# On positive predecessor density in 3n+1 dynamics

G. J. Wirsching

(preliminary version)

Mathematisch-Geographische Fakultät
Katholische Universität Eichstätt
D-85071 Eichstätt, Germany

**Abstract**

The $3n+1$ function is given by $T(n) = n/2$ for $n$ even, $T(n) = (3n+1)/2$ for $n$ odd. Given a positive integer $a$, another number $b$ is a called a *predecessor* of $a$ if some iterate $T^\nu(b)$ equals $a$. Here some ideas are described which may lead to a proof showing that the set of predecessors of $a$ has positive lower asymptotic density, for any positive integer $a \not\equiv 0 \bmod 3$. Three unbridged gaps in the argument are formulated as conjectures.

## Introduction

A dynamical system on the set $\mathbb{N} = \{1, 2, \ldots\}$ of natural numbers (positive integers) is given by the iterations of a function $f : \mathbb{N} \to \mathbb{N}$ in the following sense: If $x \in \mathbb{N}$ is an arbitrarily chosen starting number, the *trajectory* of $x$ is the sequence

$$\mathcal{T}_f(x) := \big(x, f(x), f^2(x), \ldots, f^\nu(x), \ldots\big) = (f^\nu(x))_{\nu \geq 0}\,,$$

where we denote by $f^\nu$ the $\nu$-th iterate of $f$, defined inductively by

$$f^0(x) = x, \qquad f^{\nu+1}(x) = f\left(f^\nu(x)\right).$$

A good deal of the amount of information provided for by a dynamical system is incorporated in the *predecessor sets*

$$\mathcal{P}_f(a) := \{x \in \mathbb{N} : \text{there is an index } \nu \geq 0 \text{ such that } f^\nu(x) = a\}\,.$$

The *domain of attraction* of a given trajectory $\mathcal{T}_f(x)$ is, by definition, the set of all starting numbers $y \in \mathbb{N}$ whose trajectories $\mathcal{T}_f(y)$ eventually coalesce with $\mathcal{T}_f(x)$. Such a domain of attraction is an increasing union of the predecessor sets $\mathcal{P}_f(f^\nu(x))$, where $\nu$ runs through an arbitrary infinite set of non-negative integers. So, knowing the predecessor sets is essentially the same as knowing the dynamical system generated by $f$.

One of the most interesting dynamical systems on $\mathbb{N}$ is that one which is given by the so-called $3n+1$ *function*:

$$T : \mathbb{N} \to \mathbb{N}, \qquad T(n) = \begin{cases} T_0(n) = \dfrac{n}{2} & \text{for even } n, \\[2ex] T_1(n) = \dfrac{3n+1}{2} & \text{for odd } n. \end{cases}$$

Essentially due to L. Collatz is the famous $3n+1$ *conjecture*:

*Any $3n+1$ trajectory eventually ends in the cycle* $(1,2)$.

For the time being, many mathematicians consider a proof of this conjecture as intractably hard. But, as an aphorism quoted by Lagarias [3] says, 'no problem is so intractable that something interesting cannot be said about it'.

The topic we are going to consider here are the predecessor sets of the $3n+1$ function. In this setting, the $3n+1$ conjecture is equivalent to the equation $\mathcal{P}_T(1) = \mathbb{N}$. We do not set out to prove this; rather, we are concerned with the *lower asymptotic density* of various predecessor sets $\mathcal{P}_T(a) \subset \mathbb{N}$ in the set of positive integers $\mathbb{N}$. For an arbitrary subset $S \subset \mathbb{N}$, its lower asymptotic density is defined to be the non-negative real number

$$\lambda_S = \liminf_{n\to\infty} \frac{|\{x \in S : x \leq n\}|}{n}\,.$$

Roughly speaking, $\lambda_S > 0$ means that, given a 'randomly chosen' positive integer $x$, the assertion '$x \in S$' is true with 'positive probability'. In the setting of $3n+1$ dynamics, $\lambda_{\mathcal{P}_T(a)} > 0$ means that, for a 'randomly chosen' positive integer $x$, the $3n+1$ trajectory starting in $x$ hits $a$ with 'positive probability'. Note that, for $a \notin \{1,2,4,8\}$, the relation $\lambda_{\mathcal{P}_T(a)} > 0$ would *not* immediately follow from the truth of the $3n+1$ conjecture. Moreover, note that a proof of positive density of a $3n+1$ predecessor set is beyond the reach of methods applied so far to $3n+1$ dynamics [4].

Based on technical and scientific reasons, we are not merely heading to prove positive lower asymptotic density of some predecessor set $\mathcal{P}_T(a)$. We are interested in some 'uniform' generalisation: Given a *parametrized family* $\{S(a) : a \in A\}$ of subsets $S(a) \subset \mathbb{N}$, we are interested in some kind of 'uniform lower bound' for the quantities $\lambda_{S(a)}$. In the case of predecessor sets $S(a) = \mathcal{P}_f(a)$, the following definition is natural for certain dynamical systems on $\mathbb{N}$ (including $3n+1$ dynamics):

**Definition 1** *Let $f : \mathbb{N} \to \mathbb{N}$ be given. We say that the dynamical system generated by $f$ has the property of* uniform positive predecessor density *on the set $A \subset \mathbb{N}$, if there is a real constant $c > 0$ such that*

$$\liminf_{n\to\infty} \frac{|\{x \in \mathcal{P}_f(a) : x \leq n\}|}{n} \geq \frac{c}{a} \qquad \text{for each } a \in A.$$

In the case of $3n+1$ dynamics, it is not possible to choose $A = \mathbb{N}$. Indeed, as the $T_1$-branch of the $3n+1$ function $T$ never produces a number divisible by 3, it is clear that

$$\mathcal{P}_T(3m) = \left\{2^k \cdot 3m : k = 0,1,2,\dots\right\} \qquad \text{for any positive integer } m.$$

This gives

$$\lambda_{\mathcal{P}_T(3m)} = \liminf_{n\to\infty} \frac{1}{n}\left\lfloor \log_2\left(\frac{n}{3m}\right) + 1 \right\rfloor = 0 \qquad \text{for arbitrary positive integer } m.$$

That means that we have to exclude at least the numbers divisible by 3. To be precise, our aim is to get the following result:

**Positive Density** *The predecessor sets of the $3n+1$ function have uniform positive predecessor density on the set $A = \{a \in \mathbb{N} : a \not\equiv 0 \bmod 3, a$ non-cyclic$\}$.*

As it stands, this assertion appears to be logically independent from the $3n+1$ conjecture: Neither it implies the conjecture, nor is it implied by it.

I do not have a proof of Positive Density. In this paper, several ideas to prove this are presented—leaving three unbridged gaps as precisely formulated conjectures.

## Plan of the paper

The general method to attack the problem of uniform predecessor density of $f$-predecessor sets is the following:

(A) Define appropriate predecessor counting function, and reformulate positive predecessor density in terms of these functions.

(B) Prove that these predecessor counting functions have the required property.

Now let us restrict attention to the $3n+1$ function $T$. In part (A), we arrive at a statement that roughly sounds as follows: If the predecessors of $a$ are sufficiently uniformly distributed among the residue classes modulo powers of 3, then uniform positive density holds. For various reasons coming both from the structure of the $3n+1$ function and our intention to treat distribution properties, we consider the residue classes modulo $3^\ell$ as balls in the (multiplicative) topological group $\mathbb{Z}_3^\times$. So we have a good theory of measure and integration at hands, and this turns out to be important for performing part (B).

Concerning part (B), it is a nice feature that a sufficiently sophisticated notion of *self-similarity* provides a good guide for this approach. What is meant by self-similarity? Something that stabilizes on iteration, e.g., a strongly stable Markov chain. So we are going to construct an appropriate strongly stable Markov chain, and we have to show that our predecessor counting functions may be considered as small perturbations of the successive measures generated by this Markov chain.

Technically, this attempt is divided into seven steps; step (1) corresponds to part (A), and the remaining steps (2)–(7) correspond to part (B).

