The entropy of the visible points

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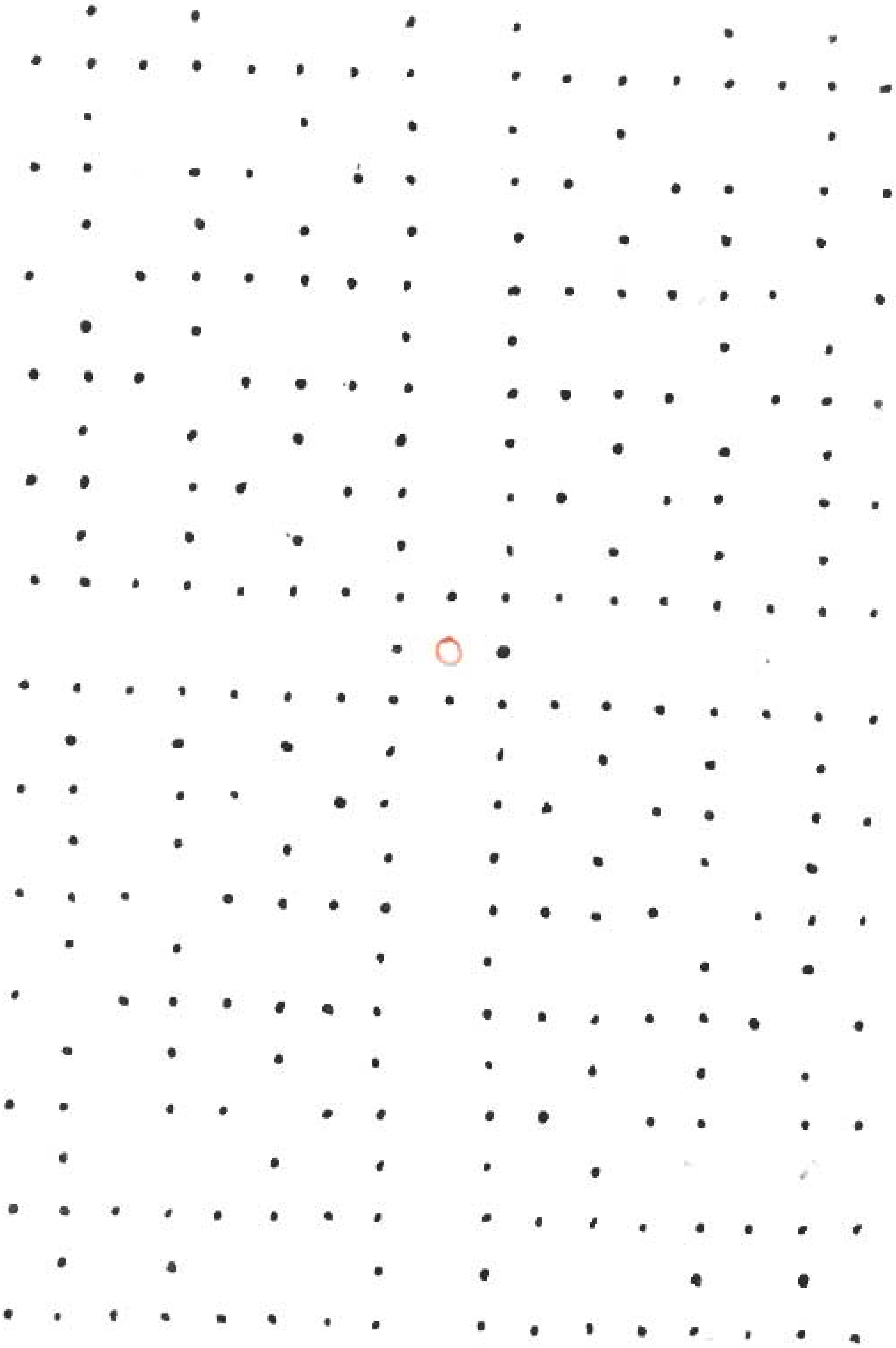
Visible points



