Aperiodic Order and Dynamical Systems I

Robert V. Moody
University of Alberta, Edmonton, Canada

MASCOS Workshop on Algebraic Dynamics

University of New South Wales, Sydney, Australia
February 14 - February 18, 2005

Aperiodic Order and Dynamical Systems

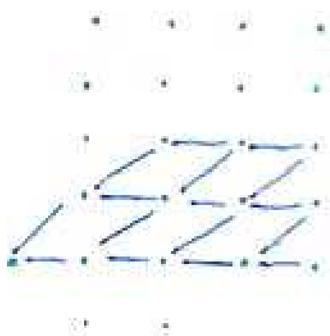
What is aperiodic order?

the context extended discrete structures in $\mathbb{R}^d$

- point sets
- tilings
- atomic structures

periodic lattice symmetry

- lattice / coset of a lattice
- tiling from a fundamental region of a lattice
- crystals a lattice of symmetries



aperiodic not fully periodic

order repetition in some form

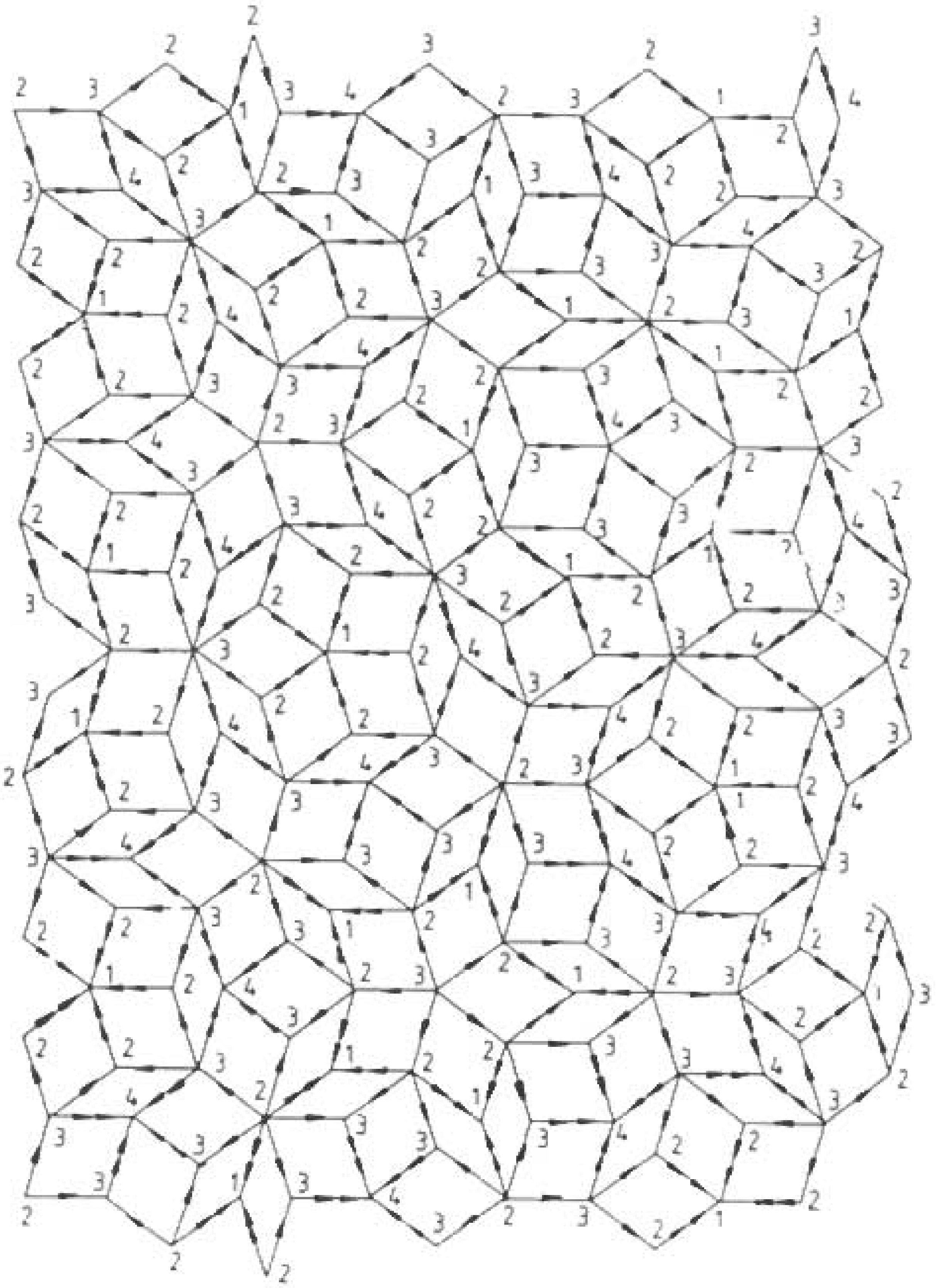
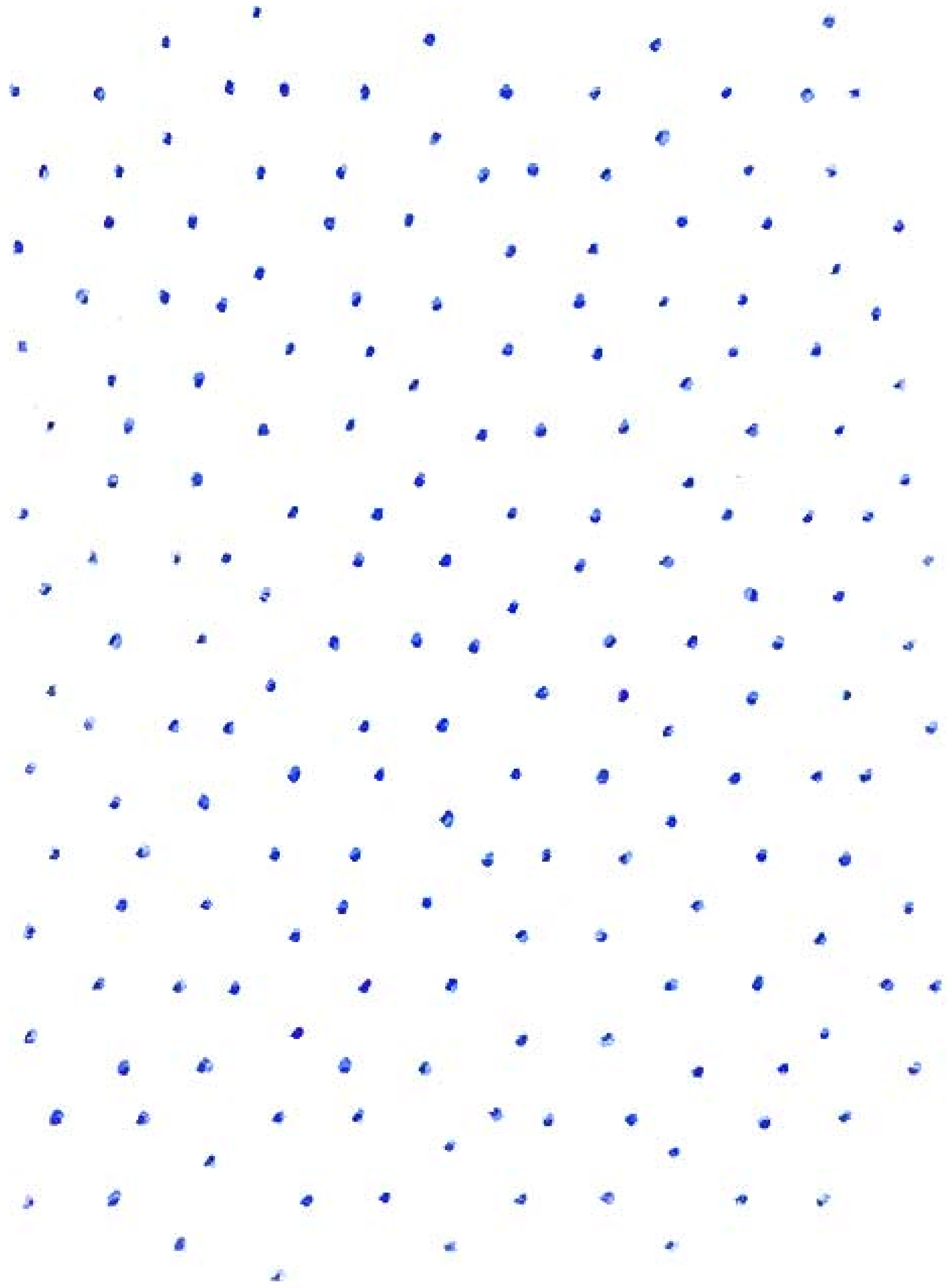
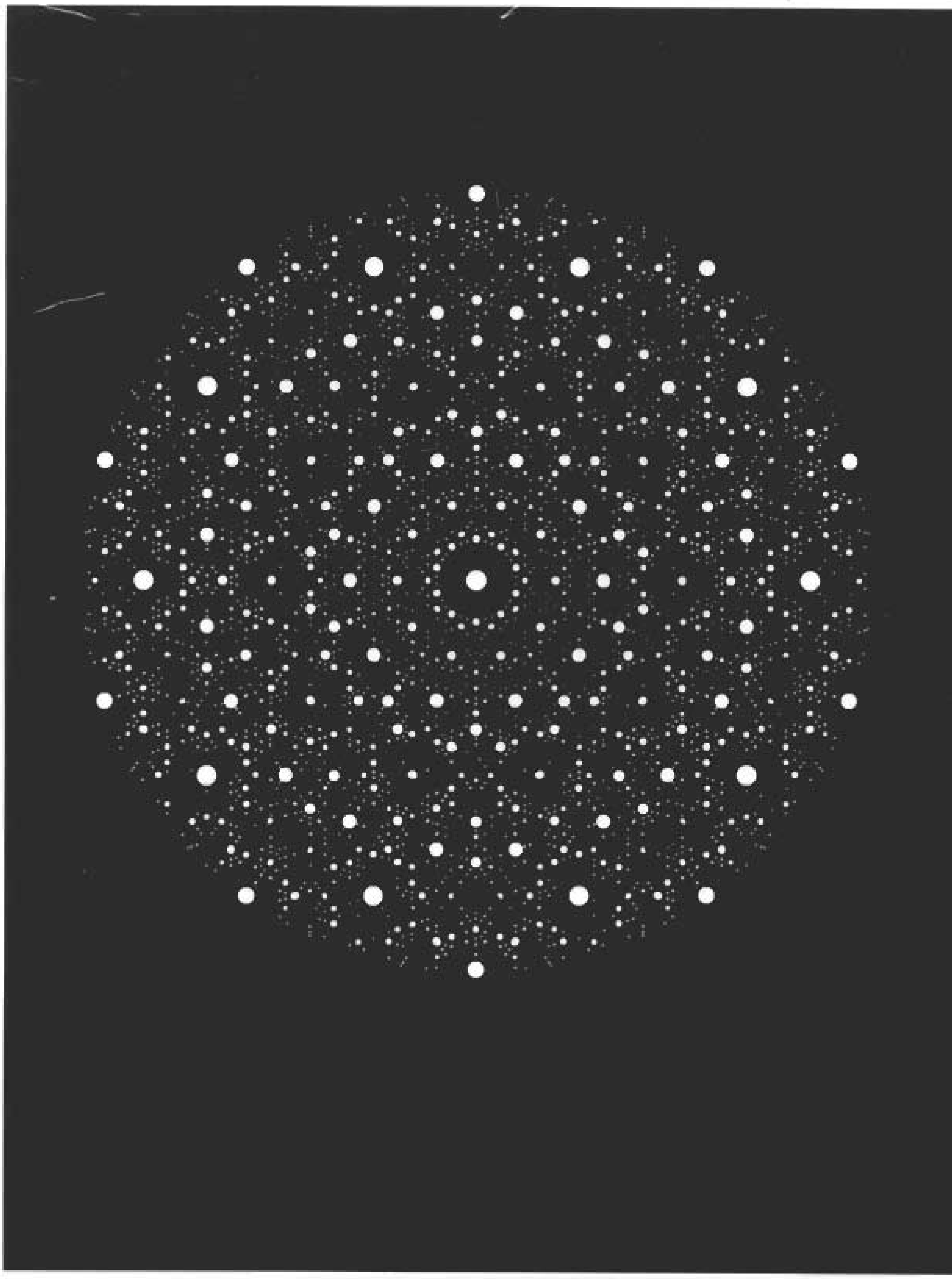
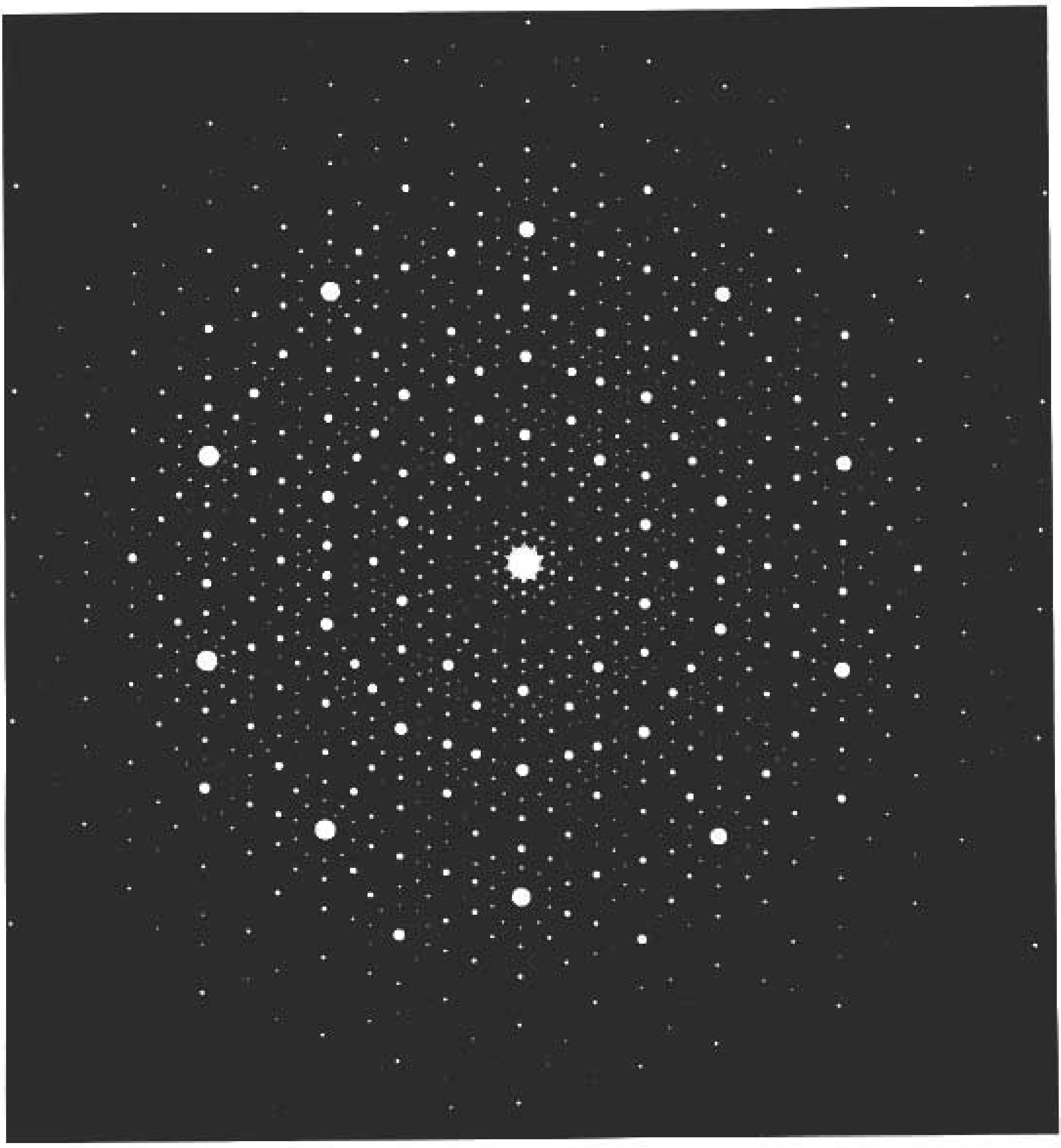


