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Optimization and Dynamics

Summer term 2007

Assignment sheet 8

(27) Consider the 2×2 Jordan block

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

(a) Prove

$$J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$

(b) Calculate J^{-1} and J^{-n}.

(c) Let $\lambda > 1$. Prove that $\lim_{n\to\infty} ||J^n x|| = \infty$ for all $x \in \mathbb{R}^2$.

(d) Let $\lambda > 1$. Prove that $\lim_{n\to-\infty} ||J^n x|| = 0$ for all $x \in \mathbb{R}^2$.

(28) Consider the twodimensional linear dynamical system $x_{n+1} = Ax_n$ given by the matrix

$$A = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}.$$

(a) Determine the fixed points and all periodic points. Discuss their stability properties.

(b) Is Sarkovskii's theorem applicable here? Why? Why not?

(29) Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = \alpha + i\beta, \alpha, \beta \in \mathbb{R}$. Let $v = x + iy, x, y \in \mathbb{R}^2$ be an eigenvector corresponding to λ.

(a) Show that $\bar{v} = x - iy$ is an eigenvector of A corresponding to the eigenvalue $\bar{\lambda}$.

(b) Show $Ax = \alpha x - \beta y$ and $Ay = \alpha y + \beta x$ *Hint: Recall that the real part of a complex number (vector, matrix, ...) c can be calculated via* $\frac{1}{2}(c + \bar{c})$, *its imaginary part via* $\frac{1}{2i}(c - \bar{c})$.

(c) Prove that x and y are linearly independent.

Turn around please!

(d) Let S be the matrix whose columns are the vectors x and y, i.e. $S = (x, y)$ Show that S transforms A into real Jordan normal form, i.e.

$$S^{-1}AS = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

Hint: Note that $S^{-1}x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $S^{-1}y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(30) Let $A = \begin{pmatrix} 2 & 4 \\ -2 & -2 \end{pmatrix}$.

(a) Transform A into real Jordan normal form.

(b) Consider the dynamical system given by $x_{n+1} = Ax_n$. Sketch the phase portrait, determine the fixed and periodic points. Discuss the stability properties of the fixed point $x = 0$.

Please hand in until 1.6.2007.