Pisot Substitutions and (Limit–) Quasiperiodic Model Sets

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Substitutions

Use a finite alphabet $\mathcal{A}$ and a rule σ how to substitute letters to generate a (two-sided) sequence (denote by $n = \operatorname{card} \mathcal{A}$).

ⓔⓧ Fibonacci-substitution $\quad \mathcal{A} = \{a, b\}, \qquad a \stackrel{\sigma}{\mapsto} ab, \quad b \stackrel{\sigma}{\mapsto} a$

$$\begin{aligned} b.a \quad & \stackrel{\sigma}{\mapsto} \sigma(b.a) = \sigma(b).\sigma(a) = a.ab \stackrel{\sigma}{\mapsto} \sigma(a.ab) = \sigma(a).\sigma(a)\sigma(b) = ab.aba \\ & \stackrel{\sigma}{\mapsto} \dots \stackrel{\sigma}{\mapsto} \dots abaaba \begin{Bmatrix} ab \\ ba \end{Bmatrix} .abaababa \dots \end{aligned}$$

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&\stackrel{\sigma}{\mapsto} \ldots \stackrel{\sigma}{\mapsto} \ldots abaaba \begin{Bmatrix} ab \\ ba \end{Bmatrix} .abaababa \ldots
\end{aligned}
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Define $(n \times n)$-substitution matrix $\boldsymbol{M}$ where $M_{ij} = \#i$'s in $\sigma(j) = \#_i(\sigma(j))$.

ex for Fibonacci-substitution $\boldsymbol{M} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

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$$a.a \quad \stackrel{\sigma}{\mapsto} \sigma(a.a) = aaba.aaba \stackrel{\sigma}{\mapsto} \ldots \stackrel{\sigma}{\mapsto} \ldots baaaaaba.aabaaaba \ldots$$

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ⓔⓧ for ♣-substitution $\boldsymbol{M} = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$

Substitutions of Pisot type

σ is *Pisot substitution* if $\boldsymbol{M}$ has exactly one dominant (simple) eigenvalue $\lambda > 1$ and all other eigenvalues λ_i satisfy $0 < |\lambda_i| < 1$ (inside unit circle).

- Characteristic polynomial $p(x) = \det(x \cdot \mathbb{1} - \boldsymbol{M})$ is irreducible over $\mathbb{Z}$. Denote by r the number of real roots, and by s the number of pairs of complex conjugate roots ($n = r + 2 \cdot s$).
- $\boldsymbol{M}$ is primitive. Let u be any fixed point of σ, denote by $\mathcal{O}(u)$ the orbit of u under shift S. Then $\mathcal{X} = \overline{\mathcal{O}(u)}$ is a compact space w.r.t. product topology and $(\mathcal{X}, S)$ is a (strictly ergodic) dynamical system ($\rightsquigarrow$ local hull).

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ex for Fibonacci-substitution $\lambda = \frac{1+\sqrt{5}}{2} \approx 1.618$ $\lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618$
$\det \boldsymbol{M} = -1$.

ex for ♣-substitution $\lambda = \frac{3+\sqrt{17}}{2} \approx 3.562$ $\lambda_2 = \frac{3-\sqrt{17}}{2} \approx -0.562$
$\det \boldsymbol{M} = -2$.

Model Sets

Cut and project scheme:

$$\begin{array}{ccccc} G & \stackrel{\pi_1}{\longleftarrow} & G \times H & \stackrel{\pi_2}{\longrightarrow} & H \\ \cup & & \cup & & \cup \text{ dense} \\ L & \stackrel{1-1}{\longleftrightarrow} & \tilde{L} & \longrightarrow & L^\star \end{array}$$

where G, H are LCAGs, $\tilde{L}$ is a lattice, π_1, π_2 are the canonical projections and $^\star : L \to H$ denotes the mapping $\pi_2 \circ (\pi_1|_{\tilde{L}})^{-1}$.

