We compute spectral densities of large sample auto-covariance matrices of stationary stochastic processes at fixed ratio $\alpha = N/M$ of matrix dimension N and sample size M. We find a remarkable scaling relation which expresses the spectral density $\rho_\alpha(\lambda)$ of sample auto-covariance matrices for processes with correlations as a continuous superposition of copies of the spectral density $\rho_\alpha^{(0)}(\lambda)$ for a sequence of uncorrelated random variables at the same value of α, rescaled in terms of the Fourier transform $\hat{C}(q)$ of the true auto-covariance function. We also obtain a closed-form approximation for the scaling function $\rho_\alpha^{(0)}(\lambda)$. Our results are in excellent agreement with numerical simulations using auto-regressive processes, and processes exhibiting a power-law decay of correlations. We discuss the analysis of DNA methylation data in the context of cancer genomics as a possible application of our methods.