in \Phi^r$ for some $m_2 \in \mathbb{N}$. Let

$$\begin{aligned}
f_1'(x) &= \varphi_1(x) \text{ for } x \in Y_1', \\
f_2'(x) &= k_1\varphi_1 k_1^{-1}(x) \text{ for } x \in k_1 Y_1', \\
\varphi_1'(x) &= \varphi_1(x) \text{ for } x \in \operatorname{dom} \varphi_1 \setminus Y_1', \\
\varphi_{m_2}'(x) &= \varphi_{m_2}(x) \text{ for } x \in \operatorname{dom} \varphi_{m_2} \setminus k_1 Y_1'.
\end{aligned}$$

Define Φ_1' as follows:

$$(\Phi_1')^r = \{\varphi_i : i \in \mathbb{N} \setminus \{1, m_2\}\} \cup \{f_1', f_2', \varphi_1', \varphi_{m_2}'\},\ (\Phi_1')^K = \Phi^K.$$

By our construction of Φ_1' and (7), we see that Φ_1' is an L-graphing of E and $C_\mu(\Phi_1') = C_\mu(\Phi)$.

We can apply this construction simultaneously to the family of functions $\{k\varphi_1 k^{-1}\}_{k \in K}$, and obtain an L-graphing Φ_1 of E with the following properties:

$$\Phi_1^K = \Phi^K \cup \{f_k :\ k \in K\}$$

where $f_k,\ k \in K$, are as in the statement of the Lemma, and

$$\Phi_1^r = \{\varphi_i :\ i \in \mathbb{N},\ i \neq 1,\ m_2, \cdots, m_{n-1}\} \cup \{\varphi_1^2, \varphi_{m_2}^2, \cdots, \varphi_{m_{n-1}}^2\}$$

$$\begin{aligned} \text{where} \quad \varphi_1^2 &= \varphi_1 \text{ for } x \in \text{ dom } \varphi_1 \backslash Y_1, \\ \varphi_{m_j}^2 &= \varphi_{m_j} \text{ for } x \in \text{ dom } \varphi_{m_j} \backslash k_j Y_1,\ 2 \leq j \leq n-1. \end{aligned}$$

Here $Y_1 \subseteq Y_1'$ is a subset of dom φ_1 as in the statement of the Lemma, with sufficiently small positive measure. It follows from our construction that $C_\mu(\Phi) = C_\mu(\Phi_1)$. $\square$

Continuing this process we obtain, after j steps, an L-graphing Φ_j with the following properties:

$$C_\mu(\Phi_j) = C_\mu(\Phi_{j-1}) = C(\Phi); \tag{8}$$

$$\Phi_{j-1}^K \subset \Phi_j^K,\ C_\mu(\Phi_{j-1}^K) < C_\mu(\Phi_j^K); \tag{9}$$

$$\Phi_j^r = \{\varphi_i^j\}_{i \in \mathbb{N}}, \tag{10}$$

where $\varphi_i^j = \varphi_i$ for $x \in \text{dom}\varphi_i^j \subseteq \text{dom}\varphi_i,\ \varphi_i \in \Phi^r$, and $\text{dom}\varphi_i^j = \text{dom}\varphi_i$ except for a finite set of integers i.

Note that this allow us to define $\varphi_i^\infty = \lim_{j\to\infty} \varphi_i^j$, where

$$\varphi_i^j \in \Phi_j^r,\ \varphi_i^\infty = \varphi_i \text{ for } x \in \text{dom}\varphi_i^\infty \subseteq \text{dom}\varphi_i^j.$$

5 Step 2: Index cocycles

We describe the structure of equivalence relations $E \subseteq F$, where $[F : E] = n < \infty$, using the theory of index cocycles developed by Feldman, Sutherland and Zimmer. The main structure theorem is as follows.

Theorem 5.1 *Let $E \subseteq F$ be aperiodic countable Borel equivalence relations on (X, μ) where $\mu(X) < \infty$ and μ is F-invariant. Assume that F is ergodic, $[F : E] = n < \infty$ and let $X = \cup_{i=1}^{t} X_i$ be the partition of X into the ergodic components of E, that is $E_i = E\big|_{X_i}$ is an ergodic component of E, for all $1 \leq i \leq t$. Then the following assertions hold:*

1. *Let $F_i = F|_{X_i}$ and $[F_i : E_i] = n_i$: then $n = \sum_{i=1}^{t} n_i$ and hence $1 \leq t \leq n$.*
2. *There is an equivalence relation N, which is a subrelation of finite index of both E and F, and which has the following properties:*
   - *(i) $N_i = N|_{X_i}$ is ergodic in $(X_i, \mu|_{X_i})$;*
   - *(ii) N_i is a strongly normal subrelation of finite index in both E_i and F_i. In particular, if $n_1 = n_2 = \cdots = n_t$ then N is strongly normal in E.*
   - *(iii) There exists $Z \subset X$, with $\mu(Z) > 0$, which is a complete Borel section of both E and F, such that $N|_Z \subseteq F|_Z$ and $N|_Z$ is a strongly normal subrelation of $F|_Z$.*

The case when E is ergodic, (i.e. $t = 1$) was dealt with by Sutherland.