The visible points V are the points $(m,n) \in \mathbb{Z}^2$ with $(m,n) = 1$

- V has pure point diffraction

(Baake, Moody, P, 2000)

- V has arbitrarily large holes

$R \subset \mathbb{R}^2$ any bounded region

Choose a prime $p(\underline{x})$ for each $\underline{x} \in R \cap \mathbb{Z}^2$.
Use Chinese Remainder Theorem to simultaneously solve the congruences

$$\underline{t} \equiv -\underline{x} \pmod{p(\underline{x})}$$

Then $R + \underline{t}$ contains no points of V.
Solutions have periods $P\mathbb{Z}^2$ where $P = \prod_{\underline{x} \in R} p(\underline{x})$.

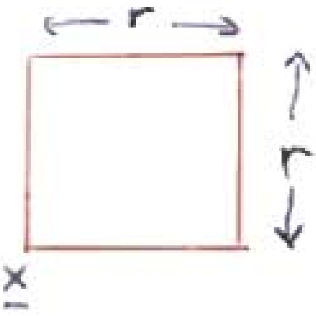
$\therefore$ +ve density of large holes

- V has density $6/\pi^2$

$$\text{density} = \lim_{r \to \infty} \frac{|V \cap B_r(\underline{0})|}{\pi r^2}$$

Not uniform: moving the centre from $\underline{0}$ to some other point can slow down the approach to the limit.

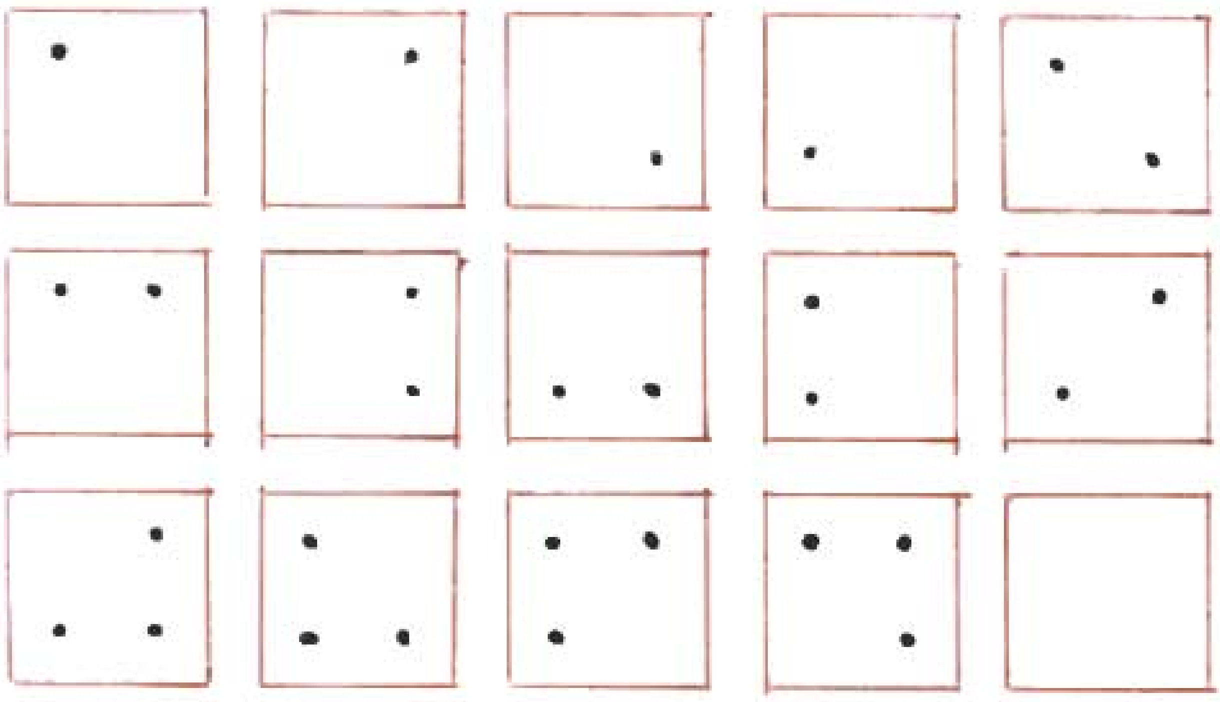
Let $S_r(\underline{x})$ be the $r \times r$ square with SW corner $\underline{x}$.



