The 3x+1 problem

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3n + 1 dynamical system

3n+1 function

$$T : \mathbb{N} \to \mathbb{N}, \qquad T(n) = \begin{cases} T_0(n) = \frac{n}{2} & \text{for } n \text{ even,} \\ T_1(n) = \frac{3n+1}{2} & \text{for } n \text{ odd.} \end{cases}$$

Collatz graph

$\Gamma_T = (V, E)$ directed graph

vertices $\quad V = \mathbb{N} = \{1, 2, \dots\}$

edges $\quad E = \{(n, T(n)) : n \in \mathbb{N}\}$

3n+1 trajectory

with starting number n:

$$\mathcal{T}_n := \left(n, T(n), T^2(n), T^3(n), \dots\right)$$

future infinite path in the Collatz graph Γ_T

3n + 1 conjecture

Any 3n+1 trajectory ends in the cycle $(1, 2)$.

If an easy proof were known, nobody would care about this. But not even a difficult proof is known.

曲阜师范大学学报自然科学版一九八六年第三期
Natural Sciences Edition
Journal of Qufu Normal University(3), 1986

关于（3n+1）一问题的起因

L. Collatz

（汉堡大学应用数学研究所）

初等数论与初等图论之间，有着许多联系。其中有一种联系是：使数论函数 $f(n)$ 与向图相结合。取整数 $n=1,2,3,\cdots$ 作为图的顶点，从 n 到 $f(n)$ 画一个箭头（或者在 n 下面写上 $f(n)$，或者与箭头一道写在下面）。1928年—1933年，在我学习期间，由于受到Edmund Landau, Oscar Perron, Issai schur a.o. 讲课的启发，使得自己在数论函数、图论及其它课题方面产生了兴趣。画了关于数论函数的图，作了如下分类：

Ⅰ．单值函数 $f(n)$

a)线

图是无穷线，例如 $f(n)=n+1$。

b)树

例如 $f(n)=n-g(n)$，这里 $g(n)$ 是除去 n 本身的关于 n 的最大因数。例如 $f(21)=21-7=14$。若 p 是素数，$f(p)=p-1$。图1表示了树的一部分，它包含从1—30的整数。

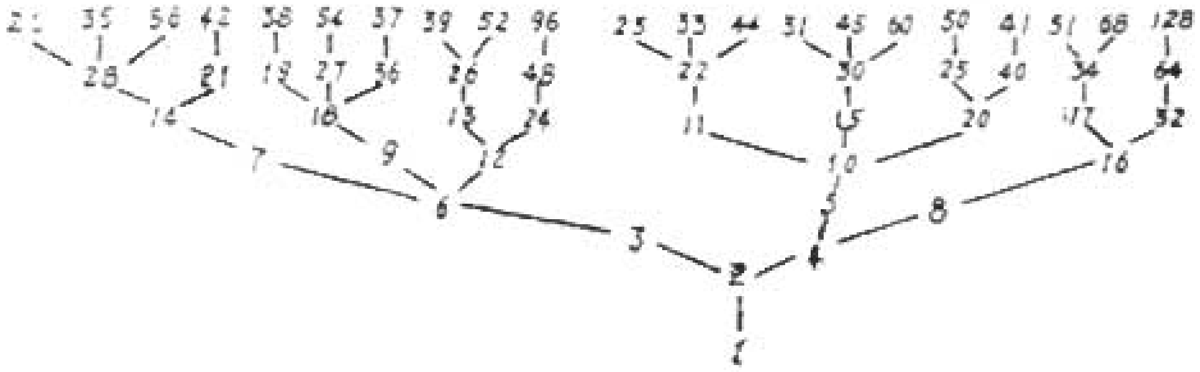


图 1

c) 森林

图由 i 棵互不连通的树构成 $(2\leq i<\infty)$，例如 $f(n)=n+n^2$。

d) 定点

图至少包含一个定点，但不包含圈。假设 $f(a)=a$，则把数 a 称为定点。含有两个或更多定点的图是不连通的。

对于定点举例如下：

3n+1 - Funktion:

$$T(n) := \begin{cases} T_0(n) = \frac{n}{2} & \text{falls } n \equiv 0 \mod 2 \\ T_1(n) = \frac{3n+1}{2} & \text{falls } n \equiv 1 \mod 2 \end{cases}$$

3n+1 - Trajektorie:

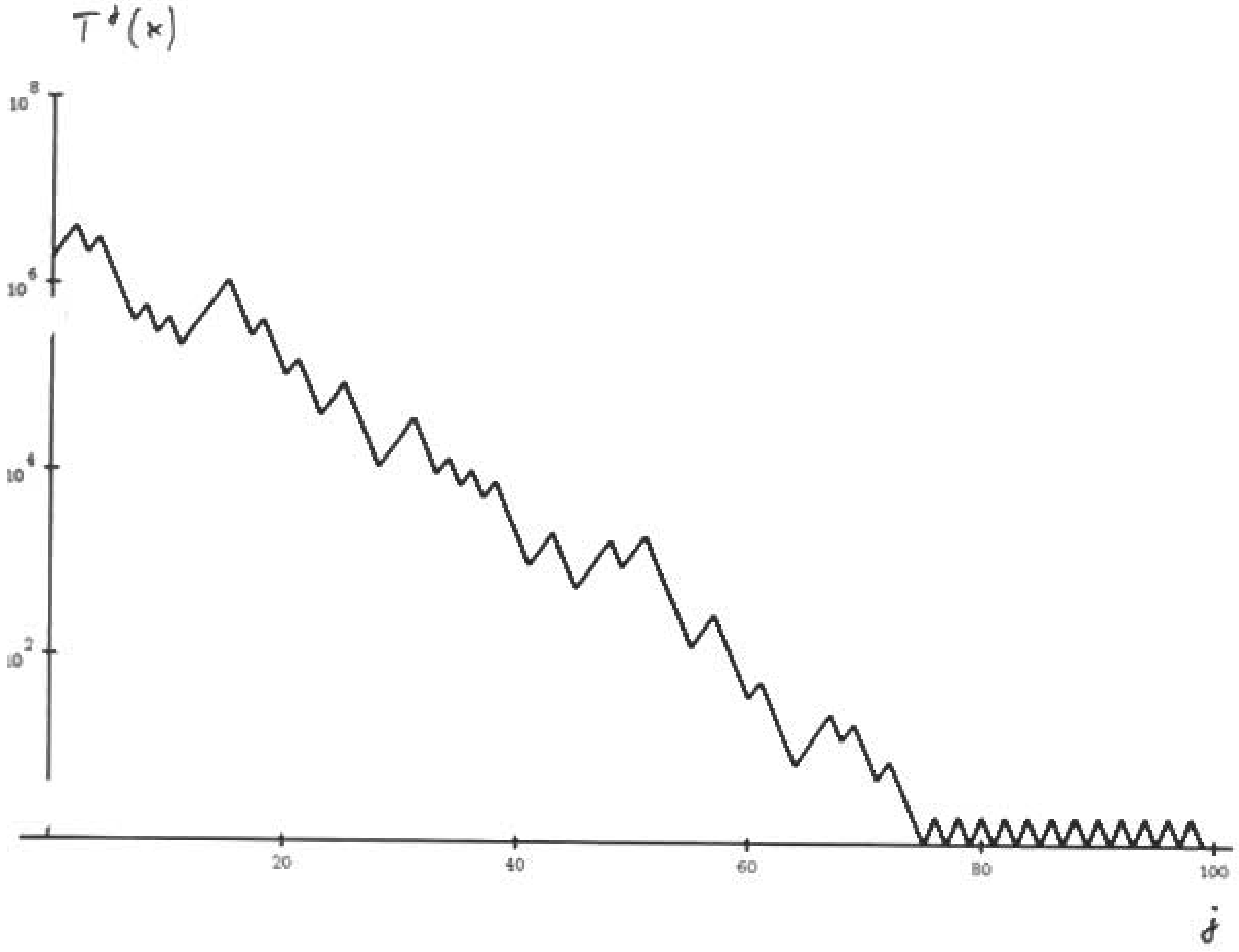
$$\mathcal{T}(n) = \left(T^k(n) : k = 0, 1, \ldots \right)$$
$$= \left(n, T(n), T^2(n), \ldots \right)$$