The general case is seen by using index cocycles for the pair $E \subseteq F$. If F is ergodic then there exist Borel functions $\varphi_j : X \to X$ such that $\varphi_j \in [[F]]$ and $\{E\varphi_j(x),\ 1 \leq j \leq n\}$ is a partition of the orbit Fx for μ-a.e. $x \in X$. The functions $\varphi_j(x)$, $1 \leq j \leq n$, are called "choice functions" for the pair $E \subseteq F$. If E is also ergodic then we can choose φ_j such that $\varphi_j \in [F]$. For every family of choice functions $\{\varphi_j(x)\}_{j=1}^{n}$, we define a cocycle $\sigma : F \to \Sigma(n)$, by

$$\sigma(x, y)(i) = j$$

if $\varphi_i(y) E \varphi_j(x)$. Then $\sigma \in Z^1(F, \Sigma(n))$, and the class of σ in $H^1(F, \Sigma(n))$ is independent of the family of choice functions $\{\varphi_j\},\ j = 1, \cdots, n$.

σ is called the **index cocycle** of the pair $E \subseteq F$.

Let F be an ergodic equivalence relation and σ a cocycle on F with values in a locally compact group G. The Mackey action of G associated with σ is always ergodic. If the action is trivial, then σ is called a cocycle with **dense range** in G.

If G is a finite group then the Mackey action associated with σ is transitive and is isomorphic to an action of G on the homogeneous space G/K where K is a subgroup of G. Zimmer showed that the cocycle σ is cohomologous to a cocycle σ' with values in K. That is, there exists a Borel function $v : X \to G$ such that

$$\sigma'(x, y) = v(x)^{-1} \sigma(x, y) v(y)$$

and $\sigma'(x, y) \in K$ for μ-a.e. $x, y \in X$. Now our group G has a transitive action on G/K but σ' has dense range in K. Let μ_K be the Haar measure of K, and define an equivalence relation F_K on $(X \times K, \mu \times \mu_K)$ by

$$(x, k_1) \sim (y, k_2)$$

if $(x, y) \in F$, with $k_1 = \sigma'(x, y) k_2$. Then F_K is ergodic, or equivalently the Mackey action of K, associated with σ', is trivial. Now let

$$N = \ker \sigma' = \{(x, y) \in X \times X : xFy, \sigma'(x, y) = e\}.$$

It follows from the ergodicity of F_K and the finiteness of K that N is an **ergodic** equivalence subrelation of F.

Indeed, it follows from the ergodicity of N that for every $k \in K$ there exists $u_k \in [F]$ such that

$$\sigma'(u_k x, x) = k, \quad x \in X,$$

and

$$u_{k_1} u_{k_2} = u_{k_1 k_2} n, \quad n \in [N].$$

Hence $\{u_k\}, k \in K$, generates a group of outer automorphisms of N mod $[N]$. Moreover, $\{u_k : k \in K\}$, and N together generate F because σ' has dense range in K. Now if xNy for $x, y \in X$ then $u_k x N u_k y$ because

$$\begin{aligned} \sigma'(u_k x, u_k y) &= \sigma'(u_k x, x)\sigma'(x, y)\sigma'(y, u_k y) \\ &= kek^{-1} = e. \end{aligned}$$

It follows that N is a strongly normal ergodic subrelation of F and hence $[F : N] = |K|$.

6 Step 3: Ergodic decomposition

Theorem 6.1 *Let F be a countable Borel equivalence relation on a standard Borel space X and assume $I_F \neq 0$. Then $\mathcal{E}I_F \neq 0$ and there is a Borel surjection $\pi : X \to \mathcal{E}I_F$ such that*

(i) π is F-invariant.

(ii) If $X_f = \{x : \pi(x) = f\}$ for $f \in \mathcal{E}I$ then $f(X_f) = 1$ (in fact, f is the unique F-ergodic invariant measure on $F|_{X_f}$).

(iii) For any $\mu \in I_F$, $\mu = \int \pi(x) d\mu(x) = \int f d\nu(f)$ where $\nu = \pi_* \mu$.

