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Optimization and Dynamics

Summer term 2007

Assignment sheet 7

(25) Consider the linear dynamical system $x_{n+1} = ax_n$ for $a > 1$.

(a) Is there any stable orbit?

(b) Has f sensitive dependence on initial conditions in $\mathbb{R}$?

(c) Is there a dense orbit on $\mathbb{R}$? On $\mathbb{R}^+$?

(d) Is f topologically transitive?

(26) Consider the dynamical system defined by the map

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x < \frac{1}{2} \\ 2x - 1 & \text{if } \frac{1}{2} \le x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Sketch the graph.

(b) Find the fixed points and the eventually fixed points.

(c) Show that every $x \in [0, 1)$ can be written as

$$x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}, \tag{1}$$

where the sequence $\{a_n\}$ is defined recursively by

$$\begin{aligned} a_1 &= [2x] \\ a_n &= \left[2^n \left(x - \sum_{m=1}^{n-1} \frac{a_m}{2^m} \right) \right]. \end{aligned} \tag{2}$$

Here $[z]$ denotes the greatest integer n such that $n \le z$. Show that $a_n \in \{0, 1\}$ for all $n \in \mathbb{N}$. Show that $a_n = b_n$ for x given by

$$x = \sum_{n=1}^{\infty} \frac{b_n}{2^n},$$

if $b_n \in \{0,1\}$ and $b_n = 0$ for an infinite number of n. Let us write

$$x = .a_1 a_2 a_3 \ldots := \sum_{n=1}^{\infty} \frac{a_n}{2^n}$$

in the following. Note that this leads to no ambiguity since $a_n \in \{0,1\}$.

(d) Use the expansion (1) to calculate $f(x)$ and $f^n(x)$.

(e) Use the results of (d) to prove that every orbit $O^+(x)$ with $x \in [0,1)$ is unstable.

(f) Show that f has sensitive dependence on initial conditions

(g) Consider the point x given by the sequence $\{a_n\}$

$$x = .0\ 1\ 00\ 01\ 10\ 11\ 000\ 001\ 010\ 011\ 100\ 101\ 110\ 111\ 0000\ \ldots$$

Show that $O^+(x)$ is dense in $[0,1]$.

(h) Show that f is topologically transitive on X. *Hint: Show that in any open interval $U \subseteq [0,1)$ there is a point $y \in U$ that is mapped on some given point $x \in [0,1)$ after sufficiently many iterations of f.*

(i) Show that there are periodic points of all periods (find appropriate sequences $\{a_n\}$). Is it possible to apply Sarkovskii's theorem here?

(j) Is f chaotic on $[0,1]$?

Please hand in until 25.05.2007.