Beispiele:

$$\mathcal{T}(1) = (1, 2, 1, 2, 1, 2, \ldots)$$

$$\mathcal{T}(100) = (100, 50, 25, 38, 19, 29, 44, 22, 11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1, \ldots)$$

$$\mathcal{T}(27) = (27, 41, 62, 31, 47, 71, 107, 161, 242, 121, 182, 91, 137, 206, 103, 155, 233, 350, 175, 263, 395, 593, 890, 445, 668, 334, 167, 251, 377, 566, 283, 425, 638, 319, 479, 719, 1079, 1619, 2429, 3644, 1822, 911, 1367, 2051, 3077, 4616, 2308, 1154, 577, 866, 433, 650, 325, 488, 244, 122, 61, 92, 46, 23, 35, 53, 80, 40, 20, 10, 5, 8, 4, 2, 1, \ldots)$$



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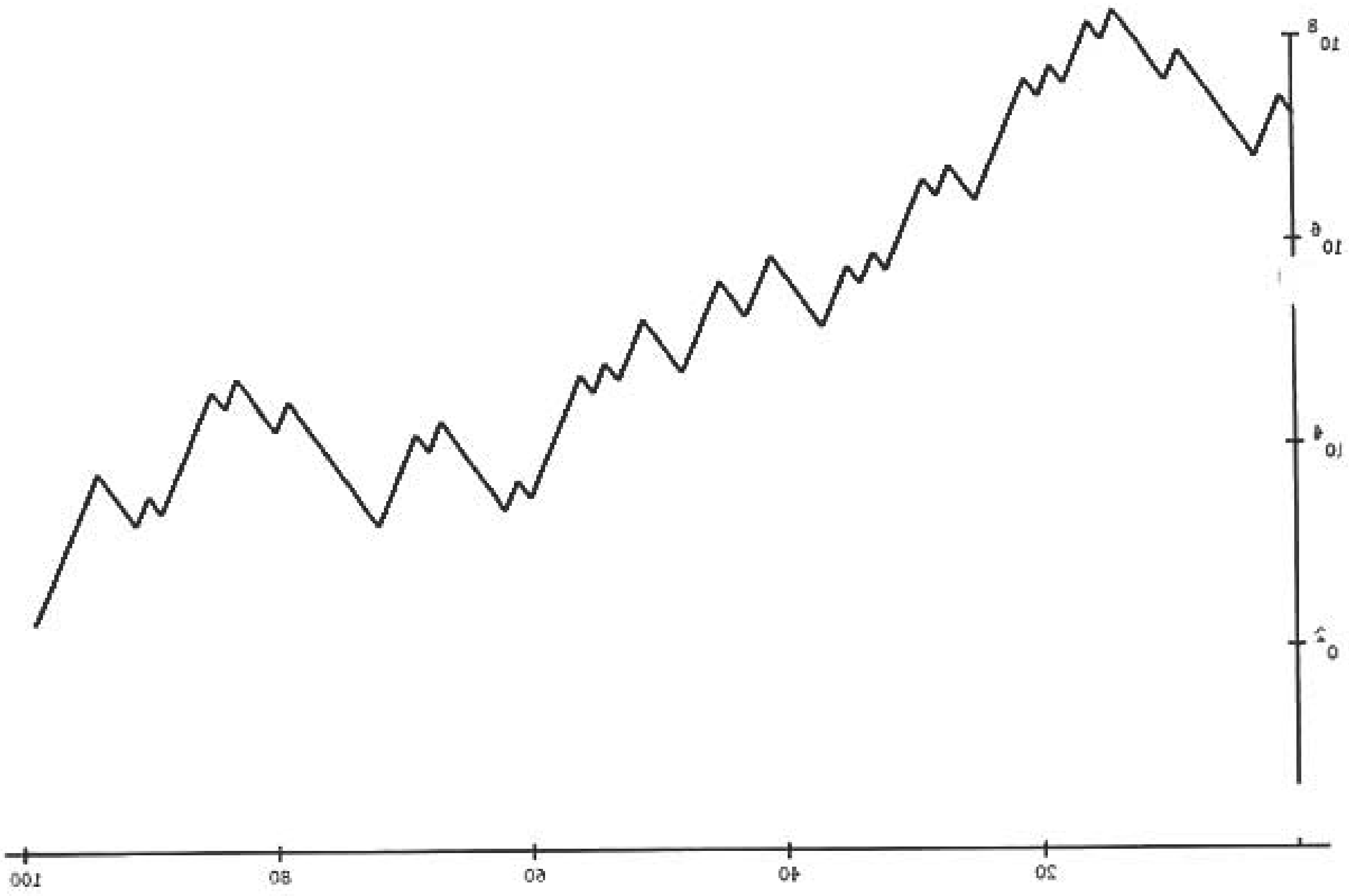

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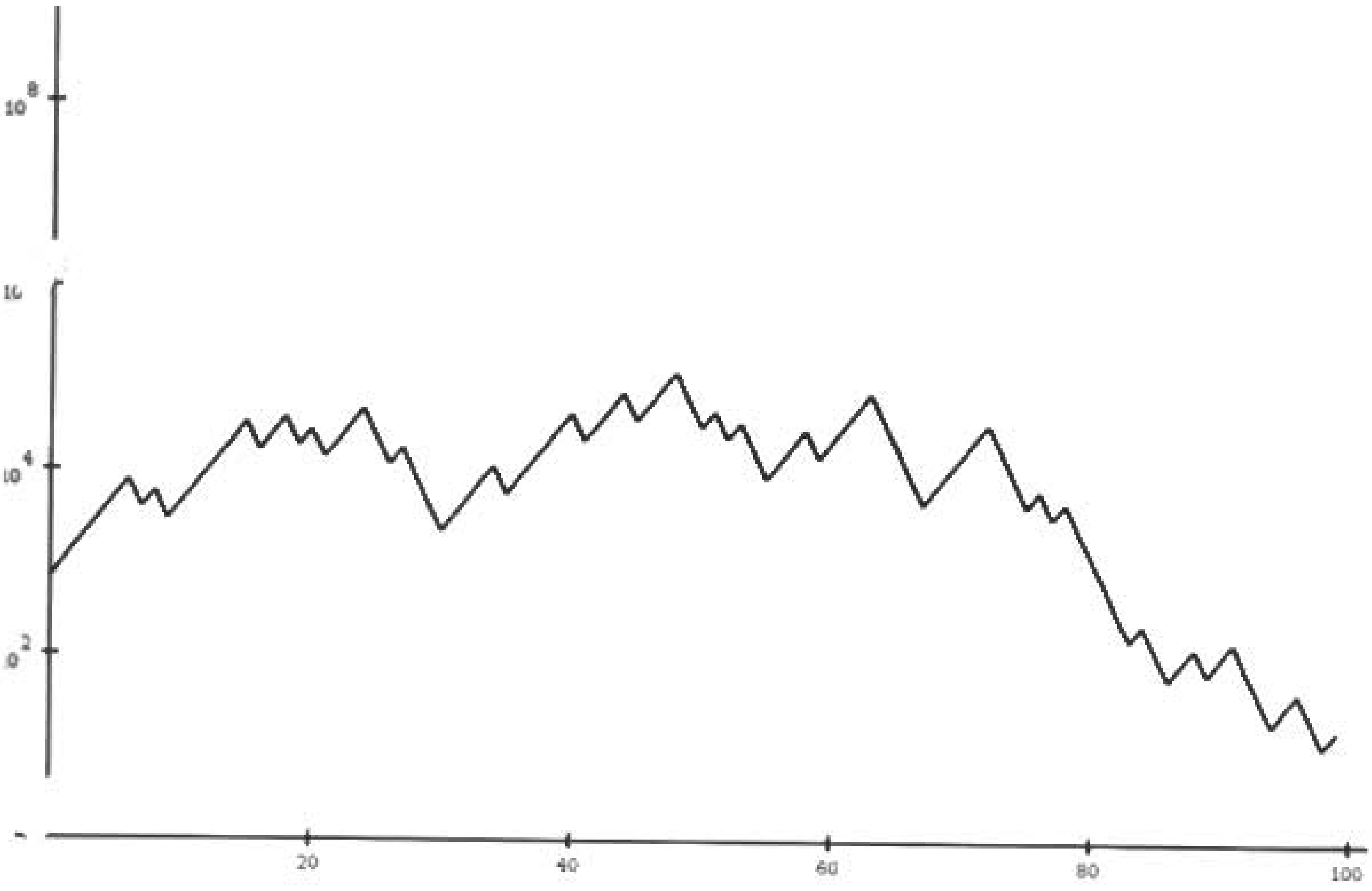