(1) The *Elka functions* $e_\ell(k,a)$ are our basic predecessor counting functions. In terms of these, a condition $(\star_1)$ is formulated which is sufficient for Positive Density.

(2) In order to make a normalization procedure possible, we investigate compactly supported *generators of Elka functions* $g_\ell(k,a)$ instead of Elka functions themselves. It is conjectured that condition $(\star_1)$ translates to $(\star_2)$.

(3) The generators $g_\ell(k,a)$ are normalized to continuous functions $\tilde{g}_\ell : \mathbb{X} \to \mathbb{R}$ with total integral 1 on a deliberately chosen common *state space* $\mathbb{X}$. Condition $(\star_2)$ translates to $(\star_3)$.

(4) A *discrete dynamical system* on $\mathbb{X}$ generating the functions $\tilde{g}_\ell$ is constructed as a sequence of integral-preserving operators $S_\ell : \mathcal{C}(\mathbb{X}) \to \mathcal{C}(\mathbb{X})$.

(5) The operators $S_\ell$ are shown to converge to a limiting operator $S_\infty$.

(6) The essential part of $S_\infty$ defines a *strongly stable Markov chain* on the unit interval $[0,1]$ with a unique invariant density $\phi$. It is conjectured that condition $(\star_3)$ translates to $(\star_4)$.

(7) A conjectured property $(\star_5)$ of the asymptotics of $\phi$ is shown to imply condition $(\star_4)$.

## 1 A first condition for positive density

We are interested in describing a predecessor set $\mathcal{P}(a)$ for the $3n+1$ function $T$, where $a \in \mathbb{N}$ is given. To this end, let us introduce the following notation for a special set of paths in the Collatz graph of the $3n+1$ function:

$$\mathcal{E}_{\ell,k}(a) := \left\{ \text{paths } b \xrightarrow{T} \ldots \xrightarrow{T} a \,\middle|\, \begin{array}{l} k \text{ times } T = T_0 \\ \ell \text{ times } T = T_1 \end{array} \right\}$$

for arbitrary non-negative integers $k, \ell$. The connection to predecessor sets is given by the following map:

$$\mathcal{E}_{\ell,k}(a) \to \mathcal{P}(a), \qquad \left(b \xrightarrow{T} \dots \xrightarrow{T} a\right) \mapsto b.$$

Piecing them together gives a surjection

$$\bigcup_{k,\ell \geq 0} \mathcal{E}_{\ell,k}(a) \to \mathcal{P}(a). \tag{1.1}$$

This map is injective if and only if $a$ is non-cyclic.

It is not really a problem to restrict attention to non-cyclic numbers $a \in \mathbb{N}$, as any predecessor set can be written as a finite union of mutually disjoint predecessor sets of non-cyclic numbers. Therefore, excluding cyclic numbers avoids some technical difficulties without producing serious restrictions in the results.

For reasons that will become clear in the sequel, we consider the counting functions for the sets $\mathcal{E}_{\ell,k}(a)$ as a sequence (indexed by $\ell$) of functions of two variables $k$ and $a$. The *Elka functions* are defined by

$$e_\ell : \mathbb{N}_0 \times \mathbb{N} \to \mathbb{N}_0, \qquad e_\ell(k,a) := \left|\mathcal{E}_{\ell,k}(a)\right|.$$

As a first set of properties of Elka function, note that (cf. [4], section II.4)

$$a \equiv b \mod 3^\ell \qquad \Longrightarrow \qquad e_\ell(k,a) = e_\ell(k,b) \quad \text{for each } k \geq 0. \tag{1.2}$$

$$a \equiv 0 \mod 3 \qquad \Longrightarrow \qquad e_\ell(k,a) = \begin{cases} 0 & \text{if } \ell \geq 1, \\ 1 & \text{if } \ell = 0. \end{cases} \tag{1.3}$$

Property (1.2) allows to admit arbitrary 3-adic numbers $a \in \mathbb{Z}_3$ as arguments for the second variable, whence we consider an Elka function as a map

$$e_\ell : \mathbb{N}_0 \times \mathbb{Z}_3 \to \mathbb{N}_0.$$

Let $\mathbb{Z}_3^\times := \{a \in \mathbb{Z}_3 : a \not\equiv 0 \bmod 3\}$ denote the multiplicative group of invertible 3-adic integers; note that $\mathbb{Z}_3^\times$ is a compact topological group. We infer from (1.3) that, for $\ell \geq 1$, the support of $e_\ell$ is contained in $\mathbb{N}_0 \times \mathbb{Z}_3^\times$.

Denoting by $da$ the normalized Haar measure on $\mathbb{Z}_3^\times$, we have the following *averages* of Elka functions (for $\ell \geq 1$):

$$\overline{e}_\ell(k) := \int_{\mathbb{Z}_3^\times} e_\ell(k,a)\, da = \frac{1}{2 \cdot 3^{\ell-1}} \sum_{\substack{a \pmod{3^\ell} \\ a \not\equiv 0 \mod 3}} e_\ell(k,a) = \frac{1}{2 \cdot 3^{\ell-1}} \binom{k+\ell}{k}. \tag{1.4}$$

The next step is to use Elka functions and the surjection (1.1) to estimate, for given $n \in \mathbb{N}$, the quantity $\left|\{x \in \mathcal{P}(a) : x \leq n\}\right|$. This is quite easy: note that the definition of the $3n+1$ function readily implies

$$a \geq \frac{3^\ell}{2^{k+\ell}} \cdot b \qquad \text{for each number } \ b \in \mathcal{E}_{\ell,k}(a).$$

In other words: For any real number $y$, we have the implication

$$(1 - \log_2 3)\ell + k \leq y \quad \& \quad b \in \mathcal{E}_{\ell,k}(a) \quad \Longrightarrow \quad b \leq 2^y a.$$

Now, using that (1.1) is surjective for non-cyclic $a$, we obtain the estimate

$$\left|\{x \in \mathcal{P}(a) : x \leq 2^y a\}\right| \geq \sum_{\ell=0}^{\infty} e_\ell\left(\lfloor(\log_2 3 - 1)\ell + y\rfloor, a\right) \qquad \text{for non-cyclic } a.$$

As we like to integrate on $\mathbb{Z}_3^\times$, we omit the $\ell = 0$ term and define the following *estimating series*:

$$s_y : \mathbb{Z}_3^\times \to \mathbb{R}, \qquad s_y(a) := \sum_{\ell=1}^{\infty} e_\ell\left(\lfloor(\log_2 3 - 1)\ell + y\rfloor, a\right).$$

In chapter III of [4] we proved the following (somewhat surprising) results:

- For each $y \in \mathbb{R}$, the function $s_y$ is (Haar-)integrable.
- $\liminf_{y\to\infty} \frac{1}{2^y} \int_{\mathbb{Z}_3^\times} s_y(a)\, da > 0$.

The proof of the following result is based on the main idea of the proof of theorem III.5.1 in [4].

