The entropy of the visible points

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MASCOS Workshop on Algebraic Dynamics
University of New South Wales, Sydney, Australia
February 14 - February 18, 2005
The visible points $V$ are the points $(m, n) \in \mathbb{Z}^2$ with $(m, n) = 1$.

1. $V$ has pure point diffraction (Baake, Moody, P, 2000)

2. $V$ has arbitrarily large holes

   $R \subset \mathbb{R}^2$ any bounded region

   Choose a prime $p(x)$ for each $x \in R \cap \mathbb{Z}$.

   Use Chinese Remainder Theorem to simultaneously solve the congruences

   $t \equiv -x \pmod{p(x)}$

   Then $R + t$ contains no points of $V$.

   Solutions have periods $P \ni \mathbb{Z}^2$ where

   $P = \prod_{x \in R} p(x)$.

   $\therefore$ $V$ has density $6/\pi^2$

   $\text{density} = \lim_{r \to \infty} \frac{|V \cap B_r(0)|}{\pi r^2}$

   Not uniform: moving the centre from $0$ to some other point can slow down the approach to the limit.
Let $S_r(x)$ be the $r \times r$ square with SW corner $x$.

$N_v(r)$ is the number of distinct patterns $V \cap S_r(x)$ for $x \in \mathbb{Z}^2$. The topological entropy of $V$ is

$$h(V) = \lim_{r \to \infty} \frac{\log_2 N_v(r)}{r^2}$$

**Example** 2x2 squares contain 4 points of $\mathbb{Z}^2$. Exactly one is divisible by 2, so invisible. The other 3 may be visible or not. 15 possibilities:

\[
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

So $\frac{\log_2 N_v(2)}{2^2} = \frac{\log_2 15}{4} = 0.976$

upper bound for $h(V)$. 
For \( p \in \mathbb{N} \) the potentially visible points \( V_p \) are the points \((m,n) \in \mathbb{Z}^2 \) with \((m,n,p) = 1\).

E.g., \( V_2 = \mathbb{Z} \setminus 2\mathbb{Z} \)

\[ V \subseteq V_p \text{ } \forall p \]

\[ p_1 \mid p_2 \implies V_{p_1} \supseteq V_{p_2} \]

**Lemma 1**

\[
|V_p \cap S_r(\mathbb{A})| = \frac{6r^2}{\pi^2} + O\left(\frac{r^2}{p}\right) + O\left(\min\left(\frac{\log r}{r}, \frac{r}{p}\right)\right)
\]

where \( p \) is the smallest prime s.t. \( p \mid p \).

Putting \( p = n! \rightarrow \infty \), then letting \( r \rightarrow \infty \) gives: density of \( V \) is \( \frac{6}{\pi^2} \).

Now take \( P = \prod_{p \leq r} p \)

**Corollary 1**

\[
|V_p \cap S_r(\mathbb{A})| = \frac{6r^2}{\pi^2} + o(r^2) \quad \text{up to } oo = O(r)
\]

Put \( L = \prod_{p \leq \log r} p = O(r) \). Then

**Corollary 2**

\[
|V_p \cap S_r(\mathbb{A})| \leq |V_L \cap S_r(\mathbb{A})| = \frac{6r^2}{\pi^2} + o(r^2)
\]
Lemma 2
Let $A \cup B$ be an arbitrary partition of $V_p \cap S_r(0)$. Then $\exists \xi \equiv 0 \pmod P$ s.t. all points in $A + \xi$ are invisible, all points in $B + \xi$ are visible.

For each $a \in A$ choose a different prime $p(a) > r$. $\exists \xi \equiv 0 \pmod P$ with $\xi \equiv -a \pmod{p(a)}$ $\forall a \in A$. Then all points in $A + \xi$ are invisible. Can also choose $\xi$ so that $S_r(\xi)$ does not overlap the $y$-axis.

Now put $Q = P \cup P(\xi)$. $\forall a \in A$

$b + \xi = (m, n) \in B + \xi \Rightarrow (m, n, P) = 1$, because $b \in V_p$.