A set $\Lambda(W) := \{x \in L \mid x^\star \in W\} \subset G$ is a *regular model set*, if $W \subset H$ is a non-empty relatively compact set with $\operatorname{cl} W = \operatorname{cl}(\operatorname{int} W)$ and boundary $\partial W = \operatorname{cl} W \setminus \operatorname{int} W$ of Haar-measure zero.

Length Representation

We want to represent a sequence $u \in \mathcal{X}$ as (Delone) point set $\varphi(u) = \Lambda \subset \mathbb{R}$ (where $\varphi : u \to \mathbb{R}$).

For this, let $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_n)$ be a left eigenvector of M to the eigenvalue λ. Let the 'distance' between $\varphi(u_i)$ and $\varphi(u_{i+1})$ be ℓ_i, and set $\varphi(u_0) = 0$.
(ℓ_i's are rationally independent)

ex   Fibonacci-substitution: $\boldsymbol{\ell} = (\lambda, 1)$

$\ldots aab.aba \ldots \quad \mapsto \quad \ldots$ $-2\lambda-1$ $\quad -\lambda-1 \quad -1 \quad 0 \quad \lambda \quad \lambda+1$ $\ldots$

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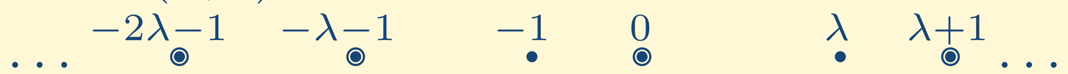

Formally, define a map $l : u \to \mathbb{Z}^n$ (Abelianization) by

$$l(u_k) = \begin{cases} (0, \ldots, 0) & \text{if } k = 0, \\ (\#_i(u_0 \ldots u_{k-1}))_{1 \le i \le n} & \text{if } k > 0, \\ (-\#_i(u_k \ldots u_{-1}))_{1 \le i \le n} & \text{if } k < 0. \end{cases}$$

Then: $\varphi(u_k) = \boldsymbol{\ell} \cdot l(u_k)$.

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ex Fibonacci-substitution: $\boldsymbol{\ell} = (\lambda, 1)$

$$\ldots aab.aba \ldots \quad \mapsto \quad \ldots \overset{-2\lambda-1}{\odot} \quad \overset{-\lambda-1}{\odot} \quad \overset{-1}{\bullet} \quad \overset{0}{\odot} \quad \overset{\lambda}{\bullet} \quad \overset{\lambda+1}{\odot} \ldots$$

- φ lines up with substitution σ: $\lambda \cdot \varphi(u) \subset \varphi(\sigma(u))$
  Especially for fixed point $u = \sigma(u) \implies \lambda \cdot \Lambda \subset \Lambda$ ($\rightsquigarrow$ Meyer set)
- Since $\ell_i \in \mathbb{Q}(\lambda)$, this representation is in fact defined on algebraic number field (of degree n).

Local Fields

Look at the completions of $\mathbb{Q}(\lambda) \cong \mathbb{Q}[x]/p(x)$:

- Recall $n = r + 2 \cdot s$, therefore there are r different embeddings into $\mathbb{R}$ and s different (non-equivalent) embeddings into $\mathbb{C}$ (w.r.t. the usual absolute value $\|\cdot\|_\infty$).

  Since we are in the Pisot case, we have $\|\lambda\|_\infty > 1$ and $\|\lambda_i\|_\infty < 1$ (by $\lambda\Lambda \subset \Lambda$: 1 expanding, $r - 1 + s$ contracting).

- Every prime ideal $\mathfrak{p}$ of the algebraic integers $\mathfrak{o}_{\mathbb{Q}(\lambda)}$ of $\mathbb{Q}(\lambda)$ yields a (complete) $\mathfrak{p}$-adic field $\mathbb{Q}_\mathfrak{p}$ (w.r.t. an ultrametric absolute value $\|\cdot\|_\mathfrak{p}$).

