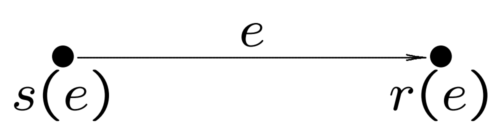
Introduction

A *directed graph* E consists of

- E^0 – countable set of vertices,
- E^1 – countable set of edges.
- maps $r, s : E^1 \to E^0$ giving direction.



For convenience we'll assume that E^0, E^1 are finite and that E is essential, i.e. every vertex receives and emits at least one edge.

$C^*(E)$ is the universal C^*-algebra generated by operators $\{S_e : e \in E^1\}$ subject to relations which encode the connectivity of E.

The edge shift associated to E is defined by

$$\mathsf{X}_E = \{x \in (E^1)^{\mathbf{Z}} : s(x_{i+1}) = r(x_i) \text{ all } i\}.$$

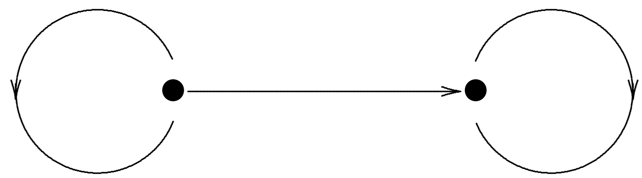
The shift map $\sigma_E : \mathsf{X}_E \to \mathsf{X}_E$ is given by $(\sigma_E x)_i = x_{i+1}$. We may also describe one-sided versions in a similar way.

Relationship between $C^*(E)$ and X_E

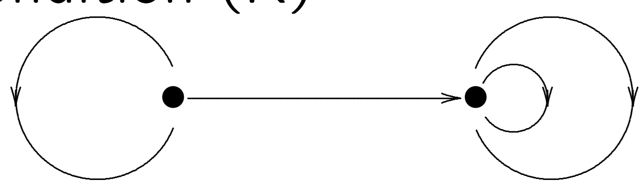
X_E	$C^*(E)$
σ_E	Canonical endomorphism $\phi_E : C^*(E) \to C^*(E)$
$h(\sigma_E)$	Voiculescu entropy $h(\phi_E)$
irreducible	simple
components	ideals
outsplitting insplitting Parry-Sullivan move	isomorphism Morita equivalence !? Morita equivalence
Krieger condition (I) no isolated points ?	(L): every loop has exit uniqueness theorem purely infinite
Bowen-Franks groups	K-groups
Lots of aperiodic points ?	(K): no vertex lies on exactly one simple loop $C^*(E)$ real rank zero Components give all ideals
mixing	AF core of $C^*(E)$ simple

We have lots of good results for essential graphs with E^0, E^1 infinite which are locally finite, i.e. every vertex emits and receives finitely many edges. In fact all $C^*(E)$ are equivalent to one of these.

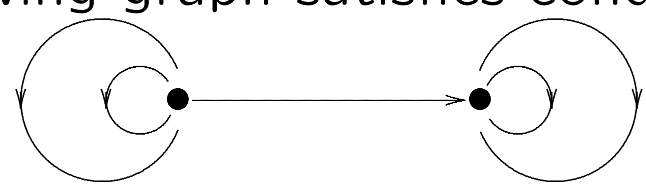
The following graph does not satisfy condition (L):



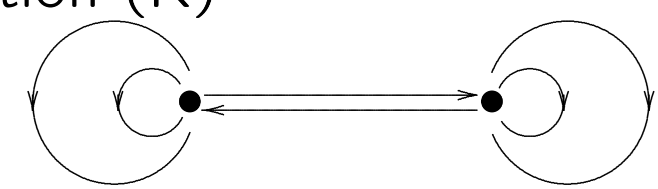
The following graph satisfies condition (L) but not condition (K)



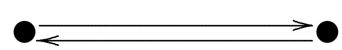
The following graph satisfies condition (K)



The following graph is irreducible and satisfies condition (K)



The following graph is irreducible but does not satisfy condition (L)



Evidently if a graph is irreducible and satisfies condition (L) then it satisfies condition (K).

Higher dimensional graphs

There is a k-dimensional analogue of a directed graph, called a k-graph, which is denoted Λ. If $k = 1$ then Λ is a directed graph.

If Λ is essential then we can construct a zero-dimensional space Λ^{Δ} which carries an expansive $\mathbf{Z}^k$ action with entropy zero.

