Definition. If \( \sigma \) is a primitive tile substitution, \( \mathcal{X}(\sigma) \) denotes the hull of \( \sigma \) (in the appropriate top, for instance local rubber top). \( \mathcal{X}(\sigma) \) aperiodic, if each member of \( \mathcal{X}(\sigma) \) is nonperiodic.

Supertile means a patch \( \sigma(T) \), or more general, \( \sigma^k(T) \), where \( T \) is some tile in \( T \in \mathcal{X}(\sigma) \). More precisely, the latter is called \( k \)-th order supertile. Edge or vertex of a supertile means some edge resp. vertex of the union of the tiles in \( \sigma^k(T) \).

Here: substitution always selfsimilar. That is, the supertile \( \sigma(T) \) is congruent to \( \lambda T \), where \( \lambda \) is the inflation factor.

Here: always tilings in \( \mathbb{R}^2 \). Let \( R \) denote the rotation through \( \pi \) about the origin.

Lemma 1. Let \( \sigma \) be a primitive tile substitution with the unique decomposition property, and let \( \mathcal{X}(\sigma) \) be aperiodic. Let \( T \in \mathcal{X}(\sigma) \) such that \( R(T) = T \). Then, for all \( k \in \mathbb{Z} \) holds \( R(\sigma^k(T)) = \sigma^k(T) \).

Proof. The claim is immediate for \( k \geq 0 \). (For \( k = 0 \) it is an assumption, and since two equal tiles/patches stay equal under \( k \)-th substitution, it is true for all \( k \geq 0 \).) So, let us assume the claim is wrong for \( k = -1 \). Then, \( R(T) = T \), but the unique tiling \( \sigma^{-1}(T) \) is not \( R \)-symmetric. But then some symmetric patch \( P \) in \( T \) corresponds to a non-symmetric patch in \( \sigma^{-1}(T) \). Thus \( P \) can be desubstituted in two ways: \( \sigma^{-1}(P) \neq \sigma^{-1}(R(P)) \). Let \( P \) be the largest such patch. Either \( P \) is finite, then no local information can tell how to desubstitute \( P \) uniquely (since the rest \( T \setminus P \) is symmetric). Or \( P \) is infinite, then again, no local information tells how to desubstitute \( P \).

Lemma 2. If 0 is contained in the interior of some tile \( T \) in a primitive substitution tiling \( T \), and if \( R(T) = T \), then \( T \) is determined uniquely by the sequence of types of the \( k \)-th order supertiles containing 0.

Proof. (Sketch) By selfsimilarity, 0 is not a vertex of any supertile. If the sequence of supertiles \( \sigma(T), \sigma^2(T), \ldots \) containing 0 fail to cover the entire plane, there is some point \( x \) which is not contained in the union \( U \) of the supertiles. By \( R(T) = T \), and by Lemma 1, \( -x \) is also not contained in \( U \). This means that all \( k \)-th order supertiles having 0 as their symmetry centres, do not contain \( x \) and \( -x \). Contradiction.

Theorem. Let \( \sigma \) be a primitive tile substitution with the unique decomposition property, and let \( \mathcal{X}(\sigma) \) be aperiodic and FLC. Then there are only finitely many elements of \( \mathcal{X}(\sigma) \) which are invariant under a rotation by \( \pi \) about the origin.

Proof. Since \( R \) fixes the entire tiling \( T \), \( R \) fixes in particular the patch \( P_0 = \{ T \in T \mid 0 \in T \} \). First, let us assume that 0 is contained in the interior of some tile \( T \) (hence \( P_0 = \{ T \} \)). Then, since \( R(T) = T \), 0 is exactly the (unique) centre of symmetry of \( T \). By the Lemma 1, 0 is
also the unique centre of each supertile containing 0. Let there be \( m \) different tile types, and let \( T \) be of type 1.

Case 1: There is more than one type of supertile containing a type 1 tile in its centre.

Case 1.1: Say, these are of type 1 and 2. Then there has to be a third supertile type containing a tile of type 2 in its centre, say, type 3. Thus we need a fourth supertile type, containing a tile of type 3 in its centre, and so on. Contradiction (at stage \( m \), by the pigeon hole principle).

Case 1.2: Say, these are of type 2 and 3. Contradiction, analogously to the last case.

Case 2: There is only one type of supertile \( \sigma(S) \) containing a type 1 tile in its centre.

