Dynkin Quivers Revisited II

Representation Theory Seminar
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Abstract

In this talk, we follow Ringel [7] to construct the indecomposable representations of quivers of type E using the magic square of Freudenthal and Tits. Moreover, we construct infinitely many pairwise non-isomorphic indecomposable representations for non-Dynkin quivers using the four-subspace quiver.

Contents

1 The indecomposable representations of a type E Dynkin quiver

1.1 The Freudenthal-Tits magic square

In the last talk we saw an elementary reasoning that every Dynkin quiver of type A or D is representation-finite. The reasoning came with a construction of the indecomposable representations using hammocks and conical representations. In the first part of this talk we wish to construct the indecomposable representations of every Dynkin quiver of type E. The construction features the *Freudenthal-Tits magic square.*

Freudenthal [2, 3, 4, 5] and Tits [8] independently gave a construction of the exceptional Lie algebras from real division algebras. We can visualize the construction by a 4×4 square matrix. In this section we would like to present a simplified construction of the exceptional Lie algebras due to Vinberg [9]. First of all, Frobenius's theorem asserts that up to isomorphism there are only three associative finite-dimensional real division algebras. The algebras are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$; their dimensions are 1, 2 and 4. Using toplogical methods Kervaire [6] and Bott-Milnor [1] independently proved that up to isomorphism there are only four (not necessarily associative) finite-dimensional real division algebras. The algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$; their dimensions are 1, 2, 4 and 8.

Suppose that $\mathfrak{a}, \mathfrak{b} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. We let $\mathrm{Der}(\mathfrak{a})$ denote the set of *derivations*, i. e. the set of $\mathbb{R}$-linear maps $D\colon \mathfrak{a} \to \mathfrak{a}$ such that the Leibniz rule $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in \mathfrak{a}$ and $D(x) = 0$ holds for all $x \in \mathbb{R}$. For example, $\mathrm{Der}(\mathbb{R}) = \mathrm{Der}(\mathbb{C}) = 0$. The vector space $\mathrm{Der}(\mathfrak{a})$ becomes a Lie algebra via the commutator. One can show that $\mathrm{Der}(\mathbb{H}) \cong \mathfrak{so}(3)$ and $\mathrm{Der}(\mathbb{O}) \cong \mathfrak{g}_2$. Furthermore, we denote by $\mathfrak{sa}_{\mathfrak{a}\otimes\mathfrak{b}}(3)$ the vector space of traceless skew-Hermitian matrices with entries in $\mathfrak{a} \otimes_{\mathbb{R}} \mathfrak{b}$. Vinberg [9] endows the vector space

$$V(\mathfrak{a}, \mathfrak{b}) = \mathrm{Der}(\mathfrak{a}) \oplus \mathrm{Der}(\mathfrak{b}) \oplus \mathfrak{sa}_{\mathfrak{a}\otimes\mathfrak{b}}(3)$$

with a Lie bracket such that $\mathrm{Der}(\mathfrak{a})$ and $\mathrm{Der}(\mathfrak{a})$ are commuting Lie subalgebras and for every derivation $D \in \mathrm{Der}(\mathfrak{a}) \cup \mathrm{Der}(\mathfrak{b})$ and every matrix $x \in \mathfrak{sa}_{\mathfrak{a}\otimes\mathfrak{b}}(3)$ the Lie bracket $[D, x]$ is given by applying D to the entries of x.

Theorem 1.1 (Vinberg)**.** The Vinberg Lie algebra $V(\mathfrak{a},\mathfrak{b})$ is isomorphic to the corresponding entry in the Freudenthal-Tits magic square in Figure 1.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sp}(3)$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{su}(3)$	$\mathfrak{su}(3)\times\mathfrak{su}(3)$	$\mathfrak{su}(6)$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{sp}(3)$	$\mathfrak{su}(6)$	$\mathfrak{so}(12)$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	A_1	A_2	C_3	F_4
$\mathbb{C}$	A_2	$A_2\times A_2$	A_5	E_6
$\mathbb{H}$	C_3	A_5	D_6	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

Figure 1: The Freudenthal-Tits magic square

1.2 An inductive construction of the indecomposable representations

Let Δ be a Dynkin diagram of type E_n with $n\in\{6,7,8\}$. We define diagrams Δ' and Δ'' according to the following submatrix of the Freudenthal-Tits magic square:

n	Δ''	Δ'	Δ
6	$A_2\times A_2$	A_5	E_6
7	A_5	D_6	E_7
8	E_6	E_7	E_8

Figure 2: Ringel's subdiagrams

Proposition 1.2. (a) There is a unique vertex y of Δ such that Δ' is obtained from Δ by removing y and all edges incident with y. The vertex y is called the *exceptional vertex* of Δ.

(b) The exceptional vertex y is adjacent to exactly one vertex z in Δ. The diagram Δ'' is obtained from Δ' by removing the vertex z together with the edges incident with z.

Let k be a field and let Q be a quiver with underlying undirected diagram Δ. We denote by $\operatorname{rep}_k(Q)$ the category of finite-dimensional representation of Q over k and by $\operatorname{ind}_k(Q)$ the set of indecomposable representations of Q over k. We consider the full subquivers Q' and Q'' of Q with underlying undirected diagrams Δ' and Δ''. When we are interested in representation-finiteness we may assume that all arrows are oriented towards the central vertex without loss of generality by a proposition in the last talk. Especially, the exceptional vertex y is a source in Q.