$N_V(r)$ is the number of distinct patterns $V \cap S_r(\underline{x})$ for $\underline{x} \in \mathbb{Z}^2$. The **topological entropy** of V is

$$h(V) = \lim_{r \to \infty} \frac{\log_2 N(r)}{r^2}$$

Example 2×2 squares contain 4 points of $\mathbb{Z}^2$. Exactly one is divisible by 2, so invisible. The other 3 may be visible or not. 15 possibilities:



So $\dfrac{\log_2 N_V(2)}{2^2} = \dfrac{\log_2 15}{4} = 0.976$

upper bound for $h(V)$.

For $P \in \mathbb{N}$ the potentially visible points V_P are the points $(m,n) \in \mathbb{Z}^2$ with $(m,n,P)=1$.

E.g. $V_2 = \mathbb{Z} \setminus 2\mathbb{Z}$

$$V \subset V_P \ \forall P$$

$$P_1 \mid P_2 \implies V_{P_1} \supseteq V_{P_2}$$

<u>Lemma 1</u>

$$|V_P \cap S_r(\underline{x})| = \frac{6r^2}{\pi^2} + O\left(\frac{r^2}{p}\right) + O\left(\min\left((|\underline{x}|+r)^{1+\varepsilon}, r P^{\varepsilon}\right)\right)$$

where p is the smallest prime s.t. $p \nmid P$.

Putting $P = n! \to \infty$, then letting $r \to \infty$ gives: density of $V = \frac{6}{\pi^2}$

Now take $P = \prod_{p \leq r} p$

<u>Corollary 1</u>

$$|V_P \cap S_r(\underline{x})| = \frac{6r^2}{\pi^2} + o(r^2) \text{ if } |\underline{x}| = O(r)$$

Put $L = \prod_{p \leq \log r} p = O(r)$. Then

<u>Corollary 2</u>

$$|V_P \cap S_r(\underline{x})| \leq |V_L \cap S_r(\underline{x})| = \frac{6r^2}{\pi^2} + o(r^2) \ \forall \underline{x}$$

Lemma 2

Let $A \cup B$ be an arbitrary partition of $V_P \cap S_r(\underline{0})$. Then $\exists\, \underline{t} \equiv \underline{0} \pmod P$ s.t.

all points in $A + \underline{t}$ are invisible,
all points in $B + \underline{t}$ are visible.

For each $\underline{a} \in A$ choose a different prime $p(\underline{a}) > r$. $\exists\, \underline{s} \equiv \underline{0} \pmod P$ with

$$\underline{s} \equiv -\underline{a} \pmod{p(\underline{a})} \quad \forall\, \underline{a} \in A.$$

Then all points in $A + \underline{s}$ are invisible.
Can also choose $\underline{s}$ so that $S_r(\underline{s})$ does not overlap the y-axis.

Now put $Q = P \prod_{\underline{a} \in A} p(\underline{a})$.

$\underline{b} + \underline{s} = (m, n) \in B + \underline{s} \Rightarrow (m, n, P) = 1$, because $\underline{b} \in V_P$.

Also $p(\underline{a}) \nmid \underline{b} + \underline{s}$, because $p(\underline{a}) \mid \underline{a} + \underline{s}$ and $|(\underline{a} + \underline{s}) - (\underline{b} + \underline{s})| < r < p(\underline{a})$.