  One has: $\|\lambda\|_\mathfrak{p} \leq 1$ for all $\mathfrak{p}$, the inequality being strict only if $\mathfrak{p}$ lies above a prime divisor of $|\det \boldsymbol{M}|$
  (because $|\det \boldsymbol{M}| = N_{\mathbb{Q}(\lambda)/\mathbb{Q}}(\lambda) = \prod_i N_{\mathbb{Q}_{\mathfrak{p}_i}/\mathbb{Q}_p}(\lambda)$)

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ex ♣-substitution: $\det \boldsymbol{M} = -2$ $\quad p(x) = x^2 - 3x - 2$ $\quad \lambda = \frac{3+\sqrt{17}}{2}$
prime ideal factorization: $(2) = (\lambda) \cdot (3 - \lambda)$
$\mathfrak{p}$-adic absolute values: $\|\lambda\|_{(\lambda)} = \frac{1}{2}$ $\quad \|\lambda\|_{(3-\lambda)} = 1$.

Visualization of $\mathfrak{p}$-adic fields

Given $\mathbb{Q}_\mathfrak{p}$, the ring $\mathbb{Z}_\mathfrak{p} = \{x \in \mathbb{Q}_\mathfrak{p} \mid \|x\|_\mathfrak{p} \le 1\}$ is a discrete valuation ring with (unique) prime ideal $\mathfrak{m} = \{x \in \mathbb{Q}_\mathfrak{p} \mid \|x\|_\mathfrak{p} < 1\}$.

Every element π s.t. $\mathfrak{m} = \pi\mathbb{Z}_\mathfrak{p}$ is called a uniformizer.

Every element $x \in \mathbb{Q}_\mathfrak{p}$ can (uniquely) be written as

$$x = \sum_{k=m}^{\infty} d_k\, \pi^k \qquad \text{with } d_k \in D \text{ and } m \in \mathbb{Z},$$

where D is a system of representatives of the residue field $\mathbb{Z}_\mathfrak{p}/\mathfrak{m}$ (including 0).

ex ♣-substitution: We can take $\pi = \lambda$ but also $\pi = 2$, i.e., $\mathbb{Q}_{(\lambda)} \cong \mathbb{Q}_2$.
Take $\pi = 2$ and $D = \{0, 1\}$, then for $x \in \mathbb{Q}$ the above sum is just the well-known binary expansion.
For $x \in \mathbb{Q}_2$ the absolute value is given by $\|x\|_2 = 2^{-\min\{k \mid d_k = 1\}}$
e.g., $\lambda = 0 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2 + 0 \cdot 2^3 + 1 \cdot 2^4 + \ldots$
$\mathbb{Z}_2$ can be visualized as Cantor set (lines up with topology).

Cut and Project Scheme

- Direct space $G = \mathbb{R}$ (“expanding”)
- Internal space $H = \mathbb{H} = \mathbb{R}^{r-1} \times \mathbb{C}^s \times \mathbb{Q}_{\mathfrak{p}_1} \times \cdots \times \mathbb{Q}_{\mathfrak{p}_k}$ (“contracting”)
- Lattice $\tilde{L} \subset \mathbb{R} \times \mathbb{H}$ given by diagonal embedding of
  $$\bigcup_{m=0}^{\infty} \frac{1}{\lambda^m} < \ell_1, \ldots, \ell_n >_{\mathbb{Z}}$$
  (if $|\det \boldsymbol{M}| = 1$ reduces to $< \ell_1, \ldots, \ell_n >_{\mathbb{Z}}$, since then λ is a unit)
- Star map $^\star : \mathbb{Q}(\lambda) \to \mathbb{H}$, $x^\star = (\sigma_2(x), \ldots, \sigma_{r+s}(x), x, \ldots, x)$, where the σ_i's denote Galois automorphisms
- $1-1$ and denseness by number field construction

$$\begin{array}{ccccc} \mathbb{R} & \overset{\pi_1}{\longleftarrow} & \mathbb{R} \times \mathbb{H} & \overset{\pi_2}{\longrightarrow} & \mathbb{H} \\ \text{dense} \cup & & \cup & & \cup \text{ dense} \\ L & \overset{1-1}{\longleftrightarrow} & \tilde{L} & \overset{1-1}{\longleftrightarrow} & L^\star \end{array}$$