Fig. 6.6 A Penrose tiling, with matching rules expressed as arrows.

L







$Al_{70}\ Co_{11}\ Ni_{19}$

Properties of a lattice $\Lambda \subset \mathbb{R}^d$ $\Lambda = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_d$

DELONE Λ is relatively dense and uniformly discrete

ALGEBRAIC STRUCTURE Λ is a group $\Lambda - \Lambda \subset \Lambda$

(for a coset $a + \Lambda$,

$(a+\Lambda) - (a+\Lambda) \subset (a+\Lambda) - a$)

REPETITION everything repeats with full perfect translational symmetry

DYNAMICAL SYSTEM $\mathbb{R}^d/\Lambda$ is a compact group

$\mathbb{R}^d \curvearrowright \mathbb{R}^d/\Lambda$

COHERENCE there is a dual object

Λ° = dual (reciprocal) lattice

$\Lambda^\circ = \{k \in \mathbb{R}^d \mid e^{2\pi i k.u} = 1$ for all $u \in \Lambda$

DIFFRACTION Λ has pure point diffraction

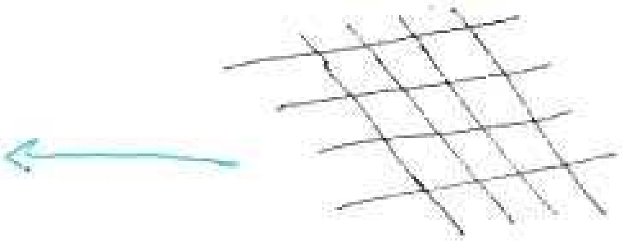
(supported on Λ°)

indexing problem
+
forbidden symmetries } suggest higher dimensions

the idea behind cut and project $\Lambda \subset \mathbb{R}^d$ - representing atomi

physical space internal space

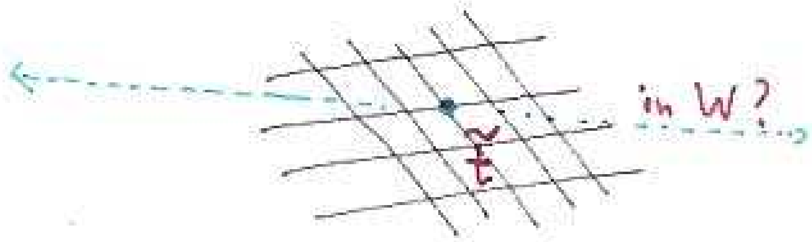
$$\mathbb{R}^d \xleftarrow{\pi_1} \mathbb{R}^d \times \mathbb{R}^m \xrightarrow{\pi_2} \mathbb{R}^m$$