To state the result, we need a further notation: For a given real number $\delta > 0$, put

$$\mathcal{A}_\delta := \left\{ (k_\ell)_{\ell\in\mathbb{N}} \subset \mathbb{Z} : \ |\ell - k_\ell| \le \delta\sqrt{\ell} \text{ for all } \ell \in \mathbb{N} \right\}. \tag{1.5}$$

**Theorem 1** *If there are real numbers $\delta, \mu > 0$ such that*

$$(\star_1) \qquad \liminf_{\ell\to\infty} \frac{e_\ell(k_\ell, a)}{\overline{e}_\ell(k_\ell)} \ge \mu \qquad \textit{uniformly for sequences } (k_\ell) \in \mathcal{A}_\delta\ ,$$

*then the $3n+1$ dynamical system has uniform positive predecessor density on the set of non-cyclic numbers $a \not\equiv 0 \bmod 3$.*

**Proof.** We are heading to define a family of sets $\{\Delta_y : y \in \mathbb{R}\}$ such that we can prove, for sufficiently large $y \in \mathbb{R}$, the following estimates:

$$\begin{aligned}
2^{-y} s_y(a) &= 2^{-y} \sum_{\ell=1}^{\infty} e_\ell\left(\lfloor(\log_2 3 - 1)\ell + y\rfloor, a\right) && \\
&= 2^{-y} \sum_{\ell=1}^{\infty} e_\ell(k_\ell(y), a) && \text{setting } k_\ell(y) := \lfloor(\log_2 3 - 1)\ell + y\rfloor, \quad (1.6)\\
&\ge 2^{-y} \sum_{\ell\in\Delta_y} e_\ell(k_\ell(y), a) && \text{for any subset } \Delta_y \subset \mathbb{N},\\
&\ge \mu_1 \sum_{\ell\in\Delta_y} 2^{-y}\, \overline{e}_\ell(k_\ell(y), a) && \text{for } \mu_1 < \mu \text{ and large } \min \Delta_y, \quad (1.7)\\
&= \mu_1 \sum_{\ell\in\Delta_y} \frac{1}{2^{y+1}\cdot 3^{\ell-1}} \binom{k_\ell(y)+\ell}{\ell} && \text{using (1.4)},\\
&= \mu_1 \sum_{\ell\in\Delta_y} \frac{M(\ell)}{\sqrt{y}} && \text{definition of } M(\ell), \quad (1.8)\\
&\ge \mu_1 \cdot c(\ell) \cdot \sum_{\ell\in\Delta_y} \frac{1}{\sqrt{y}} && \text{setting } c(y) := \min_{\ell\in\Delta_y} M(\ell), \quad (1.9)\\
&\ge \mu_1 \cdot c(y) \cdot \mu_2 && \text{using } |\Delta_y| \ge \mu_2\sqrt{y}. \quad (1.10)
\end{aligned}$$

The sets $\Delta_y \subset \mathbb{N}$ are defined by

(1.11)

$$\Delta_y := \left\{ \ell \in \mathbb{N} : |k_\ell(y) - \ell| \le \delta\sqrt{\ell} \right\} = \left\{ \ell \in \mathbb{N} : \left|\lfloor y - (2 - \log_2 3)\cdot \ell\rfloor\right| \le \delta\sqrt{\ell} \right\}$$

with $k_\ell(y)$ defined in (1.6). The obvious property

$$\min \Delta_y \to \infty \qquad \text{for} \quad y \to \infty$$

proves (1.7). There is also a constant $\mu_2 > 0$ such that

$$|\Delta_y| \geq \mu_2 \sqrt{y} \qquad \text{for large } y,$$

which proves (1.10).

The asymptotics of the binomial coefficients is well-known; we take the following form found in Feller's book (cf. [2, p. 180, formula (2.7)]):

$$\frac{1}{2^{2\nu}} \binom{2\nu}{\nu + k} \sim \frac{1}{\sqrt{\pi\nu}} \exp\left(-\frac{k^2}{\nu}\right) \qquad \text{if } \nu \to \infty \text{ with } \frac{k^3}{\nu^2} \to 0.$$

In our setting, we let $\ell \to \infty$ and use the equivalent formulation

$$\frac{1}{2^{2\nu_\ell}} \binom{2\nu_\ell}{\nu_\ell + d_\ell} = \frac{M_1(\ell)}{\sqrt{\nu_\ell}} \exp\left(-\frac{d_\ell^2}{\nu_\ell}\right), \tag{1.12}$$

where $\lim\limits_{\ell\to\infty} \dfrac{d_\ell^3}{\nu_\ell^2} = 0$ implies $\lim\limits_{\ell\to\infty} M_1(\ell) = \dfrac{1}{\sqrt{\pi}} > 0$.

Now let $y \in \mathbb{R}$ be sufficiently large such that (1.7) and (1.10) are valid. From the definition of $\Delta_y$ in (1.11) we infer that there is sequence $(k_\ell)_{\ell\in\mathbb{N}} \in \mathcal{A}_\delta$ such that $k_\ell = k_\ell(y)$ for each $\ell \in \Delta_y$. We have to put

$$\nu_\ell := \frac{\ell + k_\ell}{2} = \frac{\lfloor \log_2 3 \cdot \ell + y \rfloor}{2} \qquad \text{and} \qquad d_\ell := \frac{\ell - k_\ell}{2}.$$

This implies both

$$\frac{d_\ell^2}{\nu_\ell} \leq \frac{\delta^2}{2} \qquad \text{and} \qquad \frac{1}{\sqrt{\nu_\ell}} \geq \frac{c}{\sqrt{y}} \qquad \text{for } \ell \in \Delta_y,$$

where $c > 0$ is some constant not depending on $y$. Using (1.12), we infer that there is a constant $c_1 = c \cdot \exp(-\delta^2/2)$ such that

$$\frac{1}{2^{y+1} \cdot 3^{\ell-1}} \binom{k_\ell(y) + \ell}{\ell} \geq \frac{1}{2} \cdot \frac{1}{2^{2\nu_\ell}} \binom{2\nu_\ell}{\nu_\ell + d_\ell} \geq c_1 \cdot \frac{M_1(\ell)}{\sqrt{y}} \qquad \text{for } \ell \in \Delta_y.$$

As the quotient $d_\ell^3/\nu_\ell^2$ tends to zero when $y \to \infty$ and $\ell \in \Delta_y$, we conclude that the constant $c(y)$ defined in (1.9) is positive for large $y$. This completes the proof. $\diamondsuit$

## 2 Generators for Elka functions

The basic idea for what follows is to find and exploit something that could be called *self-similarity* of predecessor counting functions. To be more precise about what we are looking for, let us define the notion *weakly self-similar* for counting functions in the following way:

**Definition 2** *A sequence $g_1, g_2, \ldots, g_\ell, \ldots$ of mathematical objects (especially: counting functions) will be called* weakly self-similar, *if we can construct*

*(i) a measure space $\mathbb{X}$,*

*(ii) a normalization procedure $g_\ell \mapsto \mu_\ell$ assigning to each $g_\ell$ a probability measure $\mu_\ell$ on $\mathbb{X}$,*

*(iii) a topological vector space $V$ containing all the probability measures $\mu_\ell$,*

*(iv) a sequence of linear operators $S_\ell : V \to V$, and*

*(v) a Markov operator $S_\infty : V \to V$, i.e., a linear operator preserving probability measures,*

*with the following properties:*

*(a) All the information contained in $g_\ell$ is also contained in $\mu_\ell$.*

*(b) For each $\ell \geq 2$, we have $S_\ell \mu_{\ell-1} = \mu_\ell$.*

*(c) The sequence of operator $(S_\ell)$ converges to $S_\infty$ in some appropriate topology.*

Trying to apply this to Elka functions $e_\ell : \mathbb{N}_0 \times \mathbb{Z}_3^\times \to \mathbb{N}_0$, we have to seek an appropriate normalization procedure. The Elka functions do have a common domain of definition (which might facilitate the construction of $\mathbb{X}$), but they are not bounded nor integrable. Therefore, we come across a problem when trying to normalize them.

To overcome these difficulties, we are going to replace the Elka functions by other functions which we call *generators* for Elka functions. Like Elka functions themselves, their generators are defined as functions of two variables: a non-negative integer $k$ and a 3-adic number $a$. For convenience, we admit arbitrary 3-adic numbers (not only invertible 3-adic integers) as arguments for the second variable:

$$g_\ell : \mathbb{N}_0 \times \mathbb{Q}_3 \to \mathbb{N}_0.$$

The values $g_\ell(k, a)$ are defined inductively by (cf. [4], p. 103):

$$g_0(k,a) := \begin{cases} 1 & \text{if } k = 0 \text{ and } a \in \mathbb{Z}_3, \\ 0 & \text{otherwise,} \end{cases}$$

$$g_{\ell+1}(k,a) := \sum_{j=0}^{2\cdot 3^\ell - 1} g_\ell\left(k - j, \frac{2^{j+1}a - 1}{3}\right) \qquad \text{for } \ell \in \mathbb{N}_0. \tag{2.1}$$

From this definition we obtain using induction that, for each $\ell \geq 1$, the support of $g_\ell$ is contained in the compact set

$$\{0, 1, \ldots, 3^\ell - \ell - 1\} \times \mathbb{Z}_3^\times.$$

Hence the $g_\ell$ are appropriate candidates for a normalization procedure.