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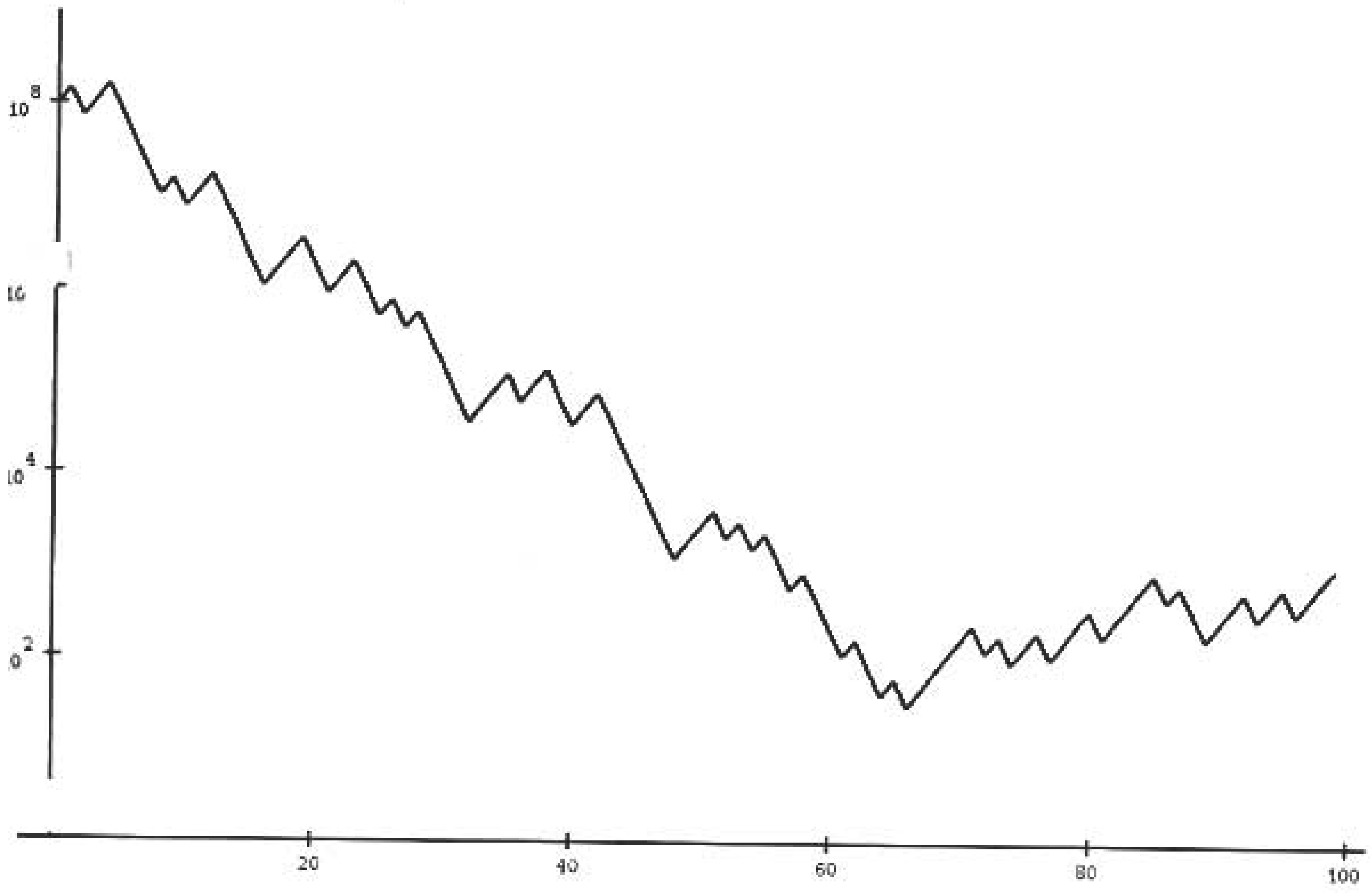
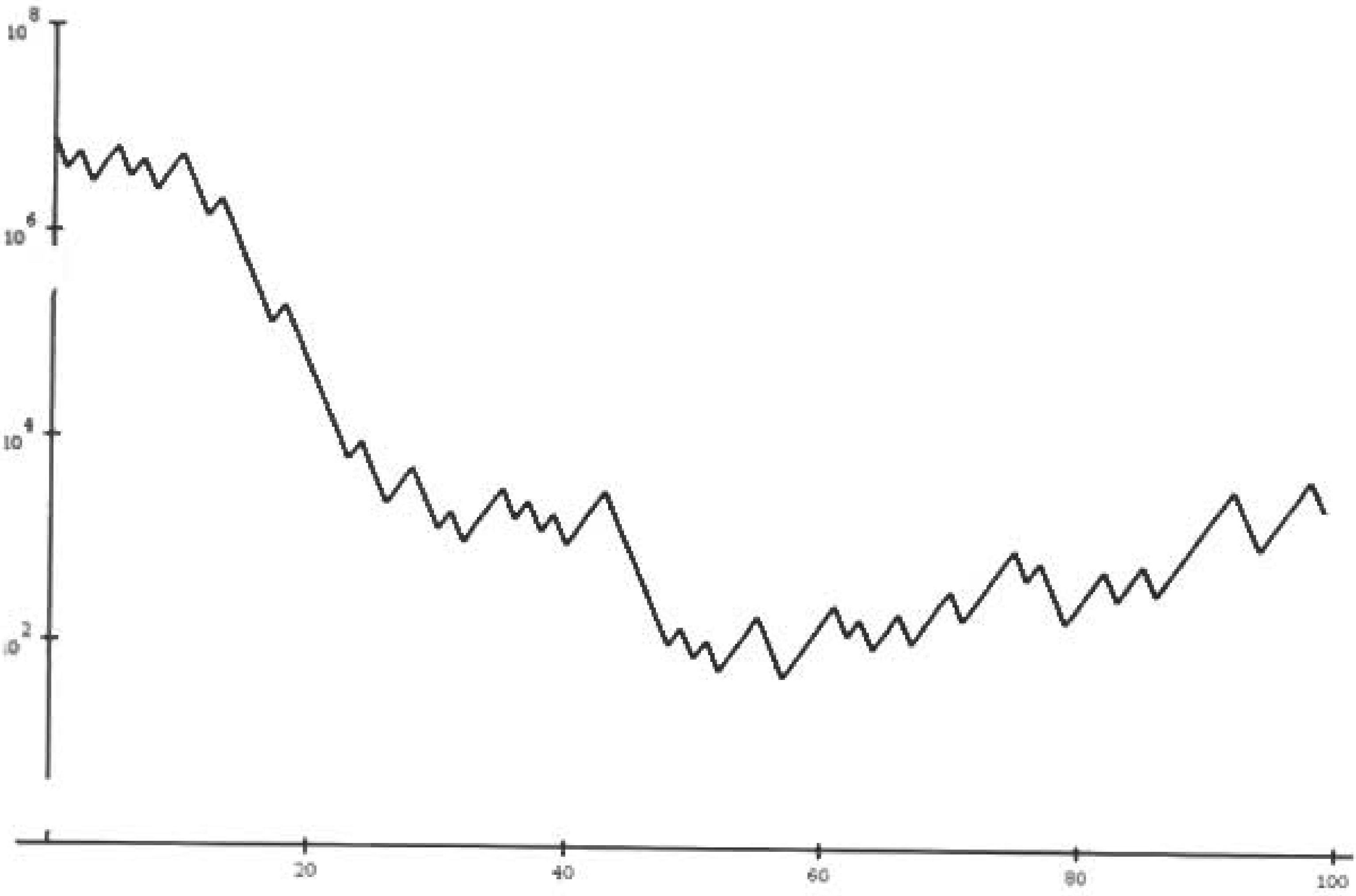


1



10^8
10^6
10^4
10^2
20
40
60
80
100

The stopping time

Define
$$\sigma(n) := \inf \{ k \in \mathbb{N} : T^k(n) < n \}$$
the stopping time of n (with $\inf \emptyset = \infty$).

Examples $\sigma(2n) = 1$,

$\sigma(1) = \infty$

Put $\theta := \frac{\log 2}{\log 3} \approx 0.63$

Theorem (Lagarias 1985): There is $c_1 > 0$ such that
$$|\{ n \in \mathbb{N} : n \le x, \sigma(n) = \infty \}| \le c_1 \, x^{H(\theta)}$$
with Shannon's function $H(\theta) = \theta \log \frac{1}{\theta} + (1-\theta) \log \frac{1}{1-\theta} \approx 0.95$

Theorem (Korec 1994): For every real $c > \frac{1}{2\theta}$, the set
$$M_c = \{ y \in \mathbb{N} : \text{there is } n \in \mathbb{N} \text{ with } T^n(y) < y^c \}$$
has asymptotic density one.

