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Optimization and Dynamics

Summer term 2007

Assignment sheet 12

(39) Let A and B be commuting $n \times n$ matrices and S an invertible $n \times n$ matrix. Prove the following formulas:

(a) $(e^A)^{-1} = e^{-A}$

(b) $(e^A)^m = e^{mA}$, $m \in \mathbb{Z}$

(c) $e^{A+B} = e^A e^B$

(d) $e^{SAS^{-1}} = Se^A S^{-1}$

(e) $\det(e^A) = e^{\operatorname{tr} A}$.

(40) Consider the linear differential equation $x' = Ax$ where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(a) Find the general solution.

(b) Find the fixed points and discuss their stability.

(c) Sketch the phase diagram.

(41) Consider the linear differential equation $x'(t) = A(t)x(t)$ with initial condition $x(t_0) = x_0$, where $A(t)$ is assumed to be continuous for all $t \in \mathbb{R}$. For $t \geq t_0$ let

$$B(t) = I + \int_{t_0}^{t} A(s)\, ds + \int_{t_0}^{t}\int_{t_0}^{s_1} A(s_1)A(s_2)\, ds_2 ds_1 + \cdots + \int_{t_0}^{t}\int_{t_0}^{s_1}\cdots\int_{t_0}^{s_{n-1}} A(s_1)\cdots A(s_n)\, ds_n \cdots ds_1 + \cdots$$

Hint: The series converges absolutely.

(a) Use Picard iteration to prove that its solution is given by

$$x(t) = B(t)x_0.$$

(b) Prove $|B_{ij}(t)| \leq e^{a(t-t_0)}$, where $a = \max_{t_0 \leq s \leq t} ||A(s)||$.

(c) Show $B(t) = e^{A(t-t_0)}$ if A is constant.

Please hand in until 29.6.2007.