Moreover:

Corollary 6.2 *For any $\mu \in I_F$*

$$C_\mu(F) = \int C_{\pi(x)}(F) d\mu(x) = \int C_f(F) d\mu(f). \qquad (11)$$

It is clear that if $[F : E] = n$ then $[F|_{X_f} : E|_{X_f}] = n$ for ν-a.e. $f \in \mathcal{E}I_F$. Furthermore, the surjection $\pi : X \to \mathcal{E}I_F$ of Theorem 6.1 is also E-invariant.

$$C_\mu(E) = \int C_{\pi(x)}(E) d\mu(x) = \int C_f(E) d\nu(f). \qquad (12)$$

7 Treeability

Theorem 7.1 *Let $E \subseteq F$ be aperiodic countable Borel equivalence relations on a standard Borel space (X, μ) with a finite F-invariant measure μ. Suppose that $[F : E] = n < \infty$. Then F is treeable if and only if E is treeable.*

The path of the proof is similar to that above. We first consider the case where F is a finite strongly normal extension of E, i.e. $E \overset{s}{\lhd} F$.

Then we establish the existence of K-invariant L-treeings of E.

Proposition 7.2 *Let $(X, \mu), E$ and K be as before, and $\Phi = \{\varphi_i\}_{i \in \mathbb{N}}$ an L-treeing of E. Then there is a K-invariant L-treeing of E.*

This proposition is an analogue of Proposition 4.7 above.

Lemma 7.3 *Suppose that the assumptions of Lemma 4.8 are satisfied, and that Φ is an L-treeing of E. Then all the conclusions of that Lemma hold and in addition Φ_1 is an L-treeing of E.*

We obtain the following.

Corollary 7.4 *Let $E \subseteq F$ be as in the statement of Theorem 7.1, with $C_\mu(E) < \infty$. Suppose that F is a strongly normal extension of E. Then F is treeable if and only if E is treeable.*

Example 7.5 *(i) Let F_n be the free group on n generators, and $\Sigma(n)$ the group of permutations of $\{1, \ldots, n\}$. Then $\Sigma(n)$ acts as a group of outer automorphisms of F_n. We can show that the semi-direct product $F_n \rtimes \Sigma(n)$ is a treeable group.*

(ii) Let $p \geq 3$ be a prime number, and let T_p be an outer automorphism of $\mathbb{Z}_p$ (for example, T_p could be multiplication by 2). Then F_n acts on $\mathbb{Z}_p$ by the automorphism T_p, so we may define the semi-direct product $G = \mathbb{Z}_p \rtimes F_n$. Then G is treeable.

(iii) We saw above that for $g > 1$, the group $\pi_1(\Sigma_g) = F_g *_{\mathbb{Z}} F_g$ has fixed price and is weakly treeable. Thus, for any finite group K, $\pi_1(\Sigma_g) \times K$ and $K \rtimes \pi_1(\Sigma_g)$ both have fixed price and are both weakly treeable.

8 Subgroups of treeable groups.

It is well known that any Borel subrelation of a measured treeable equivalence relation is also treeable. We prove an analogous result for treeable groups. This resolves another conjecture of Kechris and Miller.

Theorem 8.1 *Let G be a countable group which is treeable, and H a subgroup of G. Then H is also treeable.*

In order to prove this theorem, we show that if a countable group H as above has a treeable action α on a Lebesgue space (X, μ), and a free action on an α-invariant factor space (Y, ν) of (X, μ), then the restriction of α to (Y, ν) is also treeable. Then we construct a free action T^G_β of G from any free action of β of H, the **co-induced action**. Since G is treeable, it follows that T^G_β is treeable, and hence its restriction T^H_β to H is a treeable action of H. This action is such that it has an invariant factor action isomorphic to β.

Theorem 8.2 *Suppose that a countable group G has a free treeable action on a standard Borel space which is equipped with a G-invariant probability measure (i.e. G is weakly treeable). Then G is treeable in the sense of Gaboriau.*

Lemma 8.3 *Let G and H be as in the statement of Theorem 8.1 and let β be any measure-preserving free action of H on a Lebesgue probability space $(Y, \mathcal{C}, \mu)$. Then E^Y_H is treeable.*

Co-induction We use β to construct a free measure-preserving action of G on a Lebesgue probability space (Z, ν). Fix a section $s : H\backslash G \to G$ of the homogeneous space $H\backslash G$ with $s[e] = e$, where e is the identity of G. Consider the product space $(Z, \nu) = \prod_{H\backslash G}(Y, \mu)$, where $\nu = \otimes_{H\backslash G}\mu$. Then an element $z \in Z$ has the form $z = (z_\theta)$, where $z_\theta \in Y$ for all $\theta \in H\backslash G$. We define the **co-induced** action of G on Z by the formula

$$(gz)_\theta = s(\theta)gs(\theta g)^{-1}z_{\theta g} \qquad (13)$$

where we note that the cocycle $(g, \theta) \mapsto s(\theta)gs(\theta g)^{-1}$ takes values in H. An easy calculation shows that this action is a well-defined, free left action of G, which preserves the probability measure ν on Z. In particular, if $h \in H$, then

$$(hz)_{[e]} = h(z_{[e]}).$$

Corollary 8.4 *Let G be a treeable group and H a cheap subgroup of G. Then H is amenable.*

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