The link to Elka functions is provided by theorem IV.1.14 in [4]. To state it, let numbers $p_\ell(m)$ be defined as the coefficients in the power series expansion

$$\prod_{j=0}^{\ell} \frac{1}{1 - z^{c_j}} = \sum_{m=0}^{\infty} p_\ell(m)\, z^m, \tag{2.2}$$

where the exponents $c_j$ are defined by $c_0 = 1$ and $c_j = 2 \cdot 3^{j-1}$ for $j \geq 1$. The quantities $p_\ell(m)$ have a combinatorial meaning: $p_\ell(m)$ is the number of different ways to pay cash down an amount of $m$ units, when only coins of values $c_0, c_1, \ldots, c_\ell$ are available. Some information about the growth behaviour of the $p_\ell(m)$ for $m \to \infty$ is contained in formula (2.2), as the radius of convergence of the power series is 1, even in the limit $\ell = \infty$.

Elka functions are linked to their generators as follows: For each $k, \ell \geq 0$ and each $a \in \mathbb{Z}_3$, we have the convolution-type formula

$$e_\ell(k,a) = \sum_{j=0}^{k} p_\ell(k-j) g_\ell(j,a) =: (p_\ell * g_\ell)(j,a). \tag{2.3}$$

Taking the 3-adic averages, $\overline{g}_\ell(k) := \int_{\mathbb{Z}_3^\times} g_\ell(k,a)\,da$, we have the convolution formula

$$\overline{e}_\ell(k) = (p_\ell * \overline{g}_\ell)(k). \tag{2.4}$$

The quantities $\overline{g}_\ell(k)$ also have a combinatorial meaning, see [5]: the number of different ways to distribute $k$ indistinguishable balls into $(\ell+1)$ urns $U_0, \ldots, U_\ell$, where each urn $U_j$ has capacity $c_j$, is just $2 \cdot 3^{\ell-1} \cdot \overline{g}_\ell(k)$. It is known that, for large $\ell$ and $k$ much smaller than $3^\ell$, the function $\overline{g}_\ell(k)$ grows rapidly in $k$. Together with the information that the $p_\ell(k)$ don't grow so rapidly, and in view of formulae (2.3) and (2.4), this makes the following conjecture plausible:

**Conjecture 1** *If there are real numbers $\delta_1, \mu_1 > 0$ such that*

$$(\star_2) \qquad \liminf_{\ell\to\infty} \frac{g_\ell(k_\ell,a)}{\overline{g}_\ell(k_\ell)} \geq \mu_1 \qquad \textit{uniformly for sequences } (k_\ell) \in \mathcal{A}_{\delta_1}\ ,$$

*then condition $(\star_1)$ of Theorem 1 holds.*

# 3 Normalization and digital topology

Now the next step is to describe an appropriate procedure to normalize the functions

$$g_\ell : \left\{0, 1, \ldots, 3^\ell - \ell - 1\right\} \times \mathbb{Z}_3^\times \to \mathbb{N}_0.$$

As far as the second variable is concerned, there is no problem with the domain of definition: for each $\ell \geq 1$, the arguments for second variable are taken from the compact set $\mathbb{Z}_3^\times$. On the other hand, the arguments for the first variable are taken from a set which is rapidly increasing when $\ell$ tends to infinity. Intuitively, we shall overcome this difficulty by 'squeezing' that large set into the unit interval $[0,1]$. Formally, we shall use (a lifting of) the *expansion map*

$$[0,1] \to \left\{0, 1, \ldots, 3^\ell\right\}, \qquad x \mapsto \lfloor 3^\ell x \rfloor,$$

where $\lfloor \xi \rfloor := \max\{m \in \mathbb{Z} : m \leq \xi\}$ (for any $\xi \in \mathbb{R}$) denotes the *floor function* or *Gauß bracket*.

In step (5) we will use the concept of continuity, and even that of equi-continuity. To prepare for this, let us change the topology of the unit interval. The new topology will be called the *digital topology* (to base 3). It is induced by a metric taking information from the expansion of real numbers in base 3. To give a precise definition, let

$$x = \sum_{j=1}^{\infty} x^{(j)} \cdot 3^{-j}, \tag{3.1}$$

denote the expansion of $x \in [0,1)$ in digits $x^{(j)} \in \{0,1,2\}$ to base 3 (we use upper indices here to reserve the place for lower indices for another purpose). It is well-known that the digits $x^{(j)}$ are uniquely determined by (3.1) and the additional the condition that $x^{(j)} \neq 2$ for infinitely many indices $j$. Now let $y = \sum_{j=1}^{\infty} y^{(j)} \cdot 3^{-j}$ be the digital expansion to base 3 of another real number $y \in [0,1)$. Then the *digital distance* between $x$ and $y$ is defined by

$$d_3(x,y) := 3^{-v_3(x,y)} \qquad \text{with} \quad v_3(x,y) := \min\left\{ j \geq 1 : x^{(j)} \neq y^{(j)} \right\},$$

where the minimum of an empty set is set to $+\infty$. This is a metric on the half-open interval $[0,1)$ satisfying the ultrametric inequality

$$d_3(x,z) \leq \max\{d_3(x,y), d_3(y,z)\}.$$

The topology on $[0,1)$ generated by this metric is called the *digital topology* on $[0,1)$. The digital topology has the property that each ball is both closed and open. We call a metric space with this property a *granulated space*.

Denote by $\mathbb{I}_3$ the Cauchy completion of the interval $[0,1)$ w.r.t. the digital distance; as a set, $\mathbb{I}_3$ may be identified with the set

$$\left\{ (x^{(j)})_{j\in\mathbb{N}} : x^{(j)} \in \{0,1,2\} \right\} \cong \{0,1,2\}^{\mathbb{N}}$$

of all digit sequences to base 3. There is a natural projection

$$\mathbb{I}_3 \to [0,1], \qquad \left( x^{(j)} \right)_{j\in\mathbb{N}} \mapsto \sum_{j=1}^{\infty} \frac{x^{(j)}}{3^j}$$

which is surjective but not injective. We shall use the notation

$$\lfloor 3^\ell x \rfloor := \sum_{j=1}^{\ell} x^{(j)} \cdot 3^{\ell-j}.$$

Now the measure space required in definition 2 is the set

$$\mathbb{X} := \mathbb{I}_3 \times \mathbb{Z}_3^\times,$$

the natural probability measure $\varrho$ on $\mathbb{X}$ is the tensor product of the lifted Lebesgue measure on $\mathbb{I}_3$ and the normalized Haar measure on $\mathbb{Z}_3^\times$. Taking $\varrho$ as reference measure, the probability measure $\mu_\ell$ is defined by the density

$$\widetilde{g}_\ell : \mathbb{X} \to \mathbb{R}, \qquad \widetilde{g}_\ell(x,a) := \gamma_\ell \cdot g_\ell\left( \lfloor 3^\ell x \rfloor, a \right)$$

where $\gamma_\ell$ is a positive constant ensuring that $\mu_\ell$ is a probability measure. We have to choose $\gamma_\ell$ such that

$$\int_{\mathbb{X}} \widetilde{g}_\ell \, d\varrho = 1\,;$$

according to [4], p. 107, this gives

$$\gamma_\ell = 2^{1-\ell} \cdot 3^{-\frac{1}{2}(\ell^2 - 5\ell + 2)}.$$

The topological vector space required in definition 2 is the normed linear space

$$V := \mathcal{C}(\mathbb{X}) = \{ f : \mathbb{X} \to \mathbb{R} \text{ continuous } \}$$

with the topology induced by the sup-norm. As all the function $\widetilde{g}_\ell$ are continuous w.r.t. the (granulated) product topology on $\mathbb{X} = \mathbb{I}_3 \times \mathbb{Z}_3^\times$, the linear space $V$ contains all the probability measures $\mu_\ell$.

