Folding Transformations of the Painlevé Equations

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- $A_2$ root system
- symmetries of $P_{IV}$ system
- folding transformation for $P_{IV}$
- folding transformations for $P_{11}$ to $P_{VI}$

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Extended Affine Weyl group $A_2^{(1)}$

Geometric picture of affine Weyl group $W(A_2^{(1)}) = \langle s_0, s_1, s_2 \rangle$

triangular co-ordinate system $\alpha_0 + \alpha_1 + \alpha_2 = 1$

fundamental reflections $s_i$ ($i = 0, 1, 2$)

$$s_i(\alpha_j) = \alpha_j - \alpha_i a_{ij}, \quad (a_{ij}) = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Cartan matrix

Ex. prove the reflection formulae using geometrical reasoning.

Dynkin diagram for affine root system $A_2^{(1)}$

lattice automorphism $\pi$

$$\pi(\alpha_j) = \alpha_{j+1} \quad j \in \mathbb{Z}/3\mathbb{Z}$$

extended affine Weyl group $\widehat{W}(A_2^{(1)}) = \langle \pi, s_0, s_1, s_2 \rangle$

algebraic relations

$$s_j^2 = 1, \quad (s_js_{j+1})^3 = 1, \quad s_js_{j+1}s_j = s_{j+1}sJs_{j+1}, \quad \pi^3 = 1, \quad \pi s_j = s_{j+1}\pi$$

Ex. verify the algebraic formulae using geometrical reasoning.
Symmetries of $\text{P}_4$

fourth Painlevé equation $\text{P}_4$ with $' = d/dt, t, y \in \mathbb{C}, \alpha, \beta \in \mathbb{C}$

$$y'' = \frac{1}{2y} (y')^2 + \frac{3}{2} y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}$$

symmetric form/dressing chain

$$f'_1 = f_0(f_1 - f_2) + 2\alpha_0 \quad \alpha_0 + \alpha_1 + \alpha_2 = 1$$
$$f'_1 = f_1(f_2 - f_0) + 2\alpha_1 \quad f_0 + f_1 + f_2 = 2t$$

Ex. prove the equivalence and show the relationship of systems

$$y = -f_1, \quad \alpha = \alpha_0 - \alpha_2, \quad \beta = -2\alpha_1^2$$

Bäcklund transformations

$$s_i(f_j) = f_j + \frac{2\alpha_i}{f_i} u_{ij}$$
$$\pi (f_j) = f_{j+1}$$

orientation matrix

$$u_{ij} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Ex. prove that these transformations preserve the algebraic relations and the structure of the symmetric form.

automorphisms of Dynkin diagram

$$\text{Aut}(A_2^{(1)}) = \langle \sigma_0, \sigma_1 \rangle \cong \mathfrak{S}_3 \supset \langle \pi \rangle \cong C_3$$

Ex. find the explicit form of the transformations $\sigma_0, \sigma_1, \sigma_2$. 
<table>
<thead>
<tr>
<th>Mode</th>
<th>( \alpha_0 )</th>
<th>( \alpha_1 )</th>
<th>( \alpha_2 )</th>
<th>( f_0 )</th>
<th>( f_1 )</th>
<th>( f_2 )</th>
<th>( t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_0 )</td>
<td>(-\alpha_0)</td>
<td>(\alpha_1 + \alpha_0)</td>
<td>(\alpha_2 + \alpha_0)</td>
<td>( f_0 )</td>
<td>( f_1 + \frac{2\alpha_0}{f_0} )</td>
<td>( f_2 - \frac{2\alpha_0}{f_0} )</td>
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<td>( f_0 - \frac{2\alpha_1}{f_1} )</td>
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<td>(\alpha_1 + \alpha_2)</td>
<td>(-\alpha_2)</td>
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<tr>
<td>( \pi )</td>
<td>( \alpha_1 )</td>
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<td>( \alpha_2 )</td>
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<td>( \alpha_2 )</td>
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Translations on the $A_2$ lattice

compose shift operators/Schlesinger transformations from generators

$$T_1 := \pi s_2 s_1$$
$$T_2 := s_1 \pi s_2$$
$$T_3 := s_2 s_1 \pi$$

fundamental weights $\tilde{\omega}_j$, $j = 1, 2, 3$ of the root system $A_2^{(1)}$

$$T_1 : (\alpha_0, \alpha_1, \alpha_2) \mapsto (\alpha_0 + 1, \alpha_1 - 1, \alpha_2)$$
$$T_2 : (\alpha_0, \alpha_1, \alpha_2) \mapsto (\alpha_0, \alpha_1 + 1, \alpha_2 - 1)$$
$$T_3 : (\alpha_0, \alpha_1, \alpha_2) \mapsto (\alpha_0 - 1, \alpha_1, \alpha_2 + 1)$$

Ex. prove that the action of the shift operators corresponds to a lattice translation.

shift operator $T_3^{-1}$ generates the parameter sequence of $(\alpha_0 + n, \alpha_1, \alpha_2 - n)$ with $n \in \mathbb{Z}$, and a ladder of variables $T_3^{-n} f_j = f_j[n]$ 
Satisfy asymmetric discrete Painlevé equation $dP_1$

$$f_0 + f_0 = 2t - f_2 - \frac{2(\alpha_2 - n)}{f_2}$$
$$\overline{f}_2 + f_2 = 2t - f_0 + \frac{2(n + \alpha_0)}{f_0}$$

where $f_j = f_j[n], \overline{f}_j = f_j[n+1], f_j = f_j[n-1]$.