lattice
needs to be controlled

controlling set

t



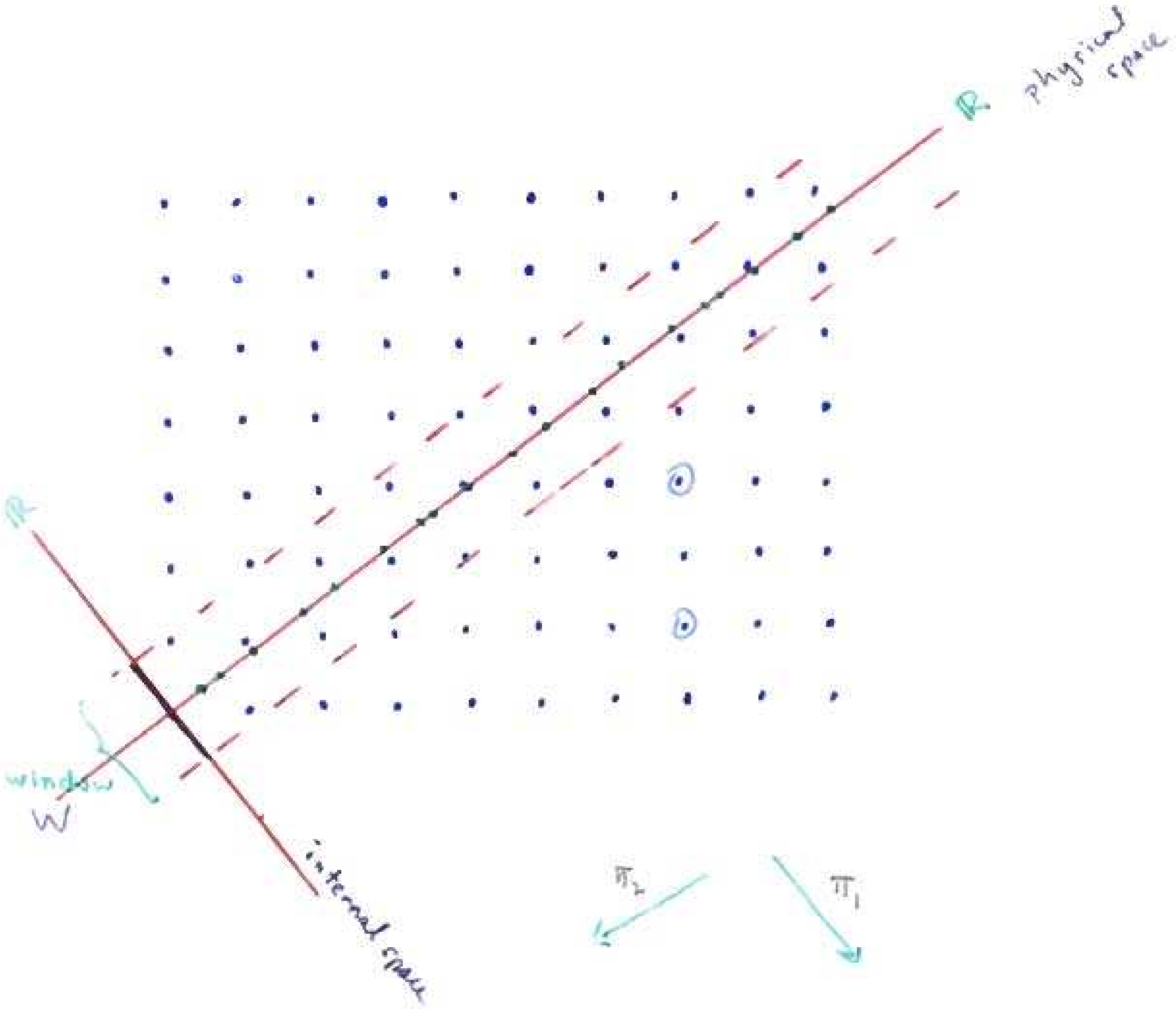
$W \subset \mathbb{R}^m$

for each lattice point $\tilde{t}$ check

$\pi_2(\tilde{t}) \in W$ $\xrightarrow{\text{NO}}$ ignore this point

YES $\downarrow$

$t = \pi_1(\tilde{t})$ is a point of our cut and project set



ℝ physical space
ℝ
window
W
internal space
π_2
π_1

Model Sets:

locally compact Abelian group

$$
\begin{tikzcd}
\mathbf{R}^m & \mathbf{R}^m \times H \arrow[l, "\pi"'] \arrow[r, "\pi_H"] & H \\
L & \tilde{L} \arrow[l, "1-1"'] \arrow[r] \arrow[u] \arrow[d, phantom, sloped, "\in"] & L^* \\
\Lambda(W) & (x, x^*) \arrow[l] \arrow[r, "?"] & W
\end{tikzcd}
$$

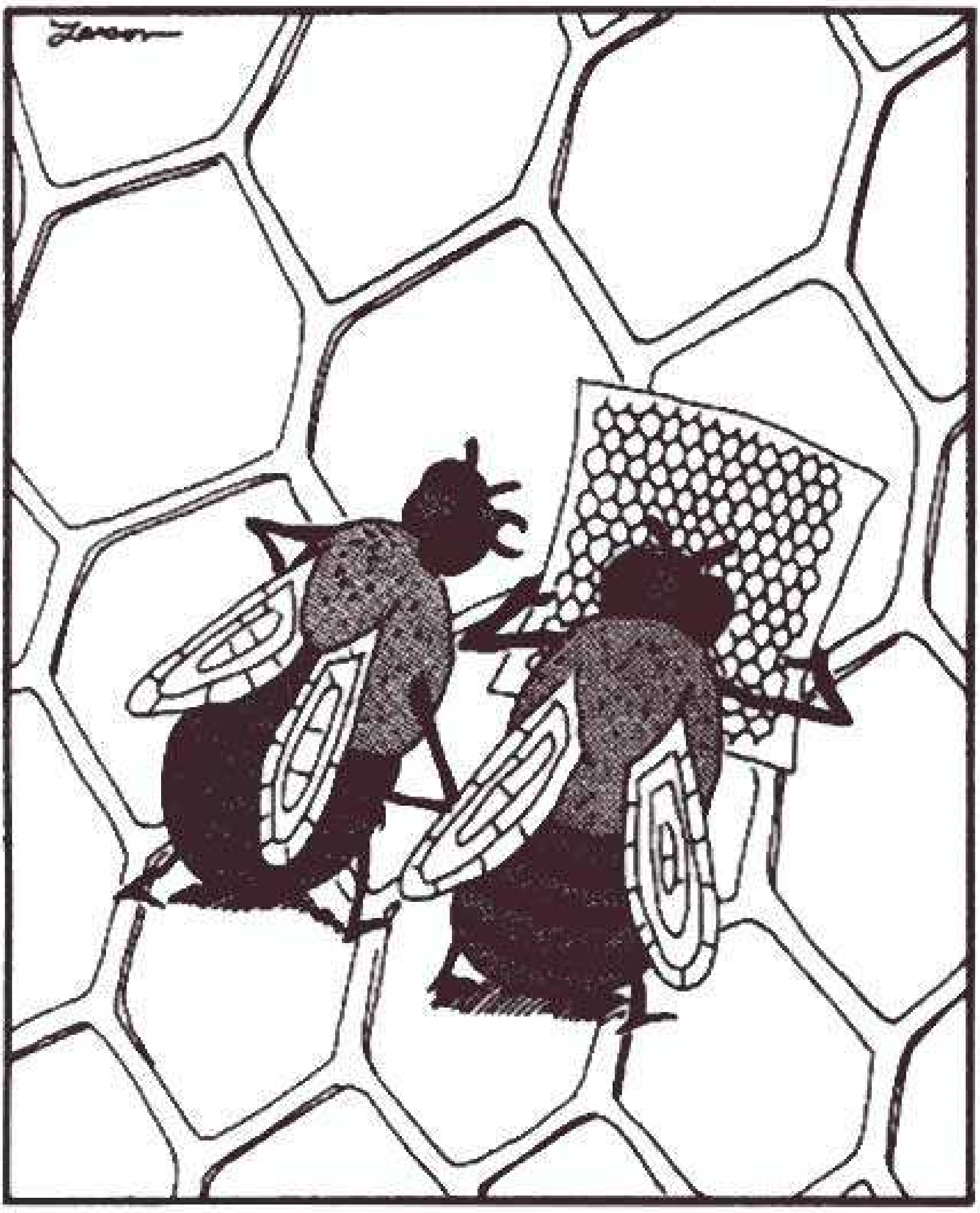
lattice

dense in H

compact closure
non-empty interior

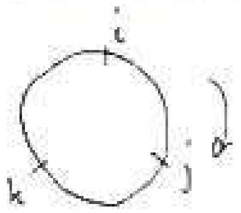
- Λ is Delone
- Λ is aperiodic (in general)
- Λ has finite local complexity
- if $L^* \cap \partial W = \emptyset$ then Λ is repetitive
- if the boundary of W has zero measure then Λ is pure point diffractive.