The remaining step for section (iii) is to translate condition ($\star_2$) into this setting. To this end, we first translate the class $\mathcal{A}_\delta$ of ceratin integer sequences, as defined in section (i), to a class of sequences in $\mathbb{I}_3$:

$$\widetilde{\mathcal{A}}_\delta = \left\{ \left( x_\ell = (x_\ell^{(j)})_{j\in\mathbb{N}} \right)_{\ell\in\mathbb{N}} : \left( \sum_{j=1}^{\ell} x_\ell^{(j)} \cdot 3^{\ell-j} \right)_{\ell\in\mathbb{N}} \in \mathcal{A}_\delta \right\}. \tag{3.2}$$

We arrive at the following translation of theorem 1:

**Theorem 2** *If there are real numbers $\delta, \mu > 0$ and in index $\ell_0$ such that*
($\star_3$)

$$\widetilde{g}_\ell(x_\ell, a) \geq \mu \int_{\mathbb{Z}_3^\times} \widetilde{g}_\ell(x_\ell, a)\, da \qquad \textit{uniformly for sequences } (x_\ell) \in \widetilde{\mathcal{A}}_\delta \textit{ and } \ell \geq \ell_0,$$

*then condition ($\star_2$) of Conjecture 1 holds.*

# 4 Transition operators

According to definition 2, the next step is to define a sequence of linear operators

$$S_\ell : \mathcal{C}(\mathbb{X}) \to \mathcal{C}(\mathbb{X})$$

performing the map $\mu_{\ell-1} \mapsto \mu_\ell$. As described in the previous section, a measure $\mu_\ell$ is represented in the vector space $V = \mathcal{C}(\mathbb{X})$ by its density $\widetilde{g}_\ell$ w.r.t. the reference measure $\varrho$ on $\mathbb{X}$. In this setting, our aim is to define the $S_\ell$ such that

$$\widetilde{g}_\ell = S_\ell(\widetilde{g}_{\ell-1}). \tag{4.1}$$

The definition of $S_\ell$ will be obtained by reformulating the recursion formula (2.1) using the normalization process of section (iii). To this end, let $(x,a) \in \mathbb{X} = \mathbb{I}_3 \times \mathbb{Z}_3^\times$ be given, and calculate (we set $\widetilde{g}_{\ell-1}(y,a) = 0$ if $y \notin \mathbb{I}_3$):

$$\begin{aligned}
\widetilde{g}_\ell(x,a) &= \gamma_\ell \cdot g_\ell\left(\left\lfloor 3^\ell x \right\rfloor, a\right) \\
&= \gamma_\ell \sum_{0\leq j<2\cdot 3^{\ell-1}} g_{\ell-1}\left(\left\lfloor 3^\ell x \right\rfloor - j, \frac{2^{j+1}a-1}{3}\right) \\
&= \gamma_\ell \sum_{0\leq j<2\cdot 3^{\ell-1}} g_{\ell-1}\left(\left\lfloor 3^{\ell-1}\left(3x - \frac{j}{3^{\ell-1}}\right)\right\rfloor, \frac{2^{j+1}a-1}{3}\right) \\
&= \frac{\gamma_\ell}{\gamma_{\ell-1}} \sum_{0\leq j<2\cdot 3^{\ell-1}} \widetilde{g}_{\ell-1}\left(3x - \frac{j}{3^{\ell-1}}, \frac{2^{j+1}a-1}{3}\right) \\
&= \frac{1}{2\cdot 3^{\ell-3}} \sum_{0\leq j<2\cdot 3^{\ell-1}} \widetilde{g}_{\ell-1}\left(3x - \frac{j}{3^{\ell-1}}, \frac{2^{j+1}a-1}{3}\right).
\end{aligned}$$

This leads to the definition of the summation operators $S_\ell : \mathcal{C}(\mathbb{X}) \to \mathcal{C}(\mathbb{X})$,

$$S_\ell f(x,a) := \frac{1}{2\cdot 3^{\ell-3}} \sum_{0\leq j<2\cdot 3^{\ell-1}} f\left(3x - \frac{j}{3^{\ell-1}}, \frac{2^{j+1}a-1}{3}\right), \tag{4.2}$$

where we set $f(y,b) = 0$ whenever $(y,b) \notin \mathbb{X}$. By construction, the operators $S_\ell$ satisfy equation (4.1).

# 5 The limiting transition operator $S_\infty$

In the sequel, the sequence $(S_\ell)_{\ell\geq 1}$ of linear operators on $V=\mathcal{C}(\mathbb{X})$ will be shown to converge to the following operator:

$$S_\infty : V \to V, \qquad f \mapsto S_\infty f : \mathbb{X} \to \mathbb{R} \quad \text{defined by:}$$

$$(S_\infty f)(x,a) := \frac{3}{2}\int_{\{t\in\mathbb{I}_3, 3x-2\leq t\leq 3x\}} \int_{\mathbb{Z}_3^\times} f(t,b)\,db\,dt\ . \tag{5.1}$$

Note that, in fact, $(S_\infty f)(x,a)$ does not depend on $a\in\mathbb{Z}_3^\times$; w.r.t. the second variable, the operator $S_\infty$ just integrates over all of $\mathbb{Z}_3^\times$.

The following theorem is 'dual' to theorem IV.4.1 in [4]; in principle, the proof of this theorem could be obtained by an appropriate 'dualization'. Probably a better idea is to consider this theorem as a variant of theorem 4.1 in [6], and taking the direct proof given there, with the necessary changes. This is elaborated here.

**Theorem 3** *For each $\ell\in\mathbb{N}$, let $S_\ell$ be given by (4.2), and let $S_\infty$ be given by (5.1).*

*(a) The sequence $(S_\ell)_{\ell\in\mathbb{N}}$ converges to $S_\infty$ in the strong operator topology. That means: For any $f\in\mathcal{C}(\mathbb{X})$*

$$\lim_{\ell\to\infty} \|S_\ell f - S_\infty f\|_\infty = 0.$$

*(b) This convergence is uniform on bounded equi-continuous families.*

Before proving the theorem on convergence, let us record the following lemma linking the summation operators $S_\ell$ to integration. This lemma has been extracted out of the proof of theorem 4.1 in [6].

**Lemma 4** *Let $f:\mathbb{X}\to\mathbb{R}$ be a digital-continuous function with modulus of continuity*

$$\omega(\delta) := \sup\{|f(x,a)-f(y,b)| : \max\{d_3'(x,y), d_3(a,b)\}\leq\delta\}\,,$$

*where $d_3'$ denotes the digital distance on $\mathbb{I}_3$ and $d_3$ is the usual 3-adic metric on $\mathbb{Z}_3^\times$. In addition, let $s\geq r>0$ and $k$ be integers with $2\leq k<3^r$ and set*

$$I_k := \left[\frac{k-2}{3^r}+\frac{1}{3^{r+s}}, \frac{k}{3^r}+\frac{1}{3^{r+s}}\right) \subset \mathbb{I}_3.$$

*Then the following inequality is valid for any $a\in\mathbb{Z}_3^\times$:*

(5.2)
$$\left|\frac{1}{3^{r+s-1}}\sum_{0\leq j<2\cdot 3^s} f\left(\frac{k}{3^r}-\frac{j}{3^{r+s}}, \frac{2^{j+1}a-1}{3}\right) - \int_{I_k\times\mathbb{Z}_3^\times} f\,d\varrho\right| \leq \frac{2}{3^r}\,\omega\left(2\cdot 3^{-r}\right).$$

**Proof:** Let us first fix some notations. For $b\in\mathbb{Z}_3^\times$ we put

$$(b \bmod 3^r) := \left\{a\in\mathbb{Z}_3^\times a\equiv b \bmod 3^r\right\}$$

for the residue class of $b$ modulo $3^r$, which is precisely the closed ball with radius $3^{-r}$ in $\mathbb{Z}_3^\times$ around $b$. Next consider the following boxes in the space $\mathbb{X}=\mathbb{I}_3\times\mathbb{Z}_3^\times$:

$$Q(b,r) := I_k\times(b \bmod 3^r)\,.$$

Then we can decompose a column $I_k\times\mathbb{Z}_3^\times\subset\mathbb{X}$ into such boxes:

$$I_k\times\mathbb{Z}_3^\times = \bigcup_{(b \bmod 3^r)} Q(b,r), \tag{5.3}$$

where the union is taken over the $2 \cdot 3^{r-1}$ prime residue classes modulo $3^r$.