To a k-graph Λ we associate a C^*-algebra $C^*(\Lambda)$. Unfortunately, we do not have a version of condition (K) which is easy to check. We propose to study a class of k-graphs which come from higher dimensional shift spaces and deduce our condition (K) from the aperiodic nature of the shift.

There is a procedure in Lind and Marcus's book which shows us how to associate a directed graph to a shift of finite type, we propose to generalise this procedure on certain higher dimensional shift spaces to produce a higher dimensional graph.

Ledrappier shift

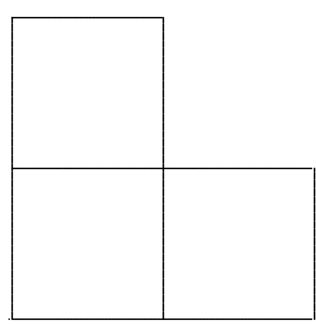
Recall the Ledrappier shift $X \subset \{0,1\}^{\mathbf{Z}^2}$ consists of those $x = (x_{i,j})_{(i,j)\in\mathbf{Z}^2}$ with

$$x_{i,j} + x_{i+1,j} + x_{i,j+1} = 0 \text{ mod } 2$$

for all $i, j \in \mathbf{Z}^2$. Equivalently $X = \widehat{M}$ where

$$M = \mathbf{Z}[s, t, s^{-1}, t^{-1}]/\langle 2, 1 + s + t\rangle.$$

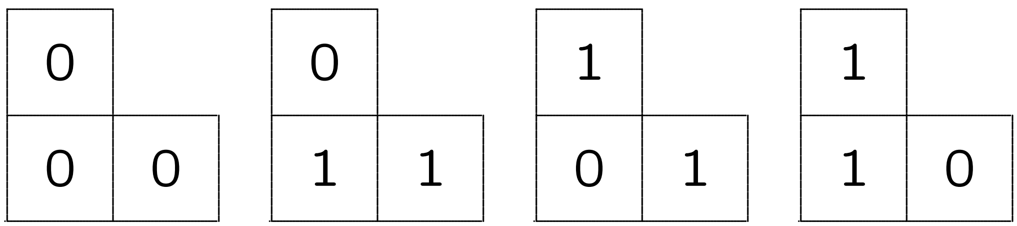
So X consists of those infinite configurations of 0's and 1's such that whenever we put down the pattern



the sum of the entries in the pattern adds to zero mod 2.

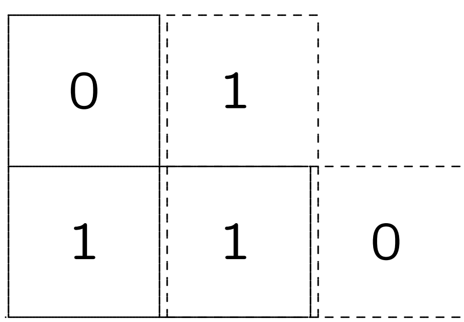
A 2-graph representing X

Our vertices are precisely the four allowed configurations in our pattern.



a b c d

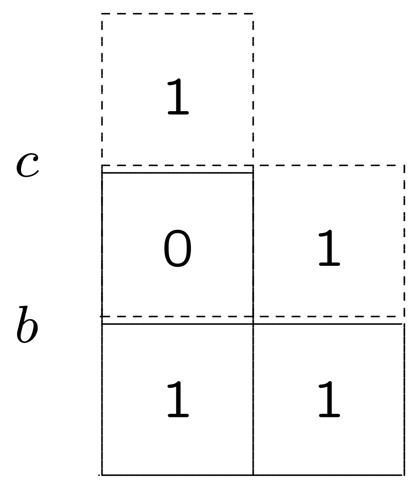
For vertices u, v there is an x-edge from u to v if u overlaps v after translation by one unit in the x-direction, otherwise there is no x-edge.



b d

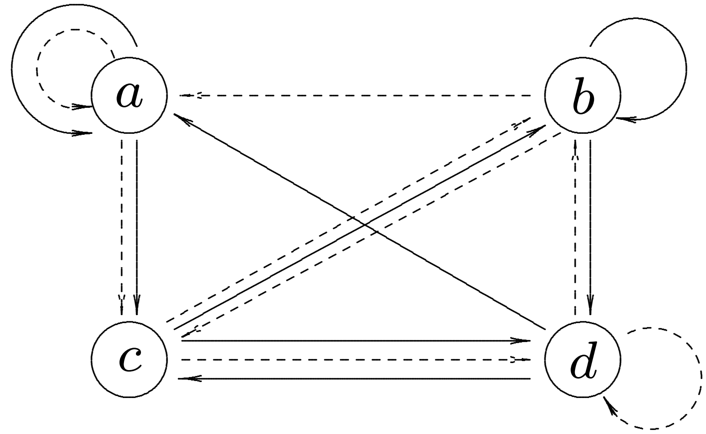
There is an x-edge from b to d. These edges are drawn as solid lines.