Window

Substitution σ induces a recursion equation for the point sets Λ_i (where Λ_i are the points of Λ associated to letter i)

$$\Lambda_i = \bigcup_{1 \le j \le n} \lambda \cdot \Lambda_j + A_{ij}$$

where A_{ij} is a set of translation vectors.

We use the star-map and look at this equation in $\mathcal{K}\mathbb{H}$, the set of non-empty compact subsets of $\mathbb{H}$, equipped with the Hausdorff metric d_H. $(\mathcal{K}\mathbb{H}, d_H)$ is a complete metric space, λ^* a contraction. By Banach's contraction principle, we get a unique solution $(\Omega_i)_{1 \le i \le n}$ of the equation (called a (graph-directed) iterated function system (IFS))

$$\Omega_i = \bigcup_{1 \le j \le n} \lambda^* \cdot \Omega_j + A^*_{ij}.$$

These are candidates for the windows, so $\Lambda_i \stackrel{?}{=} \Lambda(\Omega_i)$.

Properties of Ω_i

- By construction $\Lambda_i \subset \Lambda(\Omega_i)$ und $\Lambda \subset \Lambda(\Omega)$ (where $\Omega = \bigcup_i \Omega_i$).
- Recall the defnition of $\tilde{L}$ as diagonal embedding of

  $L = \bigcup_{m=0}^{\infty} \frac{1}{\lambda^m} < \ell_1, \ldots, \ell_n >_{\mathbb{Z}} \subset \ \{\sum_i q_i \cdot \ell_i | q_i \in \mathbb{Q}\} =: \mathcal{L}$.

  Define $\chi : \mathcal{L} \to \mathbb{Q}$, $x = \sum_i q_i \cdot \ell_i \mapsto \chi(x) = \sum_i q_i$.

  $M := \{x \in L \,|\, \chi(x) = 0\}$, then $M^\star$ is lattice in $\mathbb{H}$, $M^\star + \Omega$ is covering of $\mathbb{H}$.

Properties of Ω_i

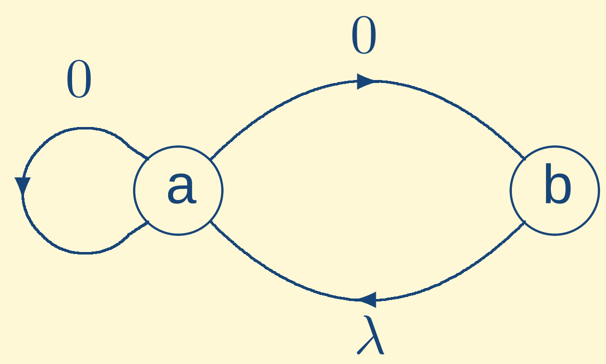
- By construction $\Lambda_i \subset \Lambda(\Omega_i)$ und $\Lambda \subset \Lambda(\Omega)$ (where $\Omega = \bigcup_i \Omega_i$).
- For $M := \{x \in L \mid \chi(x) = 0\}$, we have:
  $M^\star$ is lattice in $\mathbb{H}$, $M^\star + \Omega$ is covering of $\mathbb{H}$.
- Each Ω_i has non-empty interior and is the closure of its interior.
- The Ω_i's are the windows **iff** $M^\star + \bigcup_i \Omega_i$ is a tiling.
  (modulo a set of measure zero)
- Unions on the right side of the IFS $\Omega_i = \bigcup_{1 \leq j \leq n} \lambda^* \cdot \Omega_j + A^*_{ij}$ are disjoint in (Haar) measure.
  (But what about $\bigcup_i \Omega_i$ and $M^* + \Omega$?)
- Boundaries $\partial\Omega_i$ have Haar measure 0.

