Folding Transformations of the Painlevé Equations

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- A_2 root system
- symmetries of P_{IV} system
- folding transformation for P_{IV}
- folding transformations for P_{II} to P_{VI}

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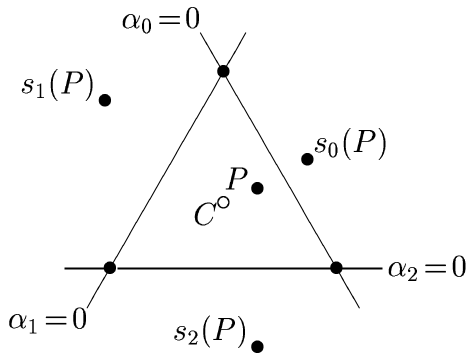
Extended Affine Weyl group $\mathrm{A}_2^{(1)}$

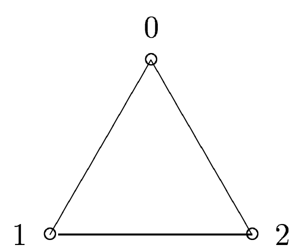
Geometric picture of affine Weyl group $W(A_2^{(1)}) = \langle s_0, s_1, s_2 \rangle$



triangular co-ordinate system $\alpha_0 + \alpha_1 + \alpha_2 = 1$
fundamental reflections s_i $(i = 0, 1, 2)$

$$s_i(\alpha_j) = \alpha_j - \alpha_i a_{ij}, \quad (a_{ij}) = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \quad \text{Cartan matrix}$$

Ex. prove the reflection formulae using geometrical reasoning.
Dynkin diagram for affine root system $A_2^{(1)}$



lattice automorphism π

$$\pi(\alpha_j) = \alpha_{j+1} \quad j \in \mathbb{Z}/3\mathbb{Z}$$

extended affine Weyl group $\widetilde{W}(A_2^{(1)}) = \langle \pi, s_0, s_1, s_2 \rangle$
algebraic relations

$$s_j^2 = 1, \quad (s_j s_{j+1})^3 = 1, \quad s_j s_{j\pm1} s_j = s_{j\pm1} s_j s_{j\pm1}, \quad \pi^3 = 1, \quad \pi s_j = s_{j+1}\pi$$

Ex. verify the algebraic formulae using geometrical reasoning.

Symmetries of $\mathrm{P_{IV}}$

fourth Painlevé equation $\mathrm{P_{IV}}$ with $' = d/dt$, $t, y \in \mathbb{C}$, $\alpha, \beta \in \mathbb{C}$

$$y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}$$

symmetric form/dressing chain

$$\begin{aligned} f_0' &= f_0(f_1 - f_2) + 2\alpha_0 \\ f_1' &= f_1(f_2 - f_0) + 2\alpha_1 \\ f_2' &= f_2(f_0 - f_1) + 2\alpha_2 \end{aligned} \qquad \begin{aligned} \alpha_0 + \alpha_1 + \alpha_2 = 1 \\ f_0 + f_1 + f_2 = 2t \end{aligned}$$

Ex. prove the equivalence and show the relationship of systems

$$y = -f_1, \quad \alpha = \alpha_0 - \alpha_2, \quad \beta = -2\alpha_1^2$$

Bäcklund transformations

$$s_i(f_j) = f_j + \frac{2\alpha_i}{f_i}u_{ij}$$

$$\pi(f_j) = f_{j+1}$$

orientation matrix

$$(u_{ij}) = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Ex. prove that these transformations preserve the algebraic relations and the structure of the symmetric form.

automorphisms of Dynkin diagram

$$\mathrm{Aut}(A_2^{(1)}) = \langle \sigma_0, \sigma_1 \rangle \cong \mathfrak{S}_3 \supset \langle \pi \rangle \cong C_3$$

Ex. find the explicit form of the transformations $\sigma_0, \sigma_1, \sigma_2$.