The parity sequence

Definition.

Given a trajectory $(n, T(n), T^2(n), \ldots)$

set $v_j \equiv T^j(n) \mod 2$, $v_j \in \{0,1\}$.

Then $(v_j)_{j\geq 0}$ is called parity sequence of n.

Put $Q_\infty(n) := (v_0, v_1, v_2, \ldots) \in \mathbb{Z}_2$ (Lagarias 1985)

As the 3n+1 function T naturally extends to 2-adic integers

$$T : \mathbb{Z}_2 \to \mathbb{Z}_2,$$

we have $Q_\infty : \mathbb{Z}_2 \to \mathbb{Z}_2$.

Theorem (Terras 1976, Lagarias 1985)

Q_∞ is continuous (w.r.t. 2-adic topology), bijective, and measure-preserving on $\mathbb{Z}_2$.

Theorem (Müller 1991, 1994)

Q_∞ is nowhere differentiable.

The $3x+1$ conjugacy map

Denote by $S : \mathbb{Z}_2 \to \mathbb{Z}_2$ the 2-adic shift map

$$(a_0, a_1, a_2, \dots) \mapsto (a_1, a_2, a_3, \dots)$$

Theorem (Bernstein, Lagarias 1996)

The diagram

$$\begin{array}{ccc} \mathbb{Z}_2 & \xrightarrow{T} & \mathbb{Z}_2 \\ \downarrow Q_\infty & & \downarrow Q_\infty \\ \mathbb{Z}_2 & \xrightarrow{S} & \mathbb{Z}_2 \end{array}$$

commutes

The inverse $\Phi := Q_\infty^{-1}$ is called 3x+1 conjugacy map. It is solenoidal (as a consequence of a result by Terras).

Theorem (Bernstein 1994) TFAE:

(a) The $3n+1$ conjecture is true

(b) $Q_\infty(\mathbb{N}) \subset \frac{1}{3}\mathbb{Z}$

(c) $\mathbb{N} \subset \Phi(\frac{1}{3}\mathbb{Z})$.

Periodicity conjecture (Lagarias 1985)

$$\Phi(\mathbb{Q} \cap \mathbb{Z}_2) = \mathbb{Q} \cap \mathbb{Z}_2$$

The Periodicity conjecture would imply that there is no divergent $3n+1$ trajectory.

Complex analysis

Theorem (Berg, Meinardus 1994, 1995) TFAE:

(a) The $3n+1$ conjecture is true.

(b) For each $n \in \mathbb{N}$,
$$g_n(w) := \sum_{j=0}^{\infty} T^j(n)\, w^j$$
is a rational function $g_n(w) = \dfrac{q_n(w)}{1-w^2}$,
where $q_n \in \mathbb{Z}[w]$.

(c) Let ζ denote a third root of unity, and consider the functional equation
$$\circledast \quad h(z^3) = h(z^6) + \frac{1}{3z}\left(h(z^2) + \zeta h(\zeta z) + \zeta^2 h(\zeta^2 z)\right).$$

Then the only solutions of $\circledast$ which are holomorphic in the complex unit disc have the form $h(z) = h_0 + \dfrac{h_1 z}{1-z^2}$ with $h_0, h_1 \in \mathbb{C}$.

Transcendence Theory

Definition:
A 3n+1 cycle with precisely m local minima is called m-cycle

Theorem (Steiner 1978)
The cycle $(1,2)$ is the only 1-cycle.

Theorem (Simons, de Weger 2004)
There is no m-cycle for $1 \leq m \leq 68$.

Method of proof:
Suppose there is an m-cycle. Denote by ℓ the number of odd members and by k the number of even members, then $T^{k+\ell}(x) = x$ for each x in the m-cycle.

Put
$$\Lambda := (k+\ell)\log 2 \overset{-}{\cancel{+}} \ell \log 3 .$$

(Motivation:
$$T^{k+\ell}(x) \approx \left(\frac{3}{2}\right)^{\ell}\left(\frac{1}{2}\right)^{k} x = \exp(-\Lambda)\cdot x$$
Consequently, $x = T^{k+\ell}(x)$ implies $\Lambda \approx 0$)

Transcendence Theory continued

Squeeze Λ between an upper and a lower bound.
If the upper bound turns out to be smaller than the lower bound, no such m-cycle is possible.

Upper bounds:

In fact, $T^{k+\ell}(x) = e^{-\Lambda} x + \text{remainder}$

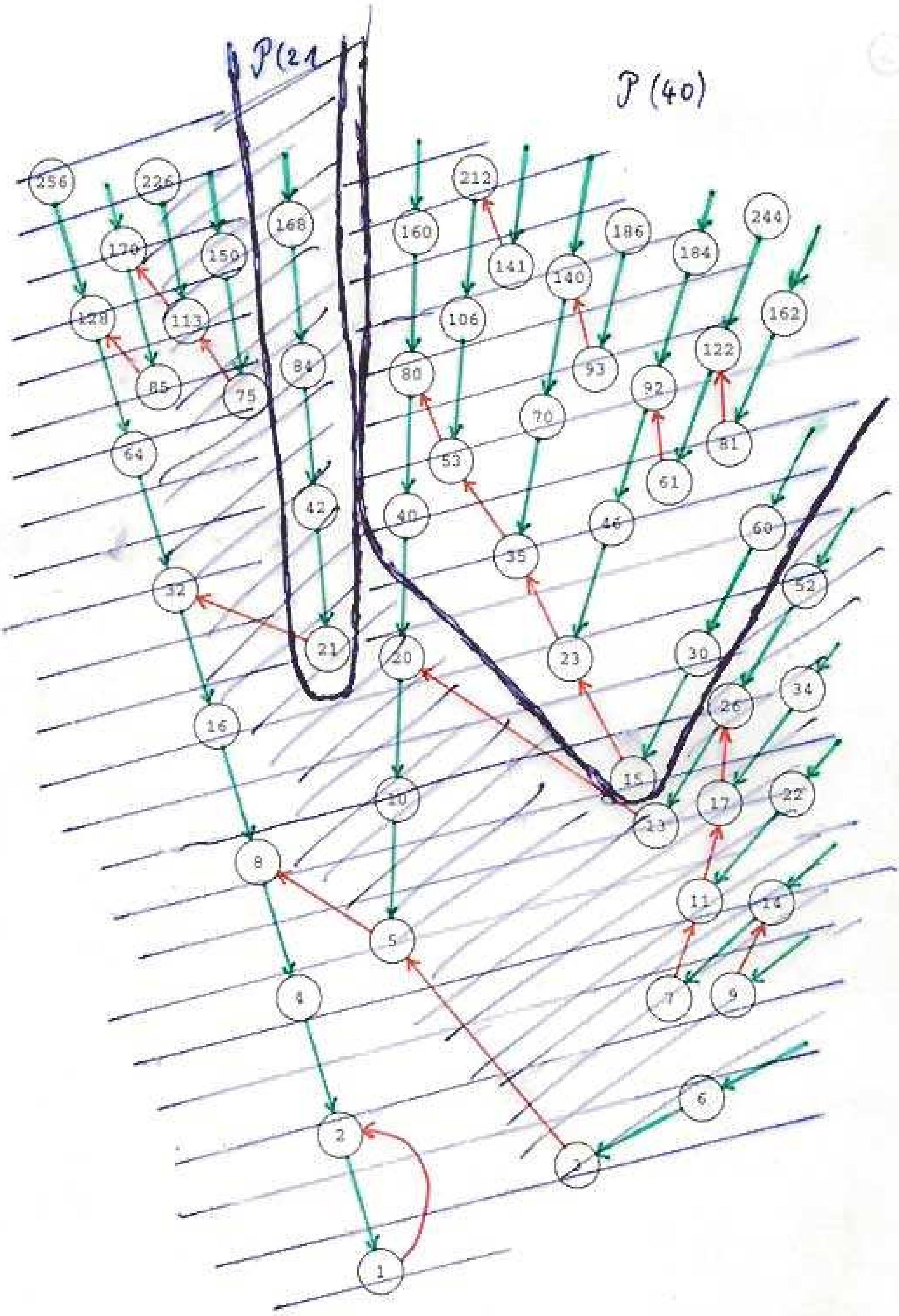
estimating this leads to upper bounds

Lower bounds:

$$\Lambda = (k+\ell)\log 2 + \ell \log 3$$

is a linear form in logarithms.