For vertices u, v there is an y-edge from u to v if u overlaps v after translation by one unit in the y-direction, otherwise there is no y-edge.



There is an y-edge from b to c. These edges are drawn as dashed lines.

We complete this procedure to get the following the graph:



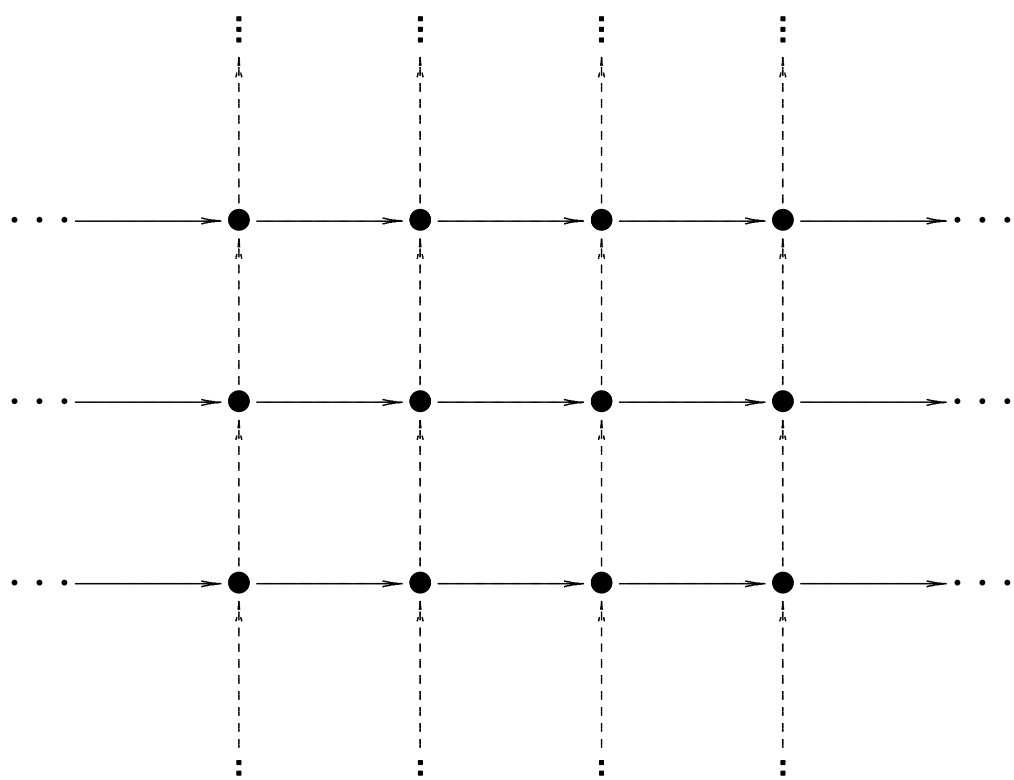
Note that for every x-edge followed by a y-edge there is precisely one y-edge followed by an x-edge between the same two vertices.

$$\begin{array}{l} c \longrightarrow d \dashrightarrow b \\ c \dashrightarrow b \longrightarrow b \end{array}$$

Moreover, there is at most one x-edge followed by a y-edge between any two vertices. This property is a key feature of a 2-graph.

How do we recover the original shift space?

Consider any infinite configuration of edges in our graph of the form:



As our vertices are valid patterns of 0's and 1's, the x- and y-edges are defined by overlapping and there is at most one x-edge followed by a y-edge between any two vertices, any such configuration uniquely determines an element of X; and conversely.

Finishing thoughts

Things worked out well in one dimension as periodic points in X_E come from loops in E. Also, one may construct aperiodic points in X_E using loops. In higher dimensions this is no longer true, as there can be points with period $(1, -1)$ for instance.

Working with 0's and 1's but with different shapes, we have found that not every pattern gives us a 2-graph. Which patterns do? It seems that "staircase" shaped patterns work.

The hope is that for those k-graphs which arise from higher dimensional shifts in this way, we can check our version of the aperiodicity condition (K). We have done this in one only example so far: $Y = \widehat{M}$ where $M = \mathbf{Z}[s, t, s^{-1}, t^{-1}]/\langle 2, 1 + s + t + st \rangle$.