Birational Canonical Transformations

	α_0	α_1	α_2	f_0	f_1	f_2	t
s_0	$-\alpha_0$	$\alpha_1+\alpha_0$	$\alpha_2+\alpha_0$	f_0	$f_1+\dfrac{2\alpha_0}{f_0}$	$f_2-\dfrac{2\alpha_0}{f_0}$	t
s_1	$\alpha_0+\alpha_1$	$-\alpha_1$	$\alpha_2+\alpha_1$	$f_0-\dfrac{2\alpha_1}{f_1}$	f_1	$f_2+\dfrac{2\alpha_1}{f_1}$	t
s_2	$\alpha_0+\alpha_2$	$\alpha_1+\alpha_2$	$-\alpha_2$	$f_0+\dfrac{2\alpha_2}{f_2}$	$f_1-\dfrac{2\alpha_2}{f_2}$	f_2	t
π	α_1	α_2	α_0	f_1	f_2	f_0	t
σ_0	α_0	α_2	α_1	if_0	if_2	if_1	it
σ_1	α_2	α_1	α_0	if_2	if_1	if_0	it

Translations on the A_2 lattice

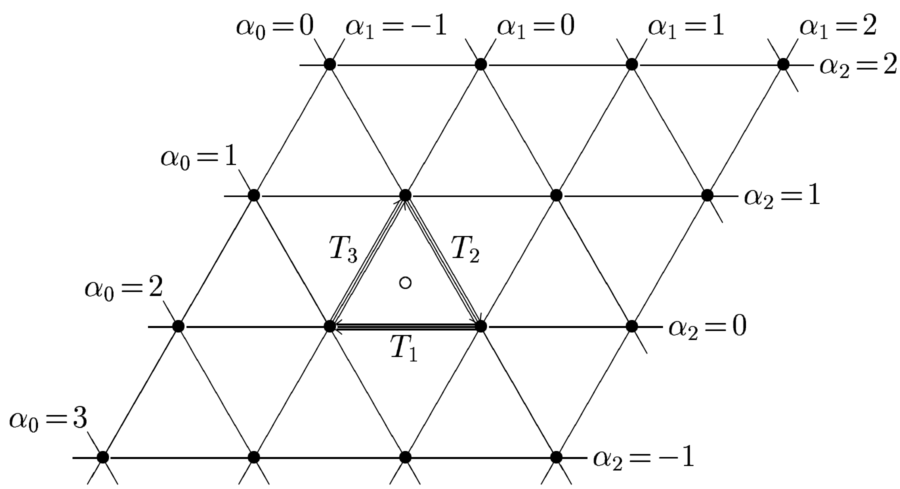
compose shift operators/Schlesinger transformations from generators

$$
\begin{aligned}
T_1 &:= \pi s_2 s_1 \\
T_2 &:= s_1 \pi s_2 \\
T_3 &:= s_2 s_1 \pi
\end{aligned}
$$

fundamental weights $\tilde{\omega}_j,\ j = 1, 2, 3$ of the root system $A_2^{(1)}$

$$
\begin{aligned}
T_1 &: (\alpha_0, \alpha_1, \alpha_2) \mapsto (\alpha_0 + 1, \alpha_1 - 1, \alpha_2) \\
T_2 &: (\alpha_0, \alpha_1, \alpha_2) \mapsto (\alpha_0, \alpha_1 + 1, \alpha_2 - 1) \\
T_3 &: (\alpha_0, \alpha_1, \alpha_2) \mapsto (\alpha_0 - 1, \alpha_1, \alpha_2 + 1)
\end{aligned}
$$

Ex. prove that the action of the shift operators corresponds to a lattice translation.



shift operator T_3^{-1} generates the parameter sequence of $(\alpha_0 + n, \alpha_1, \alpha_2 - n)$ with $n \in \mathbb{Z}$, and a ladder of variables $T_3^{-n} f_j = f_j[n]$
Satisfy *asymmetric discrete Painlevé equation* $d\mathrm{P_I}$

$$
\begin{aligned}
f_0 + \underline{f_0} &= 2t - f_2 - \frac{2(\alpha_2 - n)}{f_2} \\
\overline{f_2} + f_2 &= 2t - f_0 + \frac{2(n+\alpha_0)}{f_0}
\end{aligned}
$$

where $f_j = f_j[n]$, $\overline{f_j} = f_j[n+1]$, $\underline{f_j} = f_j[n-1]$.
Ex. Compute the action of $T_3, \underline{T_3^{-1}}$ *on the* f_j *and deduce the above coupled difference equation.*