There is a deep method in number theory founded by A. Baker and developed by several people using that $\log 2$ and $\log 3$ are transcendental and giving lower bounds for Λ.


P(21
P(40)
256 226 168 160 212 141 186 184 244
170 150 140 162
128 113 106 122
85 75 84 80 93 92
53 70 81 61
64 42 40 46 60
35
32 52
21 20 23 30
34 26
16 15
10 13 17 22
8 11 14
5
4 7 9
6
2 3
1

Predecessors

For $a \in \mathbb{N}$ put

$$\mathcal{P}(a) := \{ n \in \mathbb{N} : \text{there is } j \geq 0 \text{ such that } T^j(n) = a \}$$

and call it the 3n+1 predecessor set of a.

Consider the predecessor counting function

$$\vartheta_a(x) := | \{ n \in \mathcal{P}(a) : n \leq x \} | .$$

Known Facts

$$a \equiv 0 \mod 3 \quad \Rightarrow \quad \vartheta_a(x) = 1 + \left\lfloor \frac{\log x - \log a}{\log 2} \right\rfloor$$

Estimates of the form $\vartheta_1(x) \geq x^c$ for large x:

$c = 0.057$ (Crandall 1978)

$c = \frac{1}{4}$ (Sander 1987, 1990)

$c = \frac{3}{7}$ (Krasikov 1989)

$c = 0.48$ (W. 1993, 1995)

Using heavy computer power, better estimates are possible:

$$\left.\begin{array}{l} c = 0.654 \\ c = 0.804 \end{array}\right\} \text{(Applegate, Lagarias 1995, 1997)}$$

Positive predecessor density conjecture

For each $a \not\equiv 0 \mod 3$, there is some $\delta > 0$ such that $\vartheta_a(x) \geq \delta x$ for large x.

Elka functions

Definition

For any integer $\ell \geq 0$ put

$$e_\ell : \mathbb{N}_0 \times \mathbb{N} \longrightarrow \mathbb{N}_0$$

$$e_\ell(k,a) := \left| \left\{ \begin{array}{l} \text{paths } v_0 \xrightarrow{T} \dots \xrightarrow{T} v_{k+\ell} \text{ in Collatz graph} \\ k \text{ steps } T_0,\ \ell \text{ steps } T_1,\ v_{k+\ell} = a \end{array} \right\} \right|$$

Theorem.

(a) $e_0(k,a) = 1$ for each $k \in \mathbb{N}_0$, $a \in \mathbb{N}$

(b) $e_\ell(k,a) = 0$ whenever $\ell \geq 1$ and $a \equiv 0 \bmod 3$

(c) $e_\ell(k,a) = e_\ell(k,b)$ if $a \equiv b \bmod 3^\ell$

Consequence

A natural domain of definition is obtained by taking 3-adic integers in the second variable:

$$e_\ell : \mathbb{N}_0 \times \mathbb{Z}_3 \longrightarrow \mathbb{N}_0 .$$

Then $\operatorname{supp}(e_\ell) \subset \mathbb{N}_0 \times \mathbb{Z}_3^*$ for $\ell \geq 1$.

Estimating predecessor densities

Notation.

$$B_a(\ell,x) := e_\ell\left(\lfloor \mathrm{lb}(x) + \ell\cdot \mathrm{lb}(\tfrac{3}{2}) \rfloor, a\right)$$

$$\leq \left|\left\{ \text{paths } b \to \ldots \to a : \ell \text{ steps } T_1, \ \tfrac{x}{2a} \leq b \leq \tfrac{x}{a} \right\}\right|$$

Estimating series

$$s_x : \mathbb{Z}_3^* \to \mathbb{N}_0 \cup \{+\infty\}, \qquad s_x(a) := \sum_{\ell=1}^{\infty} B_a(\ell,x)$$

Theorem.

(a) $\quad \vartheta_a(x) \geq s_x(a) \quad$ for $a \in \mathbb{N}$

(b) There is a dense set $E \subset \mathbb{Z}_3^*$ with

$\quad s_x(a) = \infty$ for $a \in E$

(c) s_x is integrable w.r.t. Haar measure on $\mathbb{Z}_3^*$

Proof of (c):

Put $B(\ell,x) := \int_{\mathbb{Z}_3^*} B_a(\ell,x)\, da = \dfrac{1}{2\cdot 3^{\ell-1}} \dbinom{\lfloor x + \ell\cdot \mathrm{lb}3 \rfloor}{\ell}$,

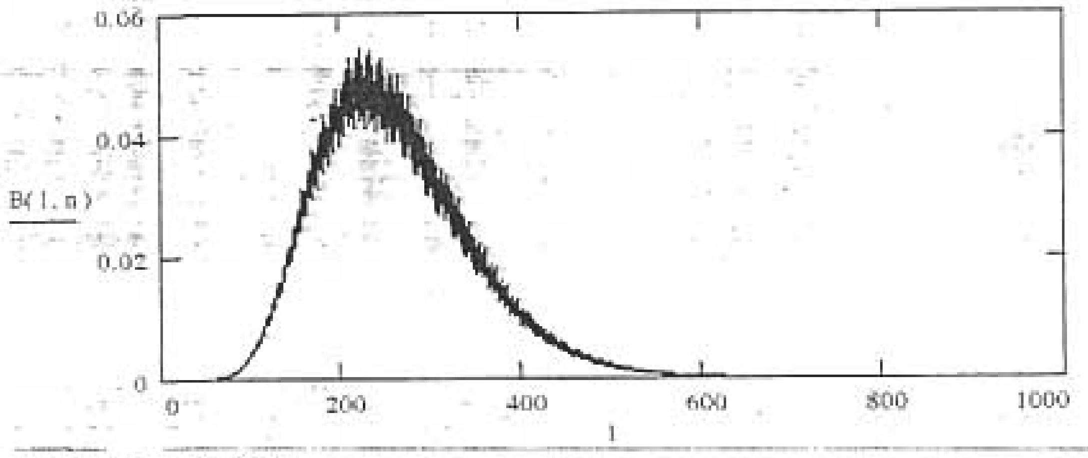
and apply Fatou's Lemma,

Stirling's Formula,

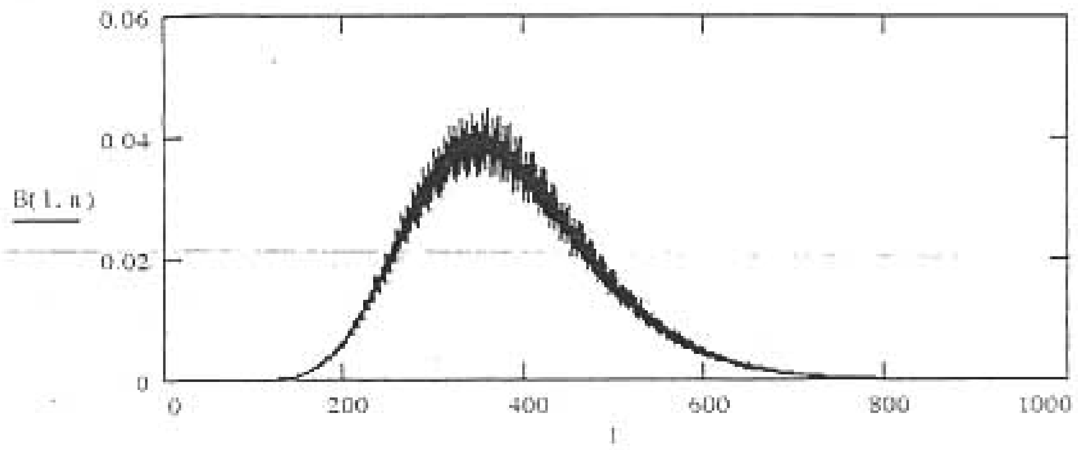
and the Root Criterion for convergence. ∎