$\therefore \quad (m, n, Q) = 1 \quad \forall\, (m, n) \in B + \underline{s}$

Can find $t \equiv 0 \pmod Q$ s.t. $(m, n + t) = 1$ by solving congruences mod $m/(m, Q)$ simultaneously for all $\underline{b} + \underline{s} \in B + \underline{s}$.

Put $\underline{t} = \underline{s} + (0, t)$.

Then all points in $B + \underline{t}$ are visible.

Lower bound

By Corollary 1 there are

$$2^{6r^2/\pi^2 + o(r^2)}$$

ways to partition $V_p \cap S_r(\underline{0})$ into 2 subsets.

By Lemma 2 each partition gives a different r-patch of V. So

$$h(V) \geq \lim_{r\to\infty} \frac{6r^2/\pi^2 + o(r^2)}{r^2} = \frac{6}{\pi^2}$$

Upper bound

By Corollary 2, for any $\underline{x} \in \mathbb{Z}^2$ there are

$$\leq 2^{6r^2/\pi^2 + o(r^2)}$$ ways of partitioning

$V_p \cap S_r(\underline{x})$ into 2 subsets.

Since $P\mathbb{Z}^2$ is a lattice of periods for V_p there are $\leq P^2$ distinct patterns $V_p \cap S_r(\underline{x})$ and

$$\log P^2 = 2\sum_{p\leq r} \log p = 2r + o(r) = O(r)$$

So

$$h(V) \leq \lim_{r\to\infty} \frac{6r^2/\pi^2 + o(r^2) + O(r)}{r^2} = \frac{6}{\pi^2}.$$

Related sets

Visible points of $\mathbb{Z}^n$: $h = \frac{1}{\zeta(n)}$

k-th-power-free numbers (subset of 1D lattice $\mathbb{Z}$): $h = \frac{1}{\zeta(k)}$

In the latter case the vertical translation argument for making $B+\underline{t}$ all visible doesn't carry over to making $B+t$ all k-free.

Argument to fill the gap

Need to find $x \in \mathbb{N}$ s.t.

$$b_i + s + Qx \text{ is } k\text{-free } \forall b_i \in B$$

From definition of Q it can be shown that

$$\forall p \; \exists x \text{ s.t. } p^k \nmid b_i + s + Qx \text{ for any } b_i \in B$$

Can find a substitution $x = ay + c$ so that

$$b_i + s + Qx = u_i (v_i y + w_i) \quad \forall i$$

where u_i, v_i, w_i are such that u_i is k-free and

$$\forall p \; \exists y \text{ s.t. } p \nmid v_i y + w_i \text{ for any } i.$$

By an improved hypothesis of Schinzel $\exists$ infinitely many y s.t.

$$v_i y + w_i \text{ are simultaneously prime}$$

So we can make

$$b_i + s + Qx \text{ simultaneously } k\text{-free.}$$

Cut-and-project schemes (model sets)

$$\begin{tikzcd}[row sep=small]
\mathbb{R}^n & \mathbb{R}^n \times H & H \\
\cup & \cup & \cup \\
M & L \arrow[l, "\text{injective}"'] \arrow[r, "\text{dense image}"'] & W
\end{tikzcd}$$

H is a locally compact Abelian group.
L is a lattice (i.e. $(\mathbb{R}^n \times H)/L$ is compact).
A scheme is **regular** if ∂W has measure 0.
Schlottmann:

Regular $\Rightarrow$ density of $M = \mu(W)/\det L$

(det L means $\mu((\mathbb{R}^n \times H)/L)$)

Square-free numbers

$$F = \{ n \in \mathbb{Z} : m > 1 \Rightarrow m^2 \nmid n \}$$

Model set description given by Baake, Moody, P.

Adeles

$\mathbb{Q}_p$ the p-adic completion of $\mathbb{Q}$

$\mathbb{Z}_p = \{\alpha \in \mathbb{Q}_p : |\alpha|_p \le 1\}$, p-adic integers.