Also $p(a) | b + \xi$, because $p(a) | a + \xi$ and $1 \leq (a + \xi, b + \xi) < r < p(a)$.

$(m, n, Q) = 1 \forall (m, n) \in B + \xi$

Can find $t \equiv 0 \pmod Q$ s.t. $(m, nt) = 1$ by solving congruences $\pmod {m/(m, Q)}$ simultaneously for all $b + \xi \in B + \xi$.

Put $\xi = \xi + (0, t)$.

Then all points in $B + \xi$ are visible.
By Corollary 1 there are
gen \frac{6r^2}{\pi^2} + o(r^2)
ways to partition \( V_p \cap S_r(x) \) into 2 subsets.

By Lemma 2 each partition gives a different
\( r \)-patch of \( V \). So
\[ h(V) \geq \lim_{r \to \infty} \frac{6r^2}{\pi^2} + o(r^2) = \frac{6}{\pi^2} \]

Upper bound

By Corollary 2, for any \( x \in \mathbb{R}^2 \) there are
gen \frac{6r^2}{\pi^2} + o(r^2)
ways of partitioning \( V_p \cap S_r(x) \) into 2 subsets.

Since \( P \subset \mathbb{Z}^2 \) is a lattice of periods for \( V_p \),
there are \( \leq P^r \) distinct patterns \( V_p \cap S_r(x) \) and
\[ \log P^r = 2 \sum_{p \in P} \log p = 2r + o(r) = o(r) \]

So
\[ h(V) \leq \lim_{r \to \infty} \frac{6r^2}{\pi^2} + o(r^2) + o(r^2) = \frac{6}{\pi^2}. \]
Related Sets

Visible points of $\mathbb{Z}^n$: $h = \frac{1}{3^n}$

$k$-th-power-free numbers
(Subset of 1D lattice $\mathbb{Z}$): $h = \frac{1}{5^k}$

In the latter case the vertical translation argument for making $B+t$ all visible doesn't carry over to making $B+t$ all $k$-free.
Need to find \( x \in \mathbb{N} \) s.t.
\[ b + s + Qx \text{ is } k\text{-free} \quad \forall b, s \in \mathbb{B} \]
From definition of \( Q \) it can be shown that
\[ \forall p \exists x \text{ s.t. } p^k + b + s + Qx \text{ for any } b, s \in \mathbb{B} \]
Can find a substitution \( x = ay + c \) so that
\[ b + s + Qx = u_i (v_i y + w_i) \quad \forall i \]
where \( u_i, v_i, w_i \) are such that \( u_i \) is \( k\)-free and
\[ \forall p \exists y \text{ s.t. } p + v_i y + w_i \text{ for any } i \]
By an improved hypothesis of Schinzel \[ \exists \text{ infinitely many } y \text{ s.t.} \]
\[ v_i y + w_i \text{ are simultaneously prime} \]
So we can make
\[ b + s + Qx \text{ simultaneously } k\text{-free.} \]
Cut-and-project schemes (model sets)

\[ \mathbb{R}^n \rightarrow \mathbb{R}^n \times H \rightarrow H \]
\[ M \overset{\text{injective}}{\rightarrow} L \overset{\text{dense image}}{\rightarrow} W \]

\( H \) is a locally compact Abelian group.
\( L \) is a lattice (i.e. \((\mathbb{R}^n \times H)/L \) is compact).
A scheme is regular if \( dW \) has measure 0.

Schottmann:

Regular \( \Rightarrow \) density of \( M = \mu(W)/\det L \)
\( \det L \) means \( \mu((\mathbb{R}^n \times H)/L) \)

Square-free numbers

\[ F = \{ n \in \mathbb{Z} : m > 1 \Rightarrow m^2 + n \} \]

Model set description given by Baake, Moody, P.
$\mathbb{Q}_p$ the $p$-adic completion of $\mathbb{Q}$
$\mathbb{Z}_p = \{ x \in \mathbb{Q}_p : 10^k / p < 1 \}$, $p$-adic integers.