For any $a \in \mathbb{Z}_3^\times$ and any integer $s > 0$, the set

$$\left\{ \frac{2^{j+1}a - 1}{3} : j = 0, \ldots, 2 \cdot 3^s - 1 \right\}$$

intersects each residue class modulo $3^s$ in exactly one point. It follows that the set of second components in the set

$$\Delta(a,k,r,s) := \left\{ \left( \frac{k}{3^r} - \frac{j}{3^{r+s}}, \frac{2^{j+1}a-1}{3} \right) : j = 0, \ldots, 2 \cdot 3^s - 1 \right\}$$

intersects each residue class modulo $3^r$ in exactly $3^{s-r}$ points, whenever $r \le s$. This implies that, for any $b \in \mathbb{Z}_3^\times$ and provided $r \le s$, the intersection

$$E(a,b,r) := Q(b,r) \cap \Delta(a,k,r,s)$$

contains precisely $3^{s-r}$ points. We conclude that the integral of $f$ on $Q(b,r)$, divided by the volume of $Q(b,r)$, can be approximated by the average of $f$ on the set $E(a,b,r)$ of evaluation points. As the boxes $Q(b,r)$ have diameter $\le 2 \cdot 3^{-r}$, we infer that the following inequality holds for any $a, b \in \mathbb{Z}_3^\times$:

$$\left| \frac{1}{3^{s-r}} \sum_{E(a,b,r)} f - \frac{1}{\varrho(Q(b,r))} \int_{Q(b,r)} f \, d\varrho \right| \le \max_{Q(b,r)} f - \min_{Q(b,r)} f \le \omega_f \left(2 \cdot 3^{-r}\right) \tag{5.4}$$

(here the arguments of $f$ are omitted to improve readability). To obtain the final form for the estimate on boxes, compute the measure

$$\varrho(Q(b,r)) = \frac{2}{3^r} \cdot \frac{1}{2 \cdot 3^{r-1}} = \frac{1}{3^{2r-1}} \; .$$

Inserting this into (5.4), and rearranging the formula to get a coefficient 1 in front of the integral, we arrive at

$$\left| \frac{1}{3^{2r-1} \cdot 3^{s-r}} \sum_{E(a,b,r)} f - \int_{Q(b,r)} f \, d\varrho \right| < \frac{\omega_f \left(2 \cdot 3^{-r}\right)}{3^{2r-1}} \; . \tag{5.5}$$

To end the proof of (5.2), we take the sum of all the inequalities (5.5) when $b$ runs through the prime residue classes modulo $3^r$. By decomposition (5.3), this gives

$$\left| \frac{1}{3^{s+r-1}} \sum_{\Delta(a,k,r,s)} f - \int_{I_k \times \mathbb{Z}_3^\times} f \, d\varrho \right| \le \varrho \left(I_k \times \mathbb{Z}_3^\times\right) \cdot \omega_f \left(2 \cdot 3^{-r}\right) = \frac{2}{3^r} \, \omega_f \left(2 \cdot 3^{-r}\right) \; .$$

This completes the proof of (5.2). $\diamondsuit$

**Proof of Theorem 3:** Let $\mathcal{F}$ be a bounded equi-continuous family of functions $f : \mathbb{X} \to \mathbb{R}$ which are continuous w.r.t. the digital topology on $\mathbb{X}$, and denote

$$M := \sup\{|f(x,a)| : (x,a) \in \mathbb{X}, f \in \mathcal{F}\}.$$

Now let $\varepsilon > 0$ be given. We will prove that there is an index $\ell_0(\varepsilon)$ such that

$$\sup_{f \in \mathcal{F}} \|S_\ell f - S_\infty f\|_\infty \le (4M+1)\varepsilon \qquad \text{for } \ell \ge \ell_0(\varepsilon). \tag{5.6}$$

Because $\mathcal{F}$ is an equi-continuous family, there exists $\delta \in (0, \varepsilon]$ such that

$$\omega_f(\delta) \leq \varepsilon \qquad \text{for all } f \in \mathcal{F}.$$

To apply lemma 4, choose an integer $r > 0$ such that $2 \cdot 3^{-r} \leq \delta$, whence

$$\omega_f\left(2 \cdot 3^{-r}\right) \leq \varepsilon \qquad \text{for all } f \in \mathcal{F}.$$

Next choose $s := \ell - r - 1$ (this gives $s \geq r$ if $\ell$ is sufficiently large) and put $I(x) := \mathbb{I}_3 \cap [3x - 2, 3x]$ and

$$K(x) := \left\{k \in \mathbb{N} : k \text{ even}, I_k \subset I(x)\right\}.$$

Then, for each $k \in K(x)$, we conclude from lemma 4 the estimate

$$\left|\frac{3}{2 \cdot 3^{\ell-2}} \sum_{0 \leq j < 2 \cdot 3^s} f\left(\frac{k \cdot 3^r - j}{3^{\ell-1}}, \frac{2^{j+1}a - 1}{3}\right) - \frac{3}{2} \int_{I_k \times \mathbb{Z}_3^\times} f \, d\varrho\right| \leq \frac{2}{3^r} \cdot \varepsilon.$$

The intervals $I_k$ where $k$ runs through the set $K(x)$ generally do not cover all of the interval $I(x)$: on the left and the right end may remain intervals of total length $< 2\delta \leq 2\varepsilon$ not covered by the $I_k$. On these ends, the sum and the integral have to be estimates by $4M$ in total. So we arrive at the final estimate

$$\left|\frac{3}{2} \sum_{0 \leq j < 2 \cdot 3^{\ell-1}} f\left(3x - \frac{j}{3^{\ell-1}}, \frac{2^{j+1}a - 1}{3}\right) - \frac{3}{2} \int_{I(x) \times \mathbb{Z}_3^\times} f \, d\varrho\right| \leq (4M + 1) \cdot \varepsilon;$$

note that the indices $j$ in the summation are just those for which the first argument of $f$ is contained in the interval $(3x - 2, 3x]$. $\diamondsuit$

**Corollary 5** *The sequence $(g_\ell)_{\ell \in \mathbb{N}}$ of generators of Elka functions is weakly self-similar in the sense of definition 2.*

# 6 A strongly stable Markov chain

Let us have a closer look at the essential part of the limiting operator $S_\infty$. To this end, consider the following operator mapping locally integrable real functions to continuous functions:

$$\text{(6.1)} \qquad W_3 : L^1_{loc}(\mathbb{R}) \to \mathcal{C}(\mathbb{R}), \qquad (W_3 f)(x) = \frac{3}{2} \int_{3x-2}^{3x} f(t) \, dt.$$

It is not very difficult to prove some basic properties of $W_3$:

**Theorem 6** *Let $W_3$ be the operator defined in* (6.1)*, and let $f$ be an $L^1$-function on $\mathbb{R}$ with* $\operatorname{supp} f \subset [0, 1]$. *Then:*

(a) $\operatorname{supp}(W_3 f) \subset [0, 1]$; *that means $W_3 : L^1([0, 1]) \to L^1([0, 1])$.*

(b) $\int_0^1 f(x) \, dx = \int_0^1 (W_3 f)(x) \, dx$; *that means $W_3$ is integral-preserving.*

(c) *If $\int_0^1 f(x) \, dx = 0$, then $\|W_3 f\|_1 \leq \frac{1}{2} \|f\|_1$.*

The proof of this theorem is contained in the proof oh theorem IV. 5.1 in [4]. The main consequences of these properties are that $W_3$ has a unique fix-point in $L^1([0, 1])$ and defines a strongly stable Markov chain on the unit interval $[0, 1]$. More precisely:

**Corollary 7** *There is unique function* $\phi \in L^1_{loc}(\mathbb{R})$ *satisfying*

$$\operatorname{supp}\phi \subset [0,1], \qquad \int_0^1 \phi(x)\,dx = 1, \qquad W_3\phi = \phi.$$

*Moreover,* $\phi$ *is a* $\mathcal{C}^\infty$*-function which is a polynomial on each interval lying outside the classical Cantor set.*

To prove this, observe that property (c) of theorem 6 means that $W_3$ is a metric contraction on each affine hyperplane

$$H_\lambda = \left\{ f \in L^1([0,1]) : \int_0^1 f(x)\,dx = \lambda \right\} \qquad (\lambda \in \mathbb{R}).$$

In addition, this implies the following convergence property.