Hamiltonian dynamics of $\mathrm{P_{IV}}$

Hamiltonian system $\mathcal{H} = \{q, p; H, t\}$ with canonical variables q, p

$$\begin{aligned} H(q,p;t) &= \frac{1}{2} f_0 f_1 f_2 + \alpha_2 f_1 - \alpha_1 f_2 \\ &= (2p - q - 2t)pq - 2\alpha_1 p - \alpha_2 q \end{aligned}$$

Hamilton's equations

$$\begin{aligned} q' = \frac{dq}{dt} &= \frac{\partial H}{\partial p} = q(4p - q - 2t) - 2\alpha_1 \\ p' = \frac{dp}{dt} &= -\frac{\partial H}{\partial q} = 2p(q - p + t) + \alpha_2 \end{aligned}$$

Ex. prove that q satisfies the $\mathrm{P_{IV}}$ ordinary differential equation and the relationships

$$f_0 = q - 2p + 2t, f_1 = -q, f_2 = 2p$$

nonconservative system $h(t) \equiv H(q(t), p(t); t)$

$$\begin{aligned} h' &= f_1 f_2 \\ h'' &= f_1 f_2 (f_2 - f_1) + 2\alpha_2 f_1 + 2\alpha_1 f_2 \end{aligned}$$

$h(t)$ satisfies the Jimbo-Miwa-Okamoto σ form

$$E : (h'')^2 - 4(th' - h)^2 + 4h'(h' + 2\alpha_1)(h' - 2\alpha_2) = 0$$

Ex. prove this.

recovery of canonical variables $\{h(t) : E = 0\} \to \mathcal{H}$ via

$$\begin{aligned} q &= \frac{h'' + 2(h - th')}{2(h' - 2\alpha_2)} \\ 2p &= \frac{h'' - 2(h - th')}{2(h' + 2\alpha_1)} \end{aligned}$$

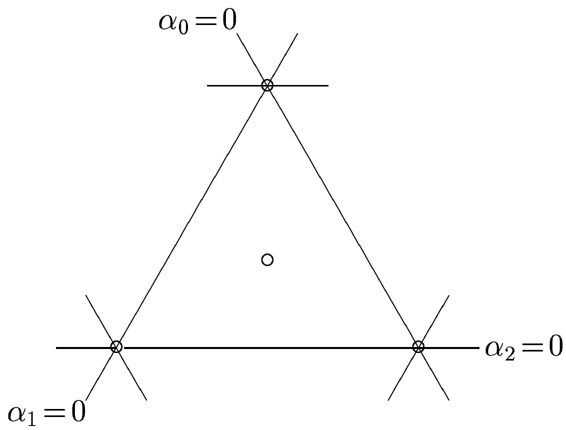
Ex. deduce this from the earlier workings.

τ-function $\tau(t)$

$$h(t) \equiv \frac{d}{dt} \log \tau(t)$$

Rational and Classical Solutions for $\mathrm{P_{IV}}$

Weyl chamber $\alpha_i > 0$ for $i = 0, 1, 2$.



$(\alpha_0, \alpha_1, \alpha_2)$	Solution Type
generic $\alpha_j \neq 0, 1$ interior of Weyl chamber	transcendental
$\alpha_j = 0$ for one j Weyl chamber wall	Hermite-Weber/parabolic-cylinder
$\alpha_j = 1$ for one j, $\alpha_{j+1} = \alpha_{j+2} = 0$ vertices of Weyl chamber	generalised Hermite polynomials
$\alpha_j = \frac{1}{3}$ for all j barycentre of Weyl chamber	Okamoto polynomials

Folding transformation for $\mathrm{P_{IV}}$

Step 1. Find fixed point of $\mathrm{Aut}(A_2^{(1)})$ that preserves t, i.e. $\langle \pi \rangle$.