p = 3

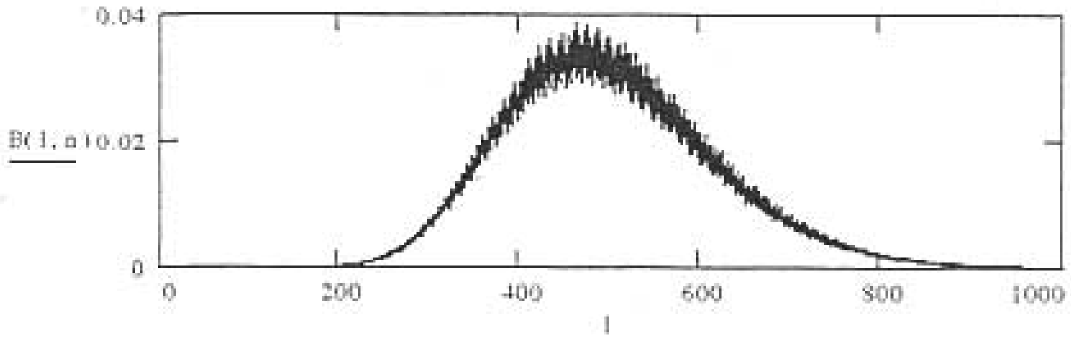
n := 100   l := 1.. 1000



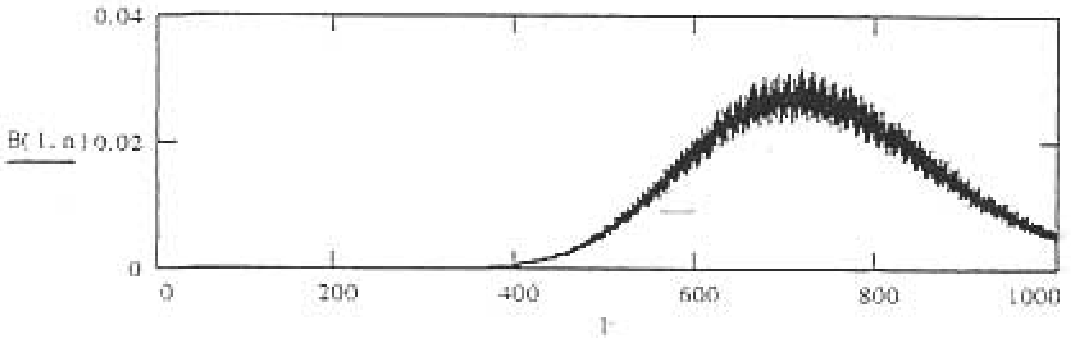
n := 150   l := 1, 2.. 1000



n := 200   l := 1.. 1000



n := 300   l := 1.. 1000



Theorem (W. 1995)

$$\liminf_{x \to \infty} \frac{1}{x} \int_{\mathbb{Z}_3^*} s_x(a)\, da > 0 .$$

Consequences

(1) If $s_x(a)$ is sufficiently close to the 3-adic average of s_x for large x and $a \in \mathbb{N}$, then the positive density conjecture follows.

(2) My paper "On the problem of positive predecessor density in 3n+1 dynamics," DCDS 9 (2003), pp. 771 - 787

Remark

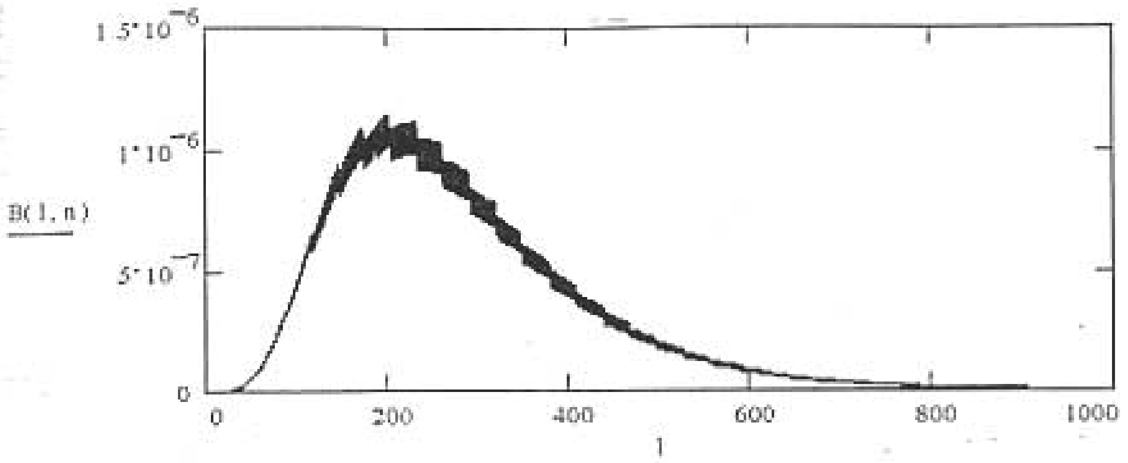
If we had put $T(n) = \begin{cases} n/2 & \text{for even } n, \\ pn+1 & \text{for odd } n, \end{cases}$

for some prime $p > 3$,

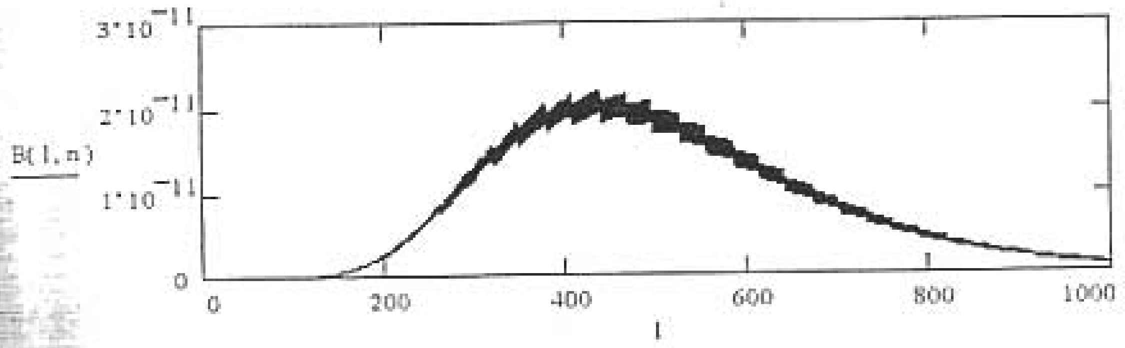
then $\lim_{x \to \infty} \frac{1}{x} \int_{\mathbb{Z}_p^*} s_x(a)\, da = 0 .$