Examples

ⓔⓧ Fibonacci-substitution: $a \stackrel{\sigma}{\mapsto} ab, \quad b \stackrel{\sigma}{\mapsto} a \qquad \boldsymbol{\ell} = (\lambda, 1)$

$$\begin{aligned} \Lambda_a &= \lambda \cdot \Lambda_a \quad \cup \quad \lambda \cdot \Lambda_b \\ \Lambda_b &= \lambda \cdot \Lambda_a + \lambda \end{aligned}$$

$$\begin{aligned} \Omega_a &= \lambda^\star \cdot \Omega_a \quad \cup \quad \lambda^\star \cdot \Omega_b \\ \Omega_b &= \lambda^\star \cdot \Omega_a + \lambda^\star \end{aligned}$$

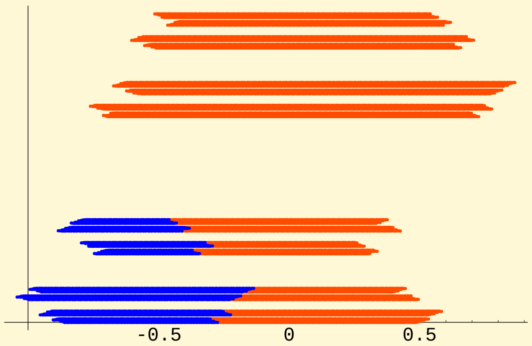


Solution: $\Omega_a = [\lambda - 2, \lambda - 1] \quad \Omega_b = [-1, \lambda - 2]$

Remark: $\Omega = [-1, \lambda - 1] \qquad M^\star = \lambda\mathbb{Z} \quad \Longrightarrow$ Fibonacci is model set

Examples

ⓔⓧ ♣-substitution: $a \overset{\sigma}{\mapsto} aaba, \quad b \overset{\sigma}{\mapsto} aa \qquad \boldsymbol{\ell} = (\frac{\lambda}{2}, 1)$



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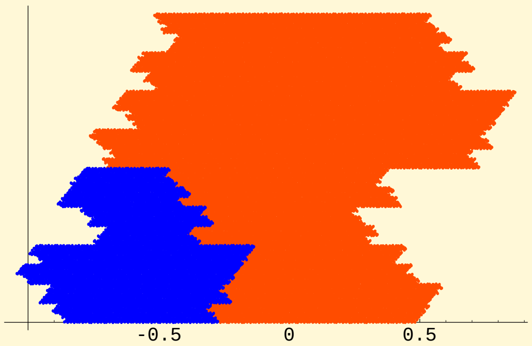
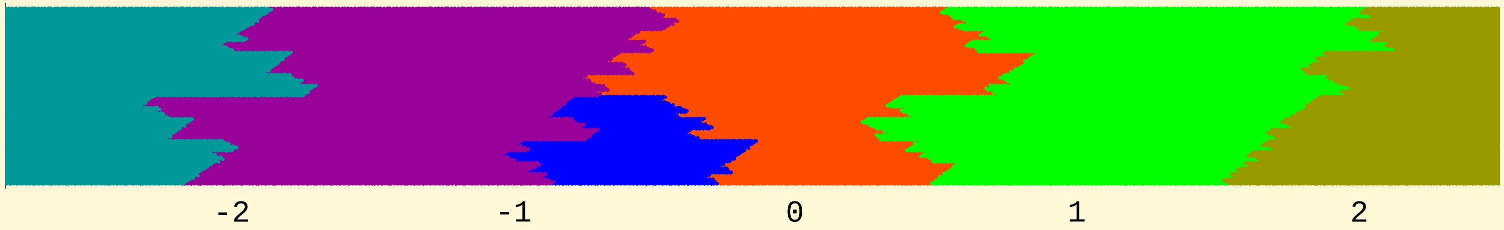


