Introduction

A directed graph $E$ consists of

- $E^0$ – countable set of vertices,
- $E^1$ – countable set of edges.
- maps $r, s : E^1 \rightarrow E^0$ giving direction.

For convenience we’ll assume that $E^0, E^1$ are finite and that $E$ is essential, i.e. every vertex receives and emits at least one edge.

$C^*(E)$ is the universal $C^*$-algebra generated by operators $\{S_e : e \in E^1\}$ subject to relations which encode the connectivity of $E$.

The edge shift associated to $E$ is defined by

$$X_E = \{x \in (E^1)^\mathbb{Z} : s(x_{i+1}) = r(x_i) \text{ all } i\}.$$

The shift map $\sigma_E : X_E \rightarrow X_E$ is given by $(\sigma_E x)_i = x_{i+1}$. We may also describe one-sided versions in a similar way.
We have lots of good results for essential graphs with $E^0, E^1$ infinite which are locally finite, i.e. every vertex emits and receives finitely many edges. In fact all $C^*(E)$ are equivalent to one of these.
The following graph does not satisfy condition (L):

\[
\begin{array}{c}
\bullet \\
\end{array}
\]

The following graph satisfies condition (L) but not condition (K)

\[
\begin{array}{c}
\bullet \\
\end{array}
\]

The following graph satisfies condition (K)

\[
\begin{array}{c}
\bullet \\
\end{array}
\]

The following graph is irreducible and satisfies condition (K)

\[
\begin{array}{c}
\bullet \\
\end{array}
\]

The following graph is irreducible but does not satisfy condition (L)

\[
\begin{array}{c}
\bullet \\
\end{array}
\]

Evidently if a graph is irreducible and satisfies condition (L) then it satisfies condition (K).
Higher dimensional graphs

There is a $k$-dimensional analogue of a directed graph, called a $k$-graph, which is denoted $\Lambda$. If $k = 1$ then $\Lambda$ is a directed graph.

If $\Lambda$ is essential then we can construct a zero-dimensional space $\Lambda^\Delta$ which carries an expansive $\mathbb{Z}^k$ action with entropy zero.

To a $k$-graph $\Lambda$ we associate a $C^*$-algebra $C^*(\Lambda)$. Unfortunately, we do not have a version of condition (K) which is easy to check. We propose to study a class of $k$-graphs which come from higher dimensional shift spaces and deduce our condition (K) from the aperiodic nature of the shift.

There is a procedure in Lind and Marcus’s book which shows us how to associate a directed graph to a shift of finite type, we propose to generalise this procedure on certain higher dimensional shift spaces to produce a higher dimensional graph.
Ledrappier shift

Recall the Ledrappier shift \( X \subset \{0, 1\}^{\mathbb{Z}^2} \) consists of those \( x = (x_{i,j})_{(i,j) \in \mathbb{Z}^2} \) with

\[
x_{i,j} + x_{i+1,j} + x_{i,j+1} = 0 \mod 2
\]

for all \( i, j \in \mathbb{Z}^2 \). Equivalently \( X = \hat{M} \) where

\[
M = \mathbb{Z}[s, t, s^{-1}, t^{-1}] / \langle 2, 1 + s + t \rangle.
\]

So \( X \) consists of those infinite configurations of 0’s and 1’s such that whenever we put down the pattern

the sum of the entries in the pattern adds to zero mod 2.
A 2-graph representing $X$

Our vertices are precisely the four allowed configurations in our pattern.

\[
\begin{array}{cccc}
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
a & b & c & d
\end{array}
\]

For vertices $u, v$ there is an $x$-edge from $u$ to $v$ if $u$ overlaps $v$ after translation by one unit in the $x$-direction, otherwise there is no $x$-edge.

\[
\begin{array}{ccc}
0 & 1 \\
1 & 1 & 0 \\
b & d
\end{array}
\]

There is an $x$-edge from $b$ to $d$. These edges are drawn as solid lines.
For vertices $u, v$ there is an $y$-edge from $u$ to $v$ if $u$ overlaps $v$ after translation by one unit in the $y$-direction, otherwise there is no $y$-edge.

There is an $y$-edge from $b$ to $c$. These edges are drawn as dashed lines.

We complete this procedure to get the following the graph:
Note that for every $x$-edge followed by a $y$-edge there is precisely one $y$-edge followed by an $x$-edge between the same two vertices.

![Diagram]

Moreover, there is at most one $x$-edge followed by a $y$-edge between any two vertices. This property is a key feature of a 2-graph.

**How do we recover the original shift space?**

Consider any infinite configuration of edges in our graph of the form:
As our vertices are valid patterns of 0’s and 1’s, the $x$- and $y$-edges are defined by overlapping and there is at most one $x$-edge followed by a $y$-edge between any two vertices, any such configuration uniquely determines an element of $X$; and conversely.

**Finishing thoughts**

Things worked out well in one dimension as periodic points in $X_E$ come from loops in $E$. Also, one may construct aperiodic points in $X_E$ using loops. In higher dimensions this is no longer true, as there can be points with period $(1, -1)$ for instance.

Working with 0’s and 1’s but with different shapes, we have found that not every pattern gives us a 2-graph. Which patterns do? It seems that ”staircase” shaped patterns work.

The hope is that for those $k$-graphs which arise from higher dimensional shifts in this way, we can check our version of the aperiodicity condition (K). We have done this in one only example so far: $Y = \hat{M}$ where $M = \mathbb{Z}[s, t, s^{-1}, t^{-1}]/\langle 2, 1 + s + t + st \rangle$. 