$p = 7$

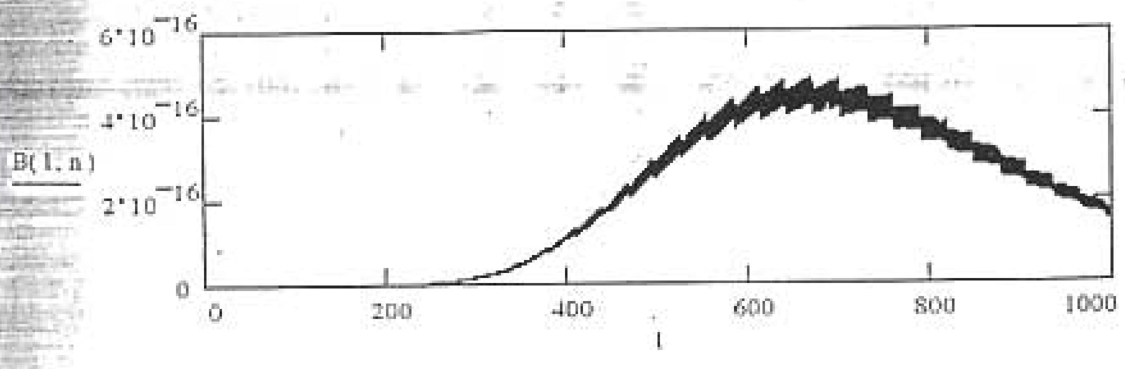
n := 100   l := 1.. 1000



n := 200   l := 1.. 1000

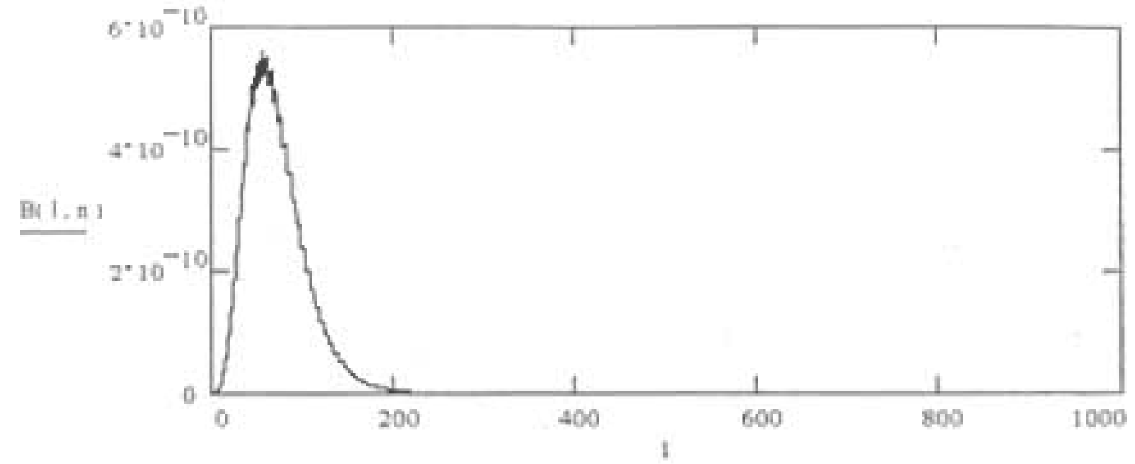


n := 300   l := 1.. 1000

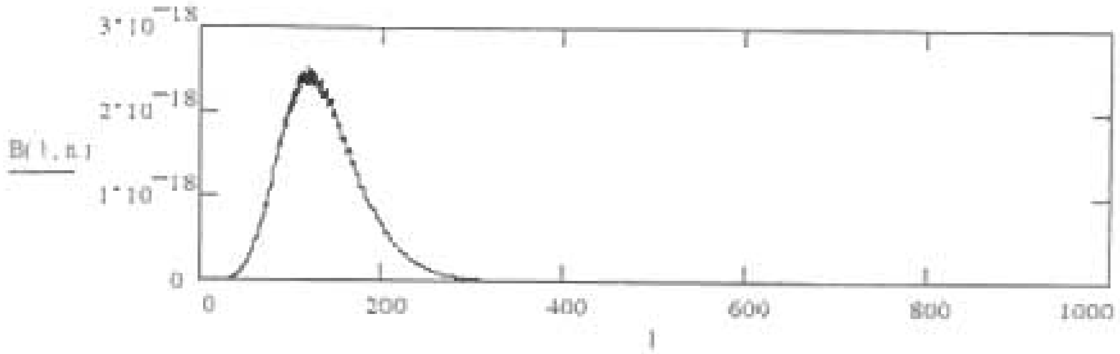


p = 11

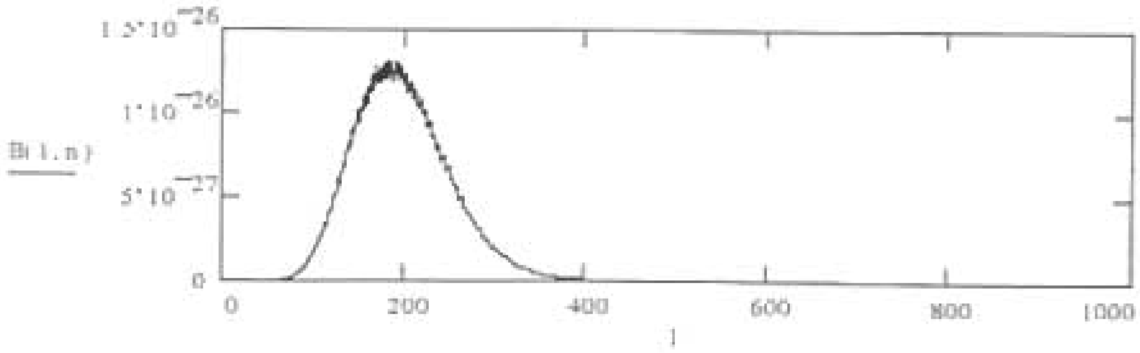
$n := 100 \quad i := 1 .. 1000$



$n := 200 \quad i := 1 .. 1000$



$n := 300 \quad i := 1 .. 1000$



Elka functions, 1. viewpoint

Definition. For any integer $\ell \geq 0$ let

$$e_\ell : \mathbb{N}_0 \times \mathbb{N} \longrightarrow \mathbb{N}_0,$$

$$e_\ell(k,a) := \left| \left\{ \begin{array}{l} \text{paths } v_0 \to \dots \to v_{k+\ell} \text{ in } \Gamma_T, \\ k \text{ steps } T_0,\ \ell \text{ steps } T_1,\ v_{k+\ell} = a \end{array} \right\} \right|$$

(a path-counting function)

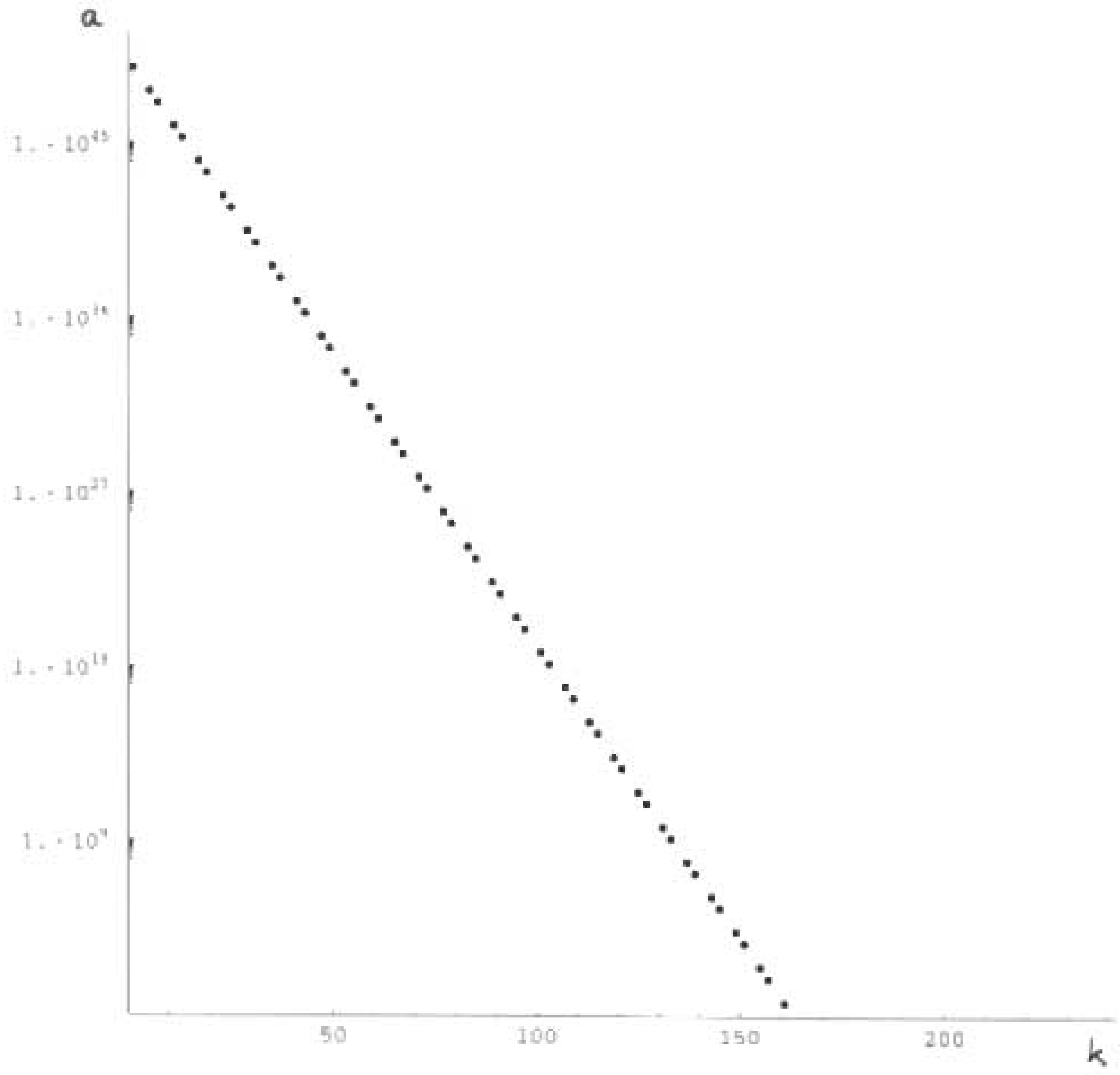
Theorem.