And $\Omega + M^{\star}$:



Dual Substitution

Instead of IFS $\Omega_i = \bigcup_{1 \le j \le n} \lambda^* \cdot \Omega_j + A^*_{ij}$

look at point set equation $X_i = \bigcup_{1 \le j \le n} (\lambda^{-1})^* \cdot X_j + (\lambda^{-1})^* \cdot A^*_{ji}$.

X_i's are model sets in the cut and project scheme

$$\begin{array}{ccccc} \mathbb{H} & \xleftarrow{\pi_1} & \mathbb{H} \times \mathbb{R} & \xrightarrow{\pi_2} & \mathbb{R} \\ \text{dense} \cup & & \cup & & \cup \text{ dense} \\ L^\star & \xleftrightarrow{1-1} & \tilde{L} & \xleftrightarrow{1-1} & L \end{array}$$

We get: $X_i = \Lambda([0, \ell_i[)$.

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look at point set equation $X_i = \bigcup_{1 \le j \le n} (\lambda^{-1})^* \cdot X_j + (\lambda^{-1})^* \cdot A^*_{ji}$.

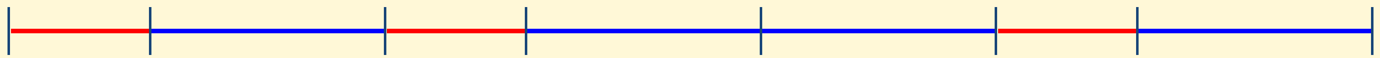
Define $J = \bigcup_i X_i + \Omega_i$, then

- the covering degree of J is constant a.e.
- J is tiling **iff** $M^\star + \Omega$ is tiling.

ex Fibonacci-substitution:

$$\begin{aligned} \Omega_a &= \lambda^\star \cdot \Omega_a \cup \lambda^\star \cdot \Omega_b \qquad & X_a &= \left(\tfrac{1}{\lambda}\right)^\star \cdot X_a \cup \left(\tfrac{1}{\lambda}\right)^\star \cdot X_b + 1 \\ \Omega_b &= \lambda^\star \cdot \Omega_a + \lambda^\star \qquad & X_b &= \left(\tfrac{1}{\lambda}\right)^\star \cdot X_a \end{aligned}$$

Get: $X_a = \{\ldots, -\lambda, 0, 1, \lambda + 1, \ldots\}$ $X_b = \{\ldots - \lambda, 0, \lambda + 1, \ldots\}$.



The Boundary

Knowledge of J also yields an IFS for the boundary $\bigcup_i \partial\Omega_i$

We have $\partial\Omega_i = \bigcup_{(i,j,x)} \Xi_{(i,j,x)}$ where $\Xi_{(i,j,x)} = \Omega_i \cap (\Omega_j + x)$ with $x \in X_j - X_i$.

ⓔⓧ Fibonacci: ($\lambda^\star = -\frac{1}{\lambda}$)

$$\begin{aligned}
\Xi_{(b,a,-\lambda)} &= -\tfrac{1}{\lambda}\, \Xi_{(a,a,1)} \\
\Xi_{(a,a,1)} &= -\tfrac{1}{\lambda}\, \Xi_{(b,a,-\lambda)} \\
\Xi_{(b,a,0)} &= -\tfrac{1}{\lambda}\, \Xi_{(a,a,-1)} \\
\Xi_{(a,a,-1)} &= -\tfrac{1}{\lambda}\, \Xi_{(b,a,0)}
\end{aligned}$$

Get: $\partial\Omega_a = \Xi_{(a,a,-1)} \cup \Xi_{(a,a,1)} = \{\lambda - 2, \lambda - 1\}$
$\partial\Omega_b = \Xi_{(b,a,-\lambda)} \cup \Xi_{(b,a,0)} = \{-1, \lambda - 2\}$.