Haar measure μ_p on $\mathbb{Q}_p$ is unique if we normalize so that $\mu_p(\mathbb{Z}_p) = 1$.

Adeles $\mathbb{A}$ are

$$\mathbb{A} = \mathbb{R} \times \prod_p \mathbb{Q}_p$$

Restricted product w.r.t. $\mathbb{Z}_p$s

$\mathbb{Q} \subset \mathbb{A}$ $\quad q \mapsto (q, q, q, \cdots)$

($q \in \mathbb{Z}_p$ except if p divides its denominator)

Give $\mathbb{A}$ the product measure and $\mathbb{A}/\mathbb{Q}$ the quotient measure. Then

$$\mu(\mathbb{A}/\mathbb{Q}) = 1$$

$\mathbb{Q}$ is a lattice of determinant 1.

Cut-and-project scheme for F

$$\begin{array}{ccccc} \mathbb{R} & & \mathbb{A} & & \prod \mathbb{Q}_p \\ \cup & & \cup & & \cup \\ F & \longleftarrow & \mathbb{Q} & \longrightarrow & W \end{array}$$

$$W = \prod_p (\mathbb{Z}_p \setminus p^2 \mathbb{Z}_p)$$

$$\mu(W) = \prod_p \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$$

$= \text{density of } F$

W is closed, $\overline{W} = W$, but $W^\circ = \emptyset$

So $\partial W = W$. $\mu(\partial W) = \frac{6}{\pi^2} \neq 0$

F is not a regular model set.

Can find an $\alpha \in \prod \mathbb{Z}_p$ such that model set derived from $W + \alpha$ is empty.

Schlottmann's proof shows that in general

$$\mu(W^\circ)/\det L \leq \underline{\operatorname{dens}}\, M \leq \overline{\operatorname{dens}}\, M \leq \mu(\overline{W})/\det L$$

For non-regular model sets density (if it exists) is confined to a known range.

Is there a similar relation between entropy and $\partial(W)$?

$$\mu(\partial W) = 0 \text{ (i.e. } M \text{ regular)} \Rightarrow h(M) = 0$$

(Kwapicz)

$$h(F) = \frac{6}{\pi^2} = \mu(\partial W)$$

$$h(M(W+\alpha)) = h(\emptyset) = 0 \neq \mu(\partial(W+\alpha)) = \frac{6}{\pi^2}$$

Can define a window entropy, $h(W)$ by counting all patterns in models derived from $W+\alpha$ for all translations α. Then

$$0 \leq h(M) \leq h(W)$$

for all model sets M derived from translates of W.

In the case of F,

$$h(F) = h(W) = \mu(\partial W) = \frac{6}{\pi^2}$$

Does $h(W) = \mu(\partial W)/\det L$ more generally?

Similar treatment works for visible points and k-th power-free numbers

Simplified cut-and-project

Baake & Moody have used instead:

$$\mathbb{B} = \mathbb{R} \times \prod_p \mathbb{Z}_p ,$$

the "integral adeles", instead of $\mathbb{A}$.

Now the product is the standard unrestricted topological product and internal space is compact.

$$\mathbb{Z} \subset \mathbb{B} \quad \text{and} \quad \mathbb{B} \cap \mathbb{Q} = \mathbb{Z}$$

$$\mathbb{B}/\mathbb{Z} \cong \mathbb{A}/\mathbb{Q}$$

so $\mathbb{Z}$ is a lattice of determinant 1 in $\mathbb{B}$.

$$\mathbb{A}/\mathbb{B} \cong \mathbb{Q}/\mathbb{Z}$$

so $\mathbb{B}$ is considerably simpler than $\mathbb{A}$.

Now F is given by

$$\begin{array}{ccccc}
\mathbb{R} & & \mathbb{B} & & \prod \mathbb{Z}_p \\
\cup & & \cup & & \cup \\
F & \text{———} & \mathbb{Z} & \text{———} & W
\end{array}$$

with the same window as before.