(a) Let $\ell \geq 0$, $k \in \mathbb{N}_0$, $a, b \in \mathbb{N}$. Then

$$a \equiv b \mod 3^\ell \Rightarrow e_\ell(k,a) = e_\ell(k,b).$$

(b) Let $\ell \geq 1$, $(k,a) \in \mathbb{N}_0 \times \mathbb{N}$. Then

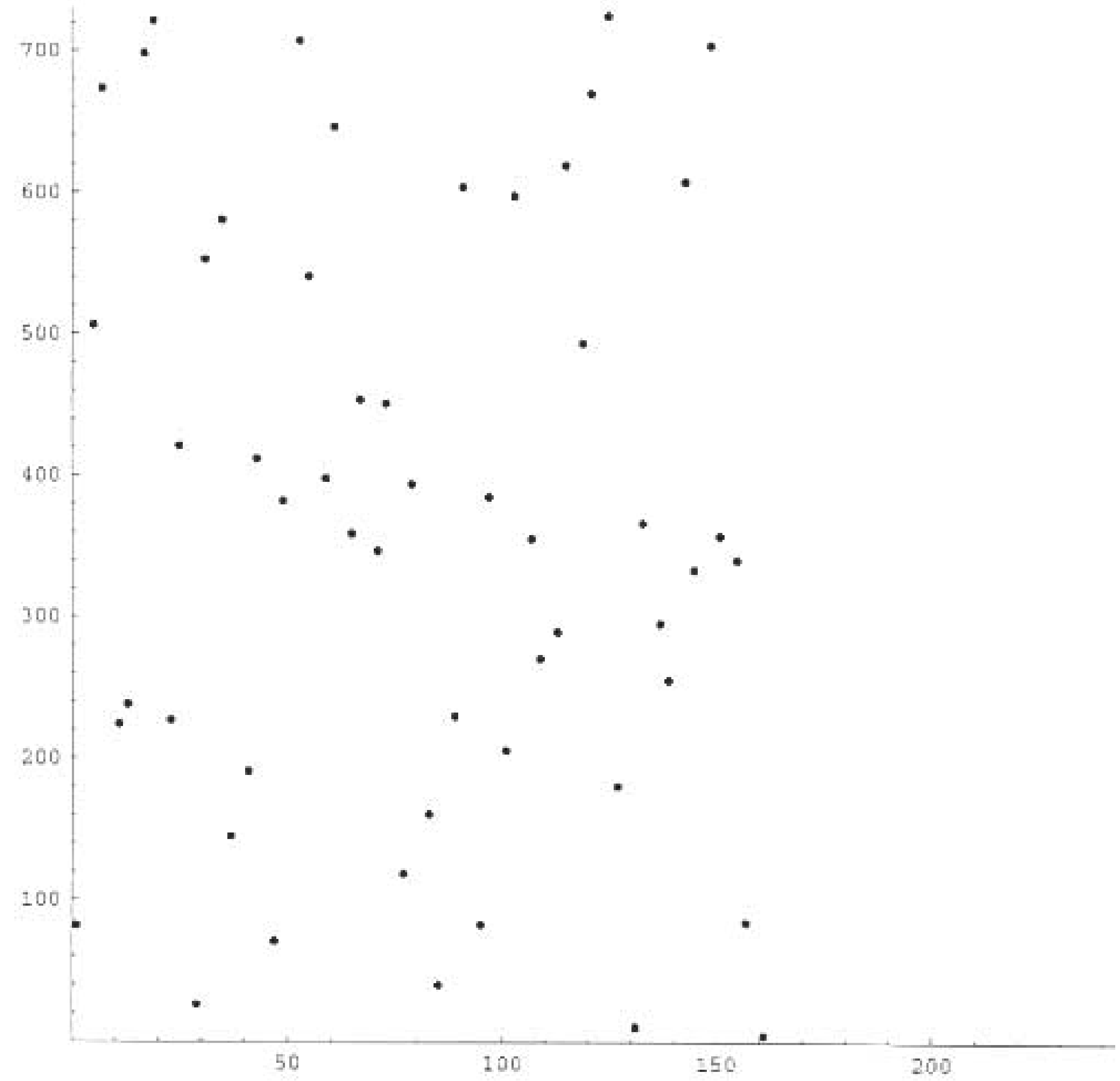
$$e_\ell(k,a) = \begin{cases} \displaystyle\sum_{j=0}^{\lfloor k/2 \rfloor} e_{\ell-1}\left(k-2j, \frac{2^{2j+1}a-1}{3}\right) & \text{if } a \equiv 2 \mod 3 \\ \displaystyle\sum_{j=0}^{\lfloor k/2 \rfloor} e_{\ell-1}\left(k-2j-1, \frac{2^{2j+2}a-1}{3}\right) & \text{if } a \equiv 1 \mod 3 \\ 0 & \text{if } a \equiv 0 \mod 3 \end{cases}$$



$$\left\{ \left(k-j,\ \frac{2^{j+1}a-1}{3} \right) :\ j = 1, 3, \ldots, 161,\quad k = 162,\ a = 4 \right\},$$

↑

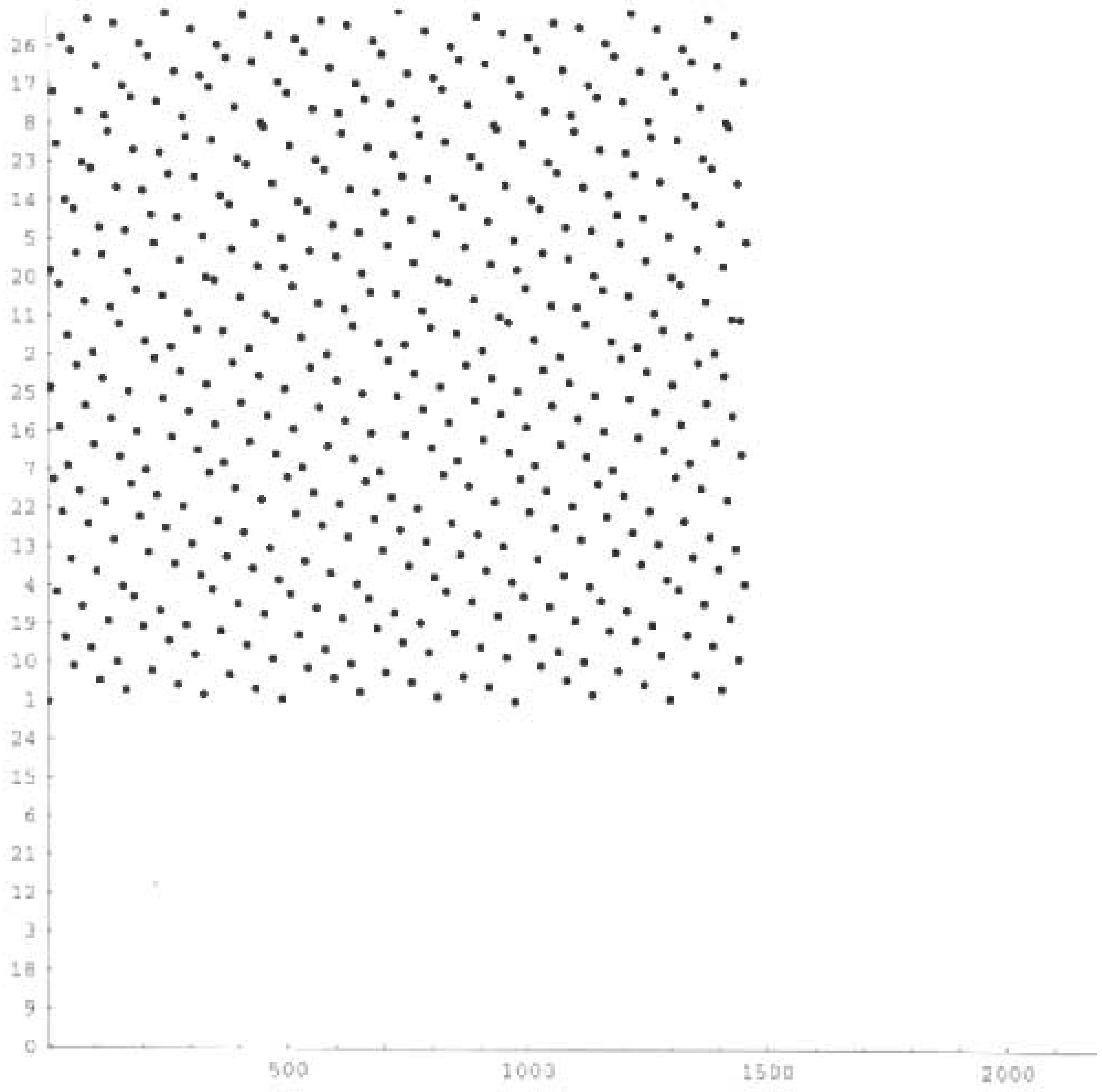
logarithmische Skala



$k = 162$

$a = 4$

$$\left\{ \left(k - j, \frac{2^{j+1} a - 1}{3} \bmod 3^6 \right) : j = 1, 3, 5, \ldots, 161 \right\}$$



k= 1458

a = 4

... und das Erzeugen von Zufallszahlen