Haar measure $\mu_p$ on $\mathbb{Q}_p$ is unique if we normalize so that $\mu_p(\mathbb{Z}_p) = 1$.

Adeles $\mathbb{A}$ are

$$\mathbb{A} = \mathbb{R} \times \prod_p \mathbb{Q}_p$$

(Revised product with $\mathbb{Z}_p$s)

$\mathcal{O} \subset \mathbb{A}$ $q \mapsto (q, q, q, \ldots)$

($q \in \mathbb{Z}_p$ except if $p$ divides its denominator)

Give $\mathbb{A}$ the product measure and $\mathbb{A} / \mathcal{O}$ the quotient measure. Then

$$\mu(\mathbb{A} / \mathcal{O}) = 1$$

$\mathcal{O}$ is a lattice of determinant 1.
Cut-and-project scheme for \( F \)

\[
\begin{align*}
R & \quad \mathbb{A} & \quad \mathbb{P} \\
V & \quad U & \quad \mathbb{W} \\
F & \quad \mathbb{Q} & \quad \mathbb{W}
\end{align*}
\]

\[
W = \mathbb{P} (\mathbb{Z}_p \setminus \mathbb{P}^2 \mathbb{Z}_p)
\]

\[
\mu(W) = \mathbb{P} (1 - \frac{1}{p^2}) = \frac{1}{3p} = \frac{6}{\pi^2}
\]

= density of \( F \)

\( W \) is closed, \( \overline{W} = W \), but \( W^0 = \emptyset \)

\( \partial W = \emptyset \). \( \mu(\partial W) = \frac{6}{\pi^2} \neq 0 \)

\( F \) is not a regular model set. Can find an \( \alpha \in \mathbb{Z}_p \) such that model set derived from \( W + \alpha \) is empty. Schottmann's proof shows that in general

\[
\mu(W^0)/\det L \leq \text{dens} M \leq \text{dens} M \leq \mu(W)/\det L
\]

For non-regular model sets density (if it exists) is confined to a known range.
Is there a similar relation between entropy and $\partial(W)$?

$$\mu(\partial W) = 0 \quad \text{(i.e. M regular)} \implies h(M) = 0 \quad \text{(Kwapić)}$$

$$h(F) = \frac{6}{\pi^2} = \mu(\partial W)$$

$$h(M(W + \alpha)) = h(\emptyset) = 0 \neq \mu(\partial(W + \alpha)) = \frac{6}{\pi^2}$$

Can define a window entropy, $h(W)$, by counting all patterns in models, derived from $W + \alpha$, for all translations $\alpha$. Then

$$0 \leq h(M) \leq h(W)$$

for all model sets $M$ derived from translations of $W$.

In the case of $F$,

$$h(F) = h(W) = \mu(\partial W) = \frac{6}{\pi^2}$$

Does $h(W) = \mu(\partial W)/d_{\text{det}}$ more generally?

Similar treatment works for visible points and $k$-th power-free numbers.
Boake and Moody have used instead

\[ B = R \times \mathbb{T} \mathbb{Z}_p, \]

the "integral adelic", instead of \( A \). Now the product is the standard unrestricted topological product and internal space is compact.

\[ \mathbb{Z} \subset B \quad \text{and} \quad B \cap \mathbb{Q} = \mathbb{Z} \]

\[ \mathbb{B}/\mathbb{Z} \simeq \mathbb{A}/\mathbb{Q} \]

so \( \mathbb{Z} \) is a lattice of determinant 1 in \( \mathbb{B} \).

\[ \mathbb{A}/\mathbb{B} \simeq \mathbb{Q}/\mathbb{Z} \]

so \( \mathbb{B} \) is considerably simpler than \( \mathbb{A} \).

Now \( \mathbb{F} \) is given by

\[
\begin{array}{ccc}
R & B & \mathbb{T} \mathbb{Z}_p \\
\mathbb{U} & \mathbb{U} & \mathbb{U} \\
\mathbb{F} & \mathbb{Z} & \mathbb{W}
\end{array}
\]

with the same window as before.