Ex. Compute the action of $T_3, T_3^{-1}$ on the $f_j$ and deduce the above coupled difference equation.
Hamiltonian dynamics of $P_{IV}$

Hamiltonian system $\mathcal{H} = \{q, p; H, t\}$ with canonical variables $q, p$

$$H(q, p; t) = \frac{1}{2} f_0 f_1 f_2 + \alpha_2 f_1 - \alpha_1 f_2$$

$$= (2p - q - 2t)pq - 2\alpha_1 p - \alpha_2 q$$

Hamilton’s equations

$$\dot{q} = \frac{dq}{dt} = \frac{\partial H}{\partial p} = q(4p - q - 2t) - 2\alpha_1$$

$$\dot{p} = \frac{dp}{dt} = -\frac{\partial H}{\partial q} = 2p(q - p + t) + \alpha_2$$

**Ex.** prove that $q$ satisfies the $P_{IV}$ ordinary differential equation and the relationships

$$f_0 = q - 2p + 2t, f_1 = -q, f_2 = 2p$$

**nonconservative system** $h(t) \equiv H(q(t), p(t); t)$

$$h' = f_1 f_2$$

$$h'' = f_1 f_2 (f_2 - f_1) + 2\alpha_2 f_1 + 2\alpha_1 f_2$$

$h(t)$ satisfies the Jimbo-Miwa-Okamoto $\sigma$ form

$$E : (h'')^2 - 4(th' - h)^2 + 4h' (h' + 2\alpha_1)(h' - 2\alpha_2) = 0$$

**Ex.** prove this.

recovery of canonical variables $\{h(t) : E = 0\} \to \mathcal{H}$ via

$$q = \frac{h'' + 2(h - th')}{2(h' - 2\alpha_2)}$$

$$2p = \frac{h'' - 2(h - th')}{2(h' + 2\alpha_1)}$$

**Ex.** deduce this from the earlier workings.

$\tau$-function $\tau(t)$

$$h(t) \equiv \frac{d}{dt} \log \tau(t)$$
Rational and Classical Solutions for $P_{IV}$

Weyl chamber $\alpha_i > 0$ for $i = 0, 1, 2$.

<table>
<thead>
<tr>
<th>$(\alpha_0, \alpha_1, \alpha_2)$</th>
<th>Solution Type</th>
</tr>
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<tbody>
<tr>
<td>generic $\alpha_j \neq 0, 1$ interior of Weyl chamber</td>
<td>transcendental</td>
</tr>
<tr>
<td>$\alpha_j = 0$ for one $j$ Weyl chamber wall</td>
<td>Hermite-Weber/parabolic-cylinder</td>
</tr>
<tr>
<td>$\alpha_j = 1$ for one $j$, $\alpha_{j+1} = \alpha_{j+2} = 0$ vertices of Weyl chamber</td>
<td>generalised Hermite polynomials</td>
</tr>
<tr>
<td>$\alpha_j = \frac{1}{3}$ for all $j$ barycentre of Weyl chamber</td>
<td>Okamoto polynomials</td>
</tr>
</tbody>
</table>
Folding transformation for $P_{IV}$

Step 1. Find fixed point of $\text{Aut}(A_2^{(1)})$ that preserves $t$, i.e. $\langle \pi \rangle$.

$$\alpha_0 = \alpha_1 = \alpha_2 = \frac{1}{3}$$

Step 2. Construct an invariant function. Try

$$x = f_0 f_1 f_2 - \frac{8}{27} t^3$$

Step 3. Compute the derivatives

$$x' = \frac{2}{3} (f_1 f_2 + f_0 f_2 + f_0 f_1) - \frac{8}{9} t^2$$

$$x - tx' = (f_0 - \frac{2}{3} t)(f_1 - \frac{2}{3} t)(f_2 - \frac{2}{3} t)$$

$$x'' = \frac{2}{3} [f_1 f_2 (f_2 - f_1) + f_0 f_1 (f_1 - f_0) + f_0 f_2 (f_0 - f_2)]$$

Step 4. Look for an algebraic relation between these. Hint - try

$$(x'')^2 + 12(x - tx')^2 + 6(x')^3$$

and this identically vanishes!

Step 5. Scale the dependent and independent variables.

$$2X = -(-3)^{3/4} x$$

$$s = (-3)^{1/4} t$$

and the scaled relation is just the Jimbo-Miwa-Okamoto $\sigma$ form with

$$\alpha_0 = 1, \alpha_1 = 0, \alpha_2 = 0$$

Conclusion. Folding transformation $\{q, p; H, t\} \mapsto \{Q, P; K, s\}$ with

$$K = -(-3)^{3/4} (H + \frac{1}{3} q + \frac{2}{3} p - \frac{4}{27} t^3)$$

and

$$dP \wedge dQ - dK \wedge ds = 3(dp \wedge dq - dH \wedge dt)$$

Ex. find the variables transformations $Q(q, p, t), P(q, p, t)$.

Ex. deduce the transformations from the barycentre to the other two vertices of the Weyl chamber and combine all three transformations in a single relation.
Folding transformations

1. are not birational transformations
2. are not canonical transformations
3. are contact transformations
4. relate rational solution parameter sets to other rational/classical solutions parameters
References

General introduction to the symmetries, algebraic aspects and combinatorics of the Painlevé equations, with an emphasis on \( P_{II} \) and \( P_{IV} \).


The Folding Transformations of the Painlevé Equations.