Y. Meyer
~ 1970



"Face it, Fred—you're lost!"

- quaternions $\mathbb{H} = \mathbb{R}1 + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$



binary icosahedral group

$$I = \begin{array}{ll} (\pm 1, 0, 0, 0) & \partial \\ \tfrac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1) & \\ \tfrac{1}{2}(0, \pm 1, \pm \tau', \pm \tau) & \partial \text{ even} \end{array}$$

$\tau = \frac{1+\sqrt{5}}{2}$

$'$ = conjugation $\sqrt{5} \to -\sqrt{5}$

I = vertices of the 600-cell $\{3,3,5\}$ $\quad \odot - \circ - \circ \overset{5}{-} \circ$

$\mathbb{I} := \mathbb{Z}[I] < \mathbb{H}$ icosian ring dense in $\mathbb{R}^4$

lift $\mathbb{I}$ into $\mathbb{R}^8$

$$\begin{array}{ccccc} \mathbb{R}^4 & \longleftarrow & \mathbb{R}^{4+4} & \longrightarrow & \mathbb{R}^4 \\ \mathbb{I} & \longleftrightarrow & \tilde{\mathbb{I}} & \longrightarrow & \mathbb{I}' \\ x & \longleftrightarrow & (x, x') & \longrightarrow & x' \end{array}$$

$$\left. \begin{array}{l} \tilde{\mathbb{I}} \text{ is a lattice in } \mathbb{R}^8 \\ \langle (x,x'), (y,y') \rangle := x \cdot y + x' \cdot y' \end{array} \right\} \rightsquigarrow E_8$$

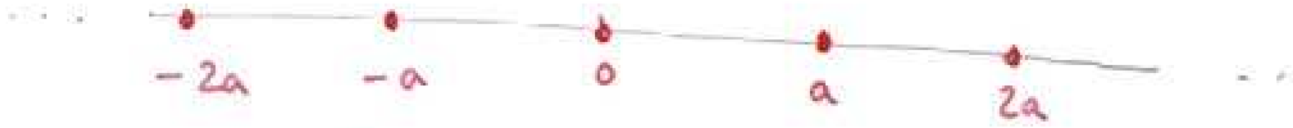
$\mathbb{I}_0$ = pure part $\quad \mathbb{I} \cap (\mathbb{R}i + \mathbb{R}j + \mathbb{R}k)$

$$\begin{array}{ccccc} \mathbb{I}_0 & \longleftarrow & \tilde{\mathbb{I}}_0 & \longrightarrow & \mathbb{I}_0' \\ x & \longmapsfrom & \tilde{x} = (x, x') & \longmapsto & x' \\ \alpha x \alpha^{-1} & \longmapsfrom & \alpha \tilde{x} \alpha^{-1} & & \end{array}$$

for all $\alpha \in I$

— COHERENCE —

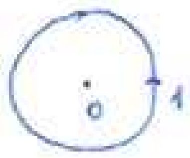
A simple lattice on the line.



$$L = \mathbb{Z}a$$

"character" $\quad \chi_{1/a} : x \longmapsto e^{2\pi i x/a}$

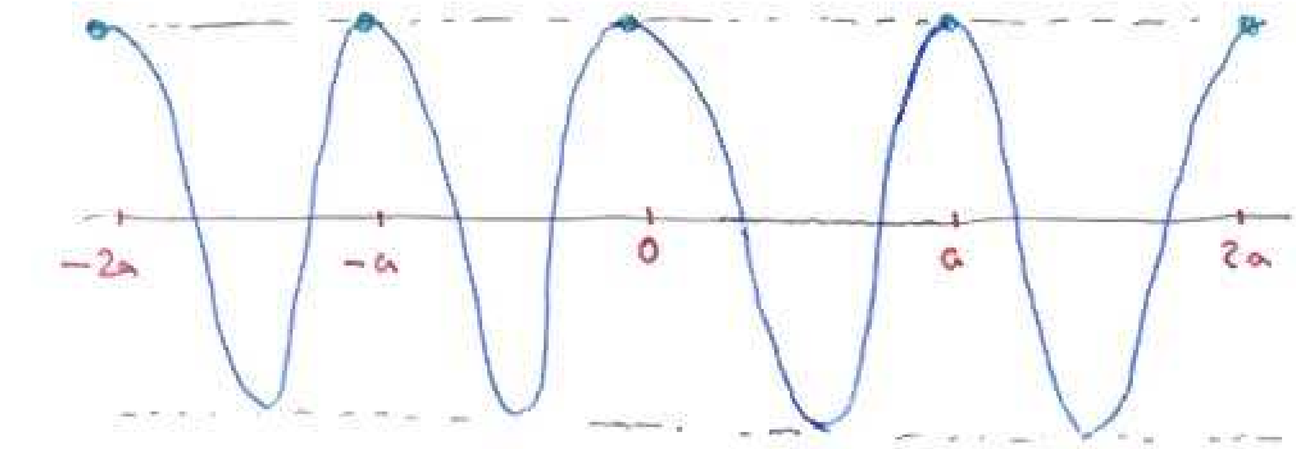
$$\mathbb{R} \longrightarrow U(1) \subset \mathbb{C}$$



$$\chi_{1/a}(ma) = e^{2\pi i ma/a} = 1$$

for all <u>integers</u> m.