Boundaries coincide at $\lambda - 2$ and line up with $M^\star = \lambda\mathbb{Z}$

$\Longrightarrow$ Fibonacci is model set

The Boundary

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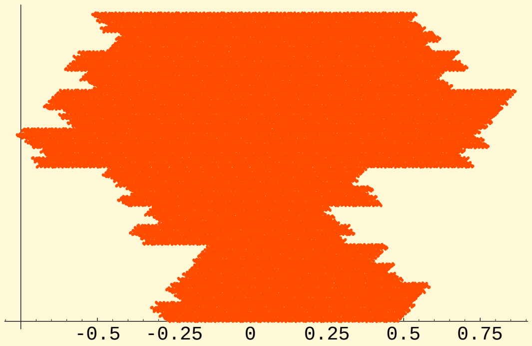
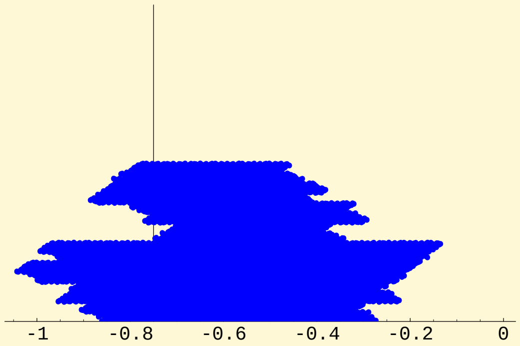
We have $\partial\Omega_i = \bigcup_{(i,j,x)} \Xi_{(i,j,x)}$ where $\Xi_{(i,j,x)} = \Omega_i \cap (\Omega_j + x)$ with $x \in X_j - X_i$.

ⓔⓧ ♣-substitution:

Do the boundaries match?





$\rightsquigarrow$ still hard to decide

SCC

Idea: Use measure-disjoint unions of IFS to decide if $\bigcup_i \Omega_i$ is disjoint in measure.

Strong coincidence condition: [Host, Arnoux-Ito]
A Pisot substitution satisfies the SCC if for every pair of vertices i, j ($i \neq j$) in the (directed) substitution graph, there exists a vertex k and walks w and $\tilde{w}$ of the same length N such that

- w ($\tilde{w}$) starts at k and ends at i (j).
- $\delta(w) = \delta(\tilde{w})$, where $\delta(e_1 \ldots e_N) = e_1 + \ldots + \lambda^{N-1} \cdot e_N$.

Remark: In this case a small copy of $\Omega_i \cup \Omega_j$ appears inside Ω_k.

SCC $\Longleftrightarrow$ the union $\bigcup_i \Omega_i$ is disjoint in measure.

[Barge-Diamond] The above condition holds for at least one pair i, j.

$\Longrightarrow$ For Pisot substitutions over 2 letters: Ω_a, Ω_b are disjoint in measure.

GCC/SuperCC

Idea: Use measure-disjoint unions of IFS to decide if J is tiling.

Geometric coincidence condition [Barge-Kwapisz]
Super coincidence condition [Ito-Rao]
A Pisot substitution satisfies the GCC/SuperCC if for every pair of vertices i, j and every walk w' ending at i and every walk $\hat{w}$ ending at j in the (directed) substitution graph, there exists a vertex k and walks w and $\tilde{w}$ of the same length N such that

- w ($\tilde{w}$) starts at k and ends in w' ($\hat{w}$)
- $\delta(w) - \lambda^N \Delta(w') = \delta(\tilde{w}) - \lambda^N \Delta(\hat{w})$, where $\Delta(e_1 \ldots e_M) = \left(\frac{1}{\lambda}\right)^M e_1 + \ldots + \left(\frac{1}{\lambda}\right) e_M$.