**Corollary 8** *Let* $f_0 \in H_\lambda$ *be given. Then the sequence* $(f_n)_{n\in\mathbb{N}}$ *defined inductively by*

$$f_{n+1} := W_3 f_n, \qquad n \geq 0,$$

*converges in* $L^1$ *to* $f_\infty := \lambda\phi$*. Moreover, we have the estimate*

$$\|f_n - f_\infty\|_1 \leq 2^{-n+1}\|f_1 - f_0\|_1 \qquad \text{for each } n \in \mathbb{N}.$$

The generators for Elka functions $g_\ell(k,a)$ are linked to the function $\phi$ of corollary 7 via the formula

$$\lim_{\ell\to\infty} \int_{\mathbb{Z}_3^\times} \widetilde{g}_\ell(x,a)\,da = \phi(x) \qquad \text{for any fixed } x \in \mathbb{I}_3.$$

This formula, together with the convergence of the transition operators $S_\ell$ proved in Theorem 3, suggest the following assertion:

**Conjecture 2** *Consider the following two characteristic functions:*

$$\chi_0(x) := \begin{cases} 1 & \text{for } 0 \leq x \leq \frac{2}{3}, \\ 0 & \text{otherwise,} \end{cases} \qquad \chi_1(x) := \begin{cases} 1 & \text{for } \frac{1}{3} \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

*If there are real numbers* $\delta, \mu > 0$ *such that*

$$(\star_4) \qquad \liminf_{\ell\to\infty} \frac{(W_3^\ell\chi_1)(x_\ell)}{(W_3^\ell\chi_0)(x_\ell)} \geq \mu \qquad \text{uniformly for sequences } (x_\ell) \in \widetilde{\mathcal{A}}_\delta,$$

*then condition* $(\star_3)$ *of Theorem 2 is fulfilled.*

# 7 The asymptotics of a quotient

Condition $(\star_4)$ concerns the behaviour of the quotient

$$\frac{(W_3^\ell\chi_1)(x_\ell)}{(W_3^\ell\chi_0)(x_\ell)} \tag{7.1}$$

for large $\ell$, where $(x_\ell)$ is a sequence in the class $\widetilde{\mathcal{A}}_\delta$. To get some feeling about what can happen, let us consider the limiting behaviour of the quotient (7.1) for some simple sequences outside the class $\widetilde{\mathcal{A}}_\delta$.

Fix a number $x$ satisfying $0 < x < 1$, and consider the constant sequence $x_\ell = x$ for all $\ell \in \mathbb{N}$. Then corollary 8 proves that

$$\lim_{\ell\to\infty} (W_3^\ell \chi_1)(x) = \lim_{\ell\to\infty} (W_3^\ell \chi_0)(x) = \tfrac{2}{3}\,\phi(x),$$

where $\phi$ denotes the function of corollary 7, which uniquely determined by

$$\operatorname{supp}\phi \subset [0,1], \qquad \int_0^1 \phi(x)\,dx = 1, \qquad W_3\phi = \phi. \tag{7.2}$$

As $\phi(x) > 0$ for any $x \in (0,1)$, we conclude that for a constant sequence $x_\ell = x \in (0,1)$ the quotient in (7.1) converges to 1.

Next consider the sequence defined by $x_\ell = 3^{-\ell-1}$ for each $\ell \in \mathbb{N}$. This sequence converges to 0 faster than any sequence in a class $\widetilde{\mathcal{A}}_\delta$. Looking at the quotient (7.1) for this sequence $(x_\ell)$, first observe that

$$\left(W_3^\ell \chi_1\right)\left(\frac{1}{3^{\ell+1}}\right) = 0 \qquad \text{for each } \ell \geq 1,$$

as one can easily prove by induction on $\ell$. On the other hand,

$$\left(W_3^\ell \chi_0\right)\left(\frac{1}{3^{\ell+1}}\right) > 0 \qquad \text{for each } \ell \geq 0.$$

We infer that for $x_\ell = 3^{-\ell-1}$, the quotient (7.1) is equal to zero for each $\ell$.

Before going into the investigation what happens for sequences $(x_\ell) \in \widetilde{\mathcal{A}}_\delta$, let us have a closer look at the Markov chain generated by our operator $W_3$. Define the *transition kernel* by

$$K(x,t) := \tfrac{3}{2}\,\chi_{[t,t+2]}(3x) = \begin{cases} \frac{3}{2} & \text{for } t \leq 3x \leq t+2, \\ 0 & \text{otherwise.} \end{cases} \qquad (x,t \in \mathbb{R}).$$

Then we have, for each $f \in L^1_{loc}(\mathbb{R})$,

$$W_3 f(x) = \int_{\mathbb{R}} K(x,t) f(t)\,dt\,.$$

Next define the *iterated transition kernels* inductively by

$$K_0(x,t) = K(x,t), \qquad K_{\ell+1}(x,t) = \int_{\mathbb{R}} K(x,y) K_\ell(y,t)\,dy\,.$$

Some properties of the iterated kernels are

$$K_\ell(x,0) = \frac{3}{2}\left(W_3^\ell \chi_0\right)(x) \qquad \text{for each } \ell \geq 0,\ x \in \mathbb{R},$$

$$K_\ell(x,1) = \frac{3}{2}\left(W_3^\ell \chi_1\right)(x) \qquad \text{for each } \ell \geq 0,\ x \in \mathbb{R},$$

$$K_\ell(x,0) = K_\ell\left(x + \frac{1}{3^{\ell+1}}, 1\right) \qquad \text{for each } x \in \mathbb{R}, \tag{7.3}$$

$$K_\ell(x,0) > \phi(x) > K_\ell(x,1) \qquad \text{for each } \ell \geq 0 \text{ and } 0 < x < \tfrac{1}{3}, \tag{7.4}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is the function defined in corollary 7. In addition, we know that

$$\lim_{\ell\to\infty} K_\ell(x,t) = \phi(x) \qquad \text{for each } t \in [0,1].$$

It follows an argument why the sequences $(x_\ell) \in \widetilde{\mathcal{A}}_\delta$ just have the 'borderline speed' making the limit of the quotient (7.1) positive. First observe the inequalities

$$\begin{aligned} K_\ell(x,1) &= K_\ell\left(x - \frac{1}{3^{\ell+1}}, 0\right) && \text{by (7.3)},\\ &\geq \phi\left(x - \frac{1}{3^{\ell+1}}\right) && \text{for small } x \text{, by (7.4)},\\ &\geq \phi\left(x + \frac{1}{3^{\ell+1}}\right) - \frac{2}{3^{\ell+1}}\phi'\left(x + \frac{1}{3^{\ell+1}}\right) && \text{for small } x. \end{aligned}$$

On the other hand, we obtain from (7.3) and (7.4) that

$$K_\ell(x,0) = K_\ell\left(x + \frac{1}{3^{\ell+1}}, 1\right) < \phi\left(x + \frac{1}{3^{\ell+1}}\right) \qquad \text{for sufficiently small } x.$$