$$\alpha_0 = \alpha_1 = \alpha_2 = 1/3$$

Step 2. Construct an invariant function. Try

$$x = f_0 f_1 f_2 - \frac{8}{27} t^3$$

Step 3. Compute the derivatives

$$\begin{aligned} x' &= \frac{2}{3}(f_1 f_2 + f_0 f_2 + f_0 f_1) - \frac{8}{9} t^2 \\ x - t x' &= (f_0 - \frac{2}{3} t)(f_1 - \frac{2}{3} t)(f_2 - \frac{2}{3} t) \\ x'' &= \frac{2}{3}[f_1 f_2 (f_2 - f_1) + f_0 f_1 (f_1 - f_0) + f_0 f_2 (f_0 - f_2)] \end{aligned}$$

Step 4. Look for an algebraic relation between these. Hint - try

$$(x'')^2 + 12(x - t x')^2 + 6(x')^3$$

and this identically vanishes!
Step 5. Scale the dependent and independent variables.

$$\begin{aligned} 2X &= -(-3)^{3/4} x \\ s &= (-3)^{1/4} t \end{aligned}$$

and the scaled relation is just the Jimbo-Miwa-Okamoto σ form with

$$\alpha_0 = 1, \alpha_1 = 0, \alpha_2 = 0$$

Conclusion. Folding transformation $\{q, p; H, t\} \mapsto \{Q, P; K, s\}$ with

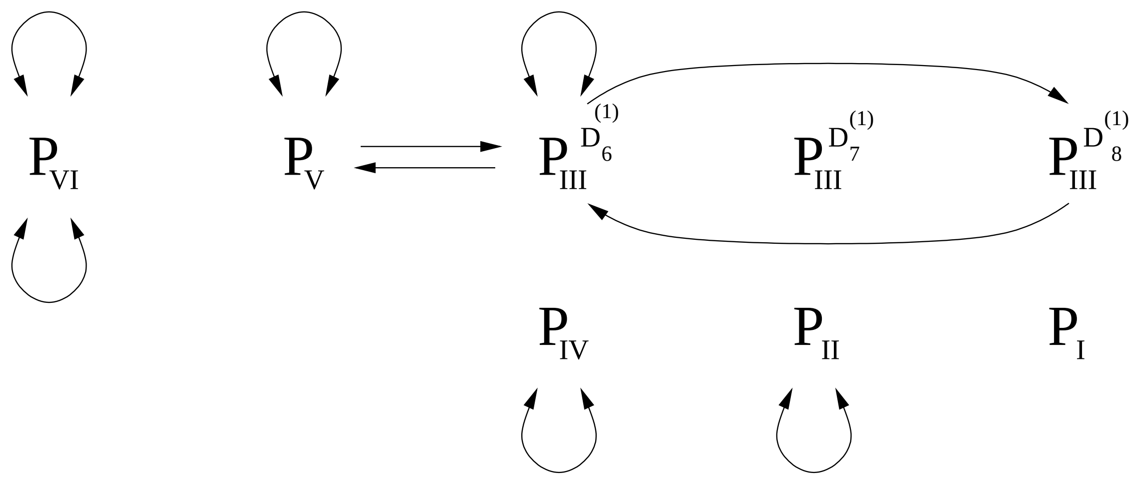
$$K = -(-3)^{3/4}(H + \frac{1}{3} q + \frac{2}{3} p - \frac{4}{27} t^3)$$

and

$$dP \wedge dQ - dK \wedge ds = 3(dp \wedge dq - dH \wedge dt)$$

Ex. find the variables transformations $Q(q,p,t), P(q,p,t)$.
Ex. deduce the transformations from the barycentre to the other two vertices of the Weyl chamber and combine all three transformations in a single relation.

$\underline{\mathrm{P_{II}\ to\ P_{VI}}}$



Folding transformations

1. are not birational transformations
2. are not canonical transformations
3. are contact transformations
4. relate rational solution parameter sets to other rational/classical solutions parameters

References

General introduction to the symmetries, algebraic aspects and combinatorics of the Painlevé equations, with an emphasis on P_{II} and P_{IV} .

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