$\mathrm{Re}\,\chi_{1/a}(x) = \cos(2\pi x/a)$



Generalizing to $\mathbb{R}^d$:

$L \subset \mathbb{R}^d$ a lattice $\qquad L \simeq \mathbb{Z}^d$

$$L^\circ := \{ k \in \mathbb{R}^d : k.L \subset \mathbb{Z} \} \qquad \text{dual lattice}$$

characters $\qquad \chi_k : \mathbb{R}^d \longrightarrow U(1) \qquad (k \in \mathbb{R}^d)$

$$x \longmapsto e^{2\pi i\, k.x}$$

$$\chi_k(L) = 1 \quad \text{iff} \quad k \in L^\circ$$

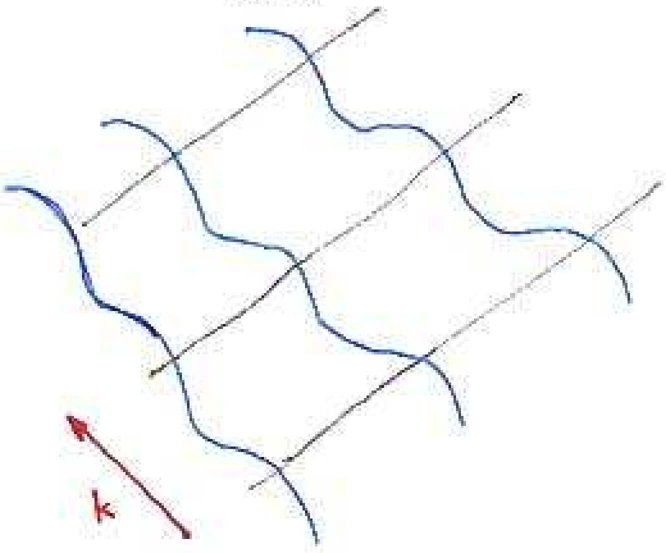
Question: $\quad S \subset \mathbb{R}^d \quad$ discrete

Can we expect to find a set $S^\circ \subset \mathbb{R}^d$ with

$\chi_k(S) = 1 \quad \forall k \in S^\circ$?

$$e^{2\pi i\, k.x} = 1$$

constrains x to an equi-spaced family of parallel hyperplanes



For $\varepsilon > 0$ define

$$\boxed{S^\varepsilon = \{ k \in \mathbb{R}^d \mid |e^{2\pi i k.x} - 1| < \varepsilon \text{ for all } x \in S \}}$$

$\varepsilon = 2 \qquad$ all k work!

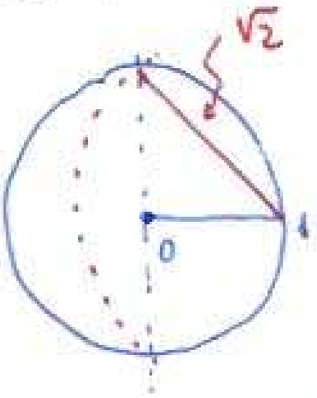
$\varepsilon = 0 \qquad$ the case of S°.

Kronecker Theorem $S \subset \mathbb{R}^d$ any **finite** set, $\varepsilon > 0$ arbitrary.

Then S^ε is relatively dense. $\square$

What about L^ε for L a lattice in $\mathbb{R}^d$?

Suppose $\varepsilon < \sqrt{2}$



$\chi \in L^\varepsilon \Rightarrow |\chi(x) - 1| < \varepsilon \quad \forall x \in L$

$\Rightarrow |\chi(nx) - 1| < \varepsilon < \sqrt{2}$

$\chi(nx) - 1 = \chi(x)^n - 1$

$\chi(x)^n$ is trapped to right hand side

$\chi(x) = 1 \quad \forall x \in L \quad \Longleftarrow$

$\Downarrow$

$\chi = \chi_k, \quad k \in L^\circ$

$$\boxed{L^\varepsilon = L^\circ \qquad \text{if } \varepsilon < \sqrt{2}}$$

Definition $\Lambda \subset \mathbb{R}^d$ is a Meyer set if

1) Λ is relatively dense

2) Λ^ε is relatively dense for some $\varepsilon < \frac{1}{2}$.

Λ Meyer $\Rightarrow$ Λ uniformly discrete

uniformly discrete + relatively dense

$\Rightarrow$ Delone

The group property has gone but

Delone and coherence have been retained.

Characterizations of Meyer sets

Suppose $\Lambda \subset \mathbb{R}^d$ is relatively dense.

- Λ is Meyer $\Leftrightarrow$ Λ^ε is relatively dense for some $\varepsilon < \frac{1}{2}$
- Λ is Meyer $\Leftrightarrow$ Λ^ε is relatively dense for all $\varepsilon > 0$
- Λ is Meyer $\Leftrightarrow$ $\Lambda - \Lambda$ is uniformly discrete (J. Lagarias)

model sets are Meyer sets: $\Lambda = \Lambda(W) = \{ x \in L \mid x^* \in W \}$

$$\Lambda - \Lambda = \Lambda(W) - \Lambda(W) \subset \Lambda(W - W)$$

non-empty interior
compact closure

uniformly discrete $\Leftarrow$ model set

- Λ is Meyer $\Leftrightarrow$ $\Lambda - \Lambda \subset \Lambda + F$ for some finite set F "almost a group"
- Λ is Meyer $\Leftrightarrow$ Λ is a subset of a model set.