Now let $(x_\ell) \in \widetilde{\mathcal{A}}_\delta$ and put $x_\ell^+ := x_\ell + \frac{1}{3^{\ell+1}}$ for abbreviation. Then we arrive at the following estimate for the quotient (7.1):

$$\frac{(W_3^\ell \chi_1)(x_\ell)}{(W_3^\ell \chi_0)(x_\ell)} = \frac{K_\ell(x_\ell,1)}{K_\ell(x_\ell,0)} \geq \frac{\phi\left(x + \frac{1}{3^{\ell+1}}\right) - \frac{2}{3^{\ell+1}}\phi'\left(x + \frac{1}{3^{\ell+1}}\right)}{\phi\left(x + \frac{1}{3^{\ell+1}}\right)} = 1 - \frac{2}{3^{\ell+1}} \cdot \frac{\phi'(x_\ell^+)}{\phi(x_\ell^+)}.$$

In order to ensure condition $(\star_4)$ of Conjecture 2, it would suffice to prove that

$$\limsup_{\ell\to\infty} \frac{2}{3^{\ell+1}} \cdot \frac{\phi'(x_\ell^+)}{\phi(x_\ell^+)} \leq 1 - \mu < 1. \tag{7.5}$$

As $(x_\ell) \in \widetilde{\mathcal{A}}_\delta$, we have the asymptotics $x_\ell \sim x_\ell^+ \sim \ell \cdot 3^{-\ell}$. This implies

$$\frac{2}{3^{\ell+1}} \sim \frac{2}{3\ell}\, x_\ell^+ \sim \frac{c\, x_\ell^+}{\ln(x_\ell^+)} \tag{7.6}$$

with the constant $c = -\frac{2}{3}\ln 3$. To compute the derivative $\phi'(x)$, we go back to the properties (7.2) which uniquely determine $\phi$. For $x \leq \frac{2}{3}$, the lower limit $3x - 2$ for integration is $\leq 0$, which implies

$$\phi(x) = \frac{3}{2}\int_0^{3x} \phi(t)\, dt \qquad \text{for } 0 \leq x \leq \frac{2}{3}.$$

Differentiating this gives

$$\phi'(x) = \frac{9}{2}\,\phi(3x) \qquad \text{for } 0 \leq x \leq \frac{2}{3}. \tag{7.7}$$

We come to the following formula for the term occurring in (7.5):

$$\frac{2}{3^{\ell+1}} \cdot \frac{\phi'(x_\ell^+)}{\phi(x_\ell^+)} = \frac{9c}{2} \cdot \frac{x_\ell^+}{\ln(x_\ell^+)} \cdot \frac{\phi(3x_\ell^+)}{\phi(x_\ell^+)}\,. \tag{7.8}$$

To prove (7.5), we are seeking information about the quotient $\phi(3x)/\phi(x)$ when $x \to 0$. For this, it would be very helpful to know something about the asymptotic behavior of $\phi(x)$ when $x \to 0$. This asymptotics has been investigated by Berg and Krüppel [1] in the following way: First they consider the differential equation

$$\lambda\phi'(t) = a(\phi(at) - \phi(at - a + 1)). \tag{7.9}$$

Differentiating the equation $W_3\phi = \phi$ we see that our function $\phi$ satisfies this equation for the parameter values $a = 3$ and $\lambda = \frac{2}{3}$. Berg and Krüppel give some relations between the solutions of (7.9) and the solutions of the *truncated equation*

$$\lambda g'(t) = a\, g(at); \tag{7.10}$$

note that the space of all solutions of this equation (without further restrictions) is an infinite-dimensional linear space. Using Laplace transformation and the saddle-point method, Berg and Krüppel determine the precise asymptotics of a special solution $g_0$ of (7.10). Moreover, they give some arguments why they expect that $\phi$ has an asymptotic as $g_0$ up to factor which is both bounded and bounded away from zero.

Let's have a look at the asymptotics of $g_0$ given in [1]:

$$g_0(t) \sim \phi_0(t) := \frac{(2\beta)^\varepsilon}{\sqrt{2\pi}} \exp\left(\gamma \ln t + \delta \ln(-\ln t) - \beta\ \ln^2\left(\frac{t}{-\ln t}\right)\right), \tag{7.11}$$

with some constants $\beta, \gamma, \delta, \varepsilon$ depending on the parameters $\lambda$ and $a$ by explicitly given formulae. Note that, moreover, $\phi_0$ is an asymptotic solution of (7.10) in the sense that

$$\lim_{t\to 0} \frac{\lambda \phi_0'(t)}{a\phi_0(at)} = 1. \tag{7.12}$$

For this function $\phi_0$, a somewhat lengthy calculation shows that (for any $\delta > 0$)

$$\lim_{\ell\to\infty} \frac{2}{3^{\ell+1}} \cdot \frac{\phi_0'(x_\ell)}{\phi_0(x_\ell)} = \frac{2}{3} < 1 \qquad \text{uniformly for sequences } (x_\ell) \in \widetilde{\mathcal{A}}_\delta. \tag{7.13}$$

This result would imply inequality (7.5), if we could replace $\phi$ by $\phi_0$ under the limit. That replacement would be possible, if, e.g., $\phi \sim c\phi_0$ for some constant $c > 0$.

It turns out that the following (slightly weaker) conjecture about the asymptotics of $\phi$ suffices to imply $(\star_4)$:

**Conjecture 3** *There are positive real constants $c$ and $\delta_5$ such that*

$$\lim_{\ell\to\infty} \frac{\phi(z_\ell)}{\phi_0(z_\ell)} = c > 0 \qquad \textit{uniformly for sequences } (z_\ell) \in \widetilde{\mathcal{A}}_{\delta_5}. \tag{$\star_5$}$$

(Note that the results given in section 9 of [1] suggest that conjecture 3 is true.)

The implication $(\star_5) \Rightarrow (\star_4)$ is shown as follows: Suppose that $(x_\ell) \in \widetilde{\mathcal{A}}_\delta$ is given. Then, for $\delta_5 > \delta$, there are sequences $(\widehat{x}_\ell), (\widehat{y}_\ell) \in \widetilde{\mathcal{A}}_{\delta_5}$ with the property

$$\widehat{x}_\ell = x_\ell^+, \qquad \widehat{y}_{\ell-1} = 3x_\ell^+ \qquad \text{for all but finitely many indices } \ell. \tag{7.14}$$

Now we are ready to calculate the limit of the sequence considered in (7.5):

$$\lim_{\ell\to\infty} \frac{2}{3^{\ell+1}} \cdot \frac{\phi'(x_\ell^+)}{\phi(x_\ell^+)} = \lim_{\ell\to\infty} \frac{2}{3^{\ell+1}} \cdot \frac{\frac{9}{2}\phi(3x_\ell^+)}{\phi(x_\ell^+)} \qquad \text{by (7.7),}$$

$$= \lim_{\ell\to\infty} \frac{2}{3^{\ell+1}} \cdot \frac{\frac{9}{2}\phi_0(\widehat{y}_{\ell-1}) \cdot \frac{\phi(\widehat{y}_{\ell-1})}{\phi_0(\widehat{y}_{\ell-1})}}{\frac{\phi(\widehat{x}_\ell)}{\phi_0(\widehat{x}_\ell)} \cdot \phi_0(\widehat{x}_\ell)} \qquad \text{by (7.14),}$$

$$= \lim_{\ell\to\infty} \frac{2}{3^{\ell+1}} \cdot \frac{\frac{9}{2}\phi_0(\widehat{y}_{\ell-1})}{\phi_0(\widehat{x}_\ell)} \qquad \text{by conjecture 3,}$$

$$= \lim_{\ell\to\infty} \frac{2}{3^{\ell+1}} \cdot \frac{\phi_0'(x_\ell^+)}{\phi_0(x_\ell^+)} \qquad \text{by (7.14) and (7.7),}$$

$$= \frac{2}{3} \qquad \text{by (7.13),}$$

which implies (7.5) and therefore $(\star_4)$. $\diamondsuit$

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