Remark: In this case a small copy of $(\Omega_i + \Delta(w')^\star) \cup (\Omega_j + \Delta(\hat{w})^\star)$ appears inside Ω_k
IFS for boundary $\rightsquigarrow$ some triples $(i, j, \Delta(\hat{w})^\star - \Delta(w')^\star)$

GCC/SuperCC $\Longleftrightarrow$ J is tiling $\Longleftrightarrow$ Λ is model set.

Remarks: It suffices to check finitely many triples $(i, j, \hat{w})$ (compact Ω_i's).
$\Longrightarrow$ ♣ is model set

The Torus

$\Upsilon \in \overline{\Lambda + \mathbb{R}}$ can also be described by the orbit of Υ under σ.

$\Longrightarrow$ Two-sided infinite walk $\hat{w}.\tilde{w}$ in the substitution graph (sofic shift).

Define: $\beta(\Upsilon) = (\Delta(\hat{w}), (\delta(\tilde{w})^\star) \in \mathbb{R} \times \mathbb{H}$.

Let $F_i := \{\hat{w}.\tilde{w} \mid \hat{w}(\tilde{w})$ ends (starts) at $i\}$.

Then $\beta(F_i) = [0, \ell_i] \times \Omega_i =: \tilde{\Omega}_i$.

$\tilde{L} + \bigcup_i \tilde{\Omega}_i$ is tiling of $\mathbb{R} \times \mathbb{H} \Longleftrightarrow J$ is tiling $\Longleftrightarrow \Lambda$ is model set.

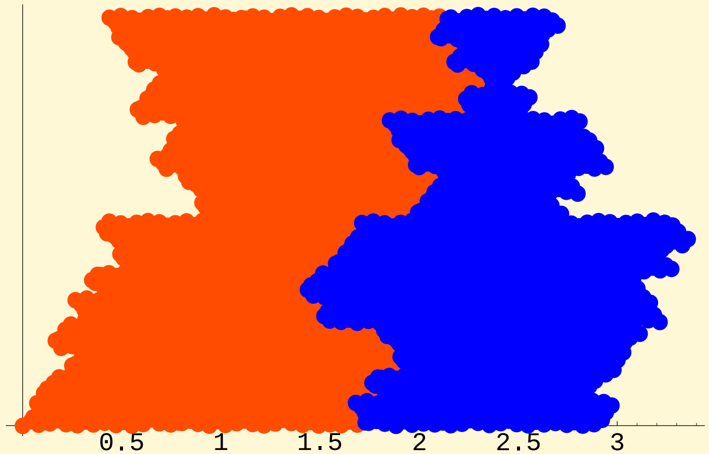
In this case: $\bigcup_i \tilde{\Omega}_i$ is torus (Markov partition),

β is $1-1$ a.e. (torus parametrization).

Conclusion

- All checked Pisot substitutions so far are model sets!
- We have explained the equivalent formulations of the conjecture. ($M^\star, J$, GCC, etc.)
- Everything also holds in the non-unimodular case.
- There are really limit-quasiperiodic model sets, i.e., model sets with mixed Euclidean and $\mathfrak{p}$-adic internal space.

$$a \stackrel{\sigma}{\mapsto} aab \qquad b \stackrel{\sigma}{\mapsto} abab$$



Connections to...

- Primitive constant-length substitutions [Dekking] /
  lattice substitution systems [Lee-Moody-Solomyak]

  Dekking-coincidence / modular coincidence ensure
  that $\bigcup_i \Omega_i$ is disjoint in measure.
  (no $M^\star$, only compact internal space!)

- Digit tiles [Vince]

  Equivalent tiling conditions there are “the same” as here.

- 2-dimensional tilings

  ex Watanabe-tilings: expect internal space $\mathbb{C} \times \mathbb{Q}_2(1+\xi_8)$.