Theorem (N. Strungaru) Λ Meyer $\Rightarrow$ Λ has a relatively dense set of Bragg peaks.

diffraction

Λ finite 

$$\delta_x * \delta_y = \delta_{x-y}$$

$$\begin{tikzcd}
\delta_\Lambda = \sum_{x\in\Lambda} \delta_x \arrow[r, squiggly] \arrow[d, squiggly, "\text{Fourier transform}"'] & \delta_\Lambda * \tilde{\delta}_\Lambda \arrow[d, squiggly, "\text{Fourier transform}"] \\
\hat{\delta}_\Lambda(k) = \sum_{x\in\Lambda} e^{-2\pi i k.x} \arrow[r, "|\cdot|^2"] & \left|\sum_{x\in\Lambda} e^{2\pi i k.x}\right|^2
\end{tikzcd}$$

$\tilde{\delta}_\Lambda = \sum_{x\in\Lambda} \delta_{-x}$

Λ infinite

assume Λ is locally finite

$$\gamma_\Lambda := \lim_{R\to\infty} \frac{1}{\mathrm{vol}\, B_R} \left(\delta_{\Lambda\cap B_R} * \tilde{\delta}_{\Lambda\cap B_R} \right)$$

the limit is a measure — taken in the vague topology

γ_Λ is the autocorrelation measure

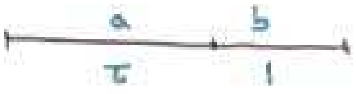
$\hat{\gamma}_\Lambda$ is the diffraction (measure)

$$\hat{\gamma}_\Lambda = (\hat{\gamma}_\Lambda)_{pp} + (\hat{\gamma}_\Lambda)_{sc} + (\hat{\gamma}_\Lambda)_{ac}$$

Bragg spectrum comes from here.

Fibonacci $\quad a \to ab \qquad b \to a$

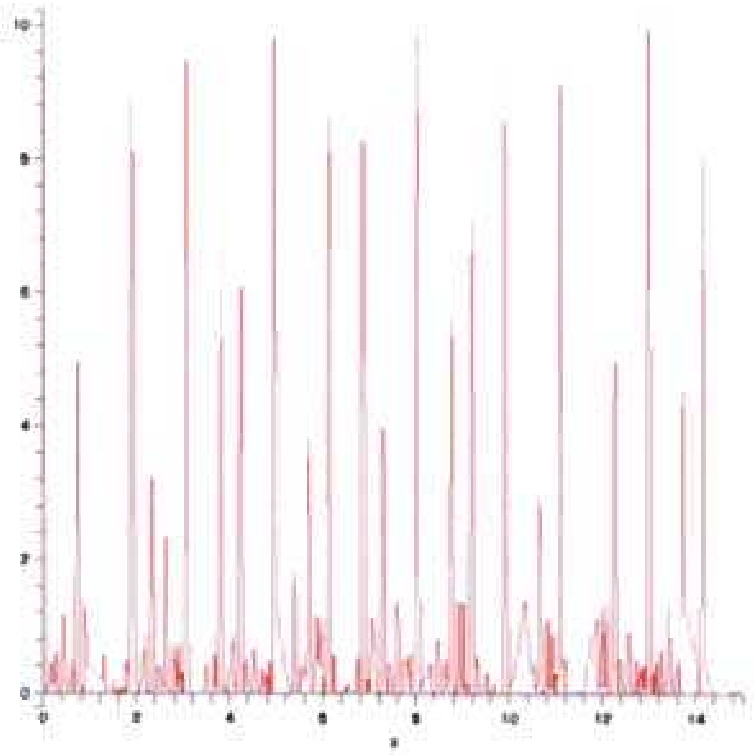
$$\underset{1}{a} \to \underset{2}{ab} \to \underset{3}{aba} \to \underset{5}{abaab} \to \underset{8}{abaababa} \to$$



Diffraction from the first 10 points of the Fibonacci substitution tiling sequence.

the graph of

$$\frac{1}{10}\left|\sum_{x \in S} e^{-2\pi i x \cdot k}\right|^2 \qquad \text{(as a function of k)}$$



$\to k$

Dynamical Systems

$G \curvearrowright X$

group — space

X compact topological space — $G \times X \to X$ continuous — topological dynamics

X measurable space, μ probability measure — $G \times X \to X$ measurable — measure theoretical dynamics

$\Lambda \subset \mathbb{R}^d$ a point set

X a set of subsets of $\mathbb{R}^d$ (with $\Lambda \in X$)

closed under translations by $\mathbb{R}^d$

$$\mathbb{R}^d \curvearrowright X$$

e.g.

- X = all $\Lambda' \subset \mathbb{R}^d$ locally indistinguishable from Λ

  e.g. all Penrose tilings

- define what is meant for two sets $\Lambda', \Lambda'' \subset \mathbb{R}^d$ to be "close"

e.g. Λ', Λ'' are close if for some large $R > 0$ and small $\varepsilon > 0$

$$\begin{cases} \Lambda' \cap B_R(0) \subset \Lambda'' + B_\varepsilon(0) \\ \Lambda'' \cap B_R(0) \subset \Lambda' + B_\varepsilon(0) \end{cases}$$

define $X = \overline{\mathbb{R}^d + \Lambda}$. (orbit closure or hull of Λ)

For model sets Λ, $\overline{\mathbb{R}^d + \Lambda}$ looks very much like $(\mathbb{R}^d \times H)/\tilde{L}$.