Case 2.1: This supertile is of type 1. Then, the next order supertile \( \sigma_2(S') \) has to be of type 1, too, and the same is true for all \( k \)-order supertiles: all are of type 1. By Lemma 2, this yields a unique tiling \( T \).

Case 2.2: This supertile is of type 2. Now, either there is more than one supertile containing a tile of type 2 in its centre, and we are in Case 1. Or there is exactly one supertile containing a tile of type 2 in its centre. It may be of type 1 (then we have a loop 1 2 1 2 ...), or of type 3. Proceeding in this manner, we will finally get into some loop 12...n of length at most \( m \).

By Lemma 2, this yields at most \( m \) different tilings \( T \) with \( R(T) = T \).

Now, assume that 0 is not contained in the interior of some tile, but in the interior of some supertile. Then, by the same arguments, 0 has to be the centre of this supertile, and the centre of all higher order supertiles containing 0; and again, this yields only finitely many (at most \( m \)) different tilings \( T \).

The remaining possibilities we have to consider is when 0 lies on the boundary of \( k \)-th order supertiles on any level \( k \). This means that in 0 two or more supertiles are meeting. If more than two are meeting on each level, then 0 is a vertex on each level, and the substitution of the vertex constellation \( P_0^{(k)} = \{ T \in \sigma^{-k}(T) \mid 0 \in T \} \) is the vertex constellation \( P_0^{(k+1)} = \sigma(P_0^{(k)}) = \{ T \in \sigma^{-(k+1)}(T) \mid 0 \in T \} \). Similar as in Lemma 2, the sequence of super-vertex constellations \( P_0, P_0^{(1)}, P_0^{(2)}, \ldots \) determines the tiling uniquely. By selfsimilarity, vertices substitute to vertices. By FLC, there are again only finitely many vertex constellations, say, \( n \). Thus, this sequence is always ultimately periodic, with period at most \( n \). This yields at most \( n \) different tilings \( T \) with \( R(T) = T \).

The last possibility to consider is that 0 lies on the boundary of exactly two supertiles, from some level \( k \) on.

Consider again \( P_0 = \{ T \in T \mid 0 \in T \} \). Now, \( P_0 \) contains exactly two tiles of the same type, and 0 is the centre of symmetry of \( P_0 \). By FLC, there are again only finitely many possibilities for \( P_0 \). Moreover, 0 is also the centre of symmetry for each constellation of the two supertiles \( P_0^{(k)} = \{ \sigma^k(T) \mid 0 \in \sigma^k(T) \} \). By considering all the centres of the \( P_0 \)s as artificial vertices, we are in the situation of the last case: 0 is a vertex on each level, vertices substitute to vertices, there are only finitely many of them, say, \( n \). Thus the possible sequence of super-vertices is ultimately periodic, yielding finitely many tilings. 

\( \Box \)
Remarks: The proof generalises immediately to any rotation $R$ about the origin. However, it does not work for mirror reflections: One can construct infinitely many pinwheel tilings which are mirror symmetric.

The proof indicates that all $R$-symmetric tilings are of the form $\bigcup_{k\geq 0} \sigma^{nk}(P)$ for some symmetric legal patch $P$ in $T$.

Let $a$ be the number of $R$-symmetric tiles, $b$ be the number of $R$-symmetric vertex constellations and $c$ be the number of $R$-symmetric pairs of adjacent tiles, then $a + b + c$ is an upper bound for the number of $R$-symmetric tilings in $X(\sigma)$.

A more detailed study how vertices substitute to vertices etc. yields the exact number of symmetric tilings. For instance, there are exactly four pinwheel tilings which are $R$-symmetric: One with vertex constellation $V7$ (Fig. 6 in [1]) in its centre, (the tiling being $T_7 := \bigcup_{k\geq 0} \sigma^{2k}(V7)$, where $\sigma^2(T_7) = T_7$) one with vertex constellation 11 in its centre (also fixed by $\sigma^2$), one with a domino $D$ in its centre (see Figure 4 in [1], it is $T_D = \bigcup_{k\geq 0} \sigma^{2k}(D)$), and its substitution $T_s := \sigma(T_D)$. Again, we have $T_D = \sigma(T_s)$ and vice versa, thus $\sigma^2(T_D) = T_D$ and $\sigma^2(T_s) = T_s$.

References