Given a representation X of Q. Recall that the objects in the *hammock category* $\mathcal{H}(X,Q)$ are the representations of Q; for two representations $M,N\in\operatorname{rep}_k(Q)$ the set of morphisms is given by $\operatorname{Hom}_{\mathcal{H}(X,Q)}(M,N)=\operatorname{Hom}_{kQ}(M,N)/\simeq$ where we define $\varphi\simeq\varphi'$ if $\operatorname{Hom}_{kQ}(X,\varphi-\varphi')=0$. Note that $M\in\mathcal{H}(X,Q)$ is zero if and only if $\operatorname{Hom}_{kQ}(X,M)=0$.

Theorem 1.3 (Ringel)**.** Suppose that $M\in\operatorname{rep}_k(Q)$ is indecomposable. Then exactly one of the following six statements is true:

(1) The support of M is contained in Q''. In this case we may view M as an element in $\operatorname{ind}_k(Q'')$.

(2) The support of M is contained in Q', but not in Q''. In this case $0\neq\dim_k(M_z)=\dim_k\operatorname{Hom}_{kQ}(P(z),M)$. Hence we may view M as an element in the hammock category $\mathcal{H}(P(z),Q')$.

(3) We have $\dim_k(M_y)=1$ and the restriction $N=M|_{Q'}$ of M to Q' is an indecomposable object in the hammock category $\mathcal{H}(P(z),Q')$.

(4) We have $\dim_k(M_y)=1$ and the restriction $N=M|_{Q'}$ of M to Q' is isomorphic to a direct sum $N=N_1\oplus N_2$ of two indecomposables object N_1,N_2 in the hammock category $\mathcal{H}(P(z),Q')$ such that $\operatorname{Hom}_{kQ'}(N_1,N_2)=0=\operatorname{Hom}_{kQ'}(N_2,N_1)$.

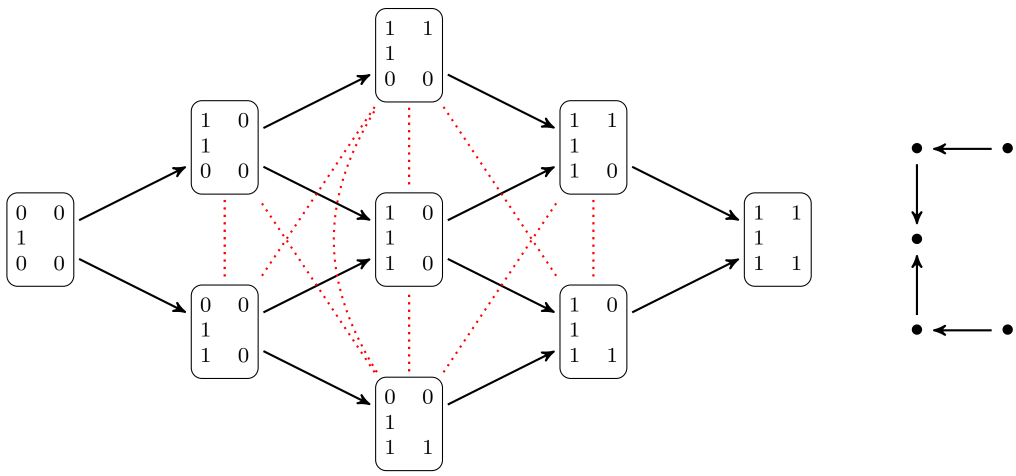

Figure 3: The hammock category $\mathcal{H}(P(z), Q')$ for Q of type E_6

(5) We have $\dim_k(M_y) \in \{1, 2\}$ and the restriction $N = M|_{Q'}$ of M to Q' is isomorphic to a direct sum $N = N_1 \oplus N_2 \oplus N_3$ of three indecomposables object N_1, N_2, N_3 in the hammock category $\mathcal{H}(P(z), Q')$ such that $\mathrm{Hom}_{kQ'}(N_i, N_j) = 0$ for all $i \neq j$. In the case the triple (N_1, N_2, N_3) is called a *special antichain triple*. Furthermore, up to isomorphism and reordering there is only one special antichain triple in the hammock category $\mathcal{H}(P(z), Q')$. The special antichain triple obeys the relation $\mathrm{Ext}_{kQ'}(N_i, N_j) = 0$ for all $i \neq j$.

(6) The representation M is isomorphic to the simple representation $S(y)$.

In particular, $\mathrm{rep}_k(Q)$ is representation finite. Figures 3 and 5 illustrate Ringel's theorem in the case $\Delta = E_6$ and $\Delta' = A_5$. The hammock category $\mathcal{H}(P(z), Q')$ contains 9 indecomposable objects; a red edge indicates when there are no non-zero morphisms between the indecomposable objects. In Figure 5 indecomposable objects are colored red (case 1), green (case 2), blue (case 3), yellow (case 4), grey (case 5) and orange (case 6).

2 Representation-finite quivers are Dynkin

2.1 Cross ratios

Let k be an infinite field and let Q be the *four subspace quiver* of type $\tilde{D}_4$ as defined in Figure 4. We consider the dimension vector $\mathbf{d} = (1, 1, 1, 1, 2) \in \mathbb{N}^5$. It is easy to see that if a representation $M \in \mathrm{rep}_k(Q, \mathbf{d})$ is indecomposable, then the linear maps M_a with $\varphi \in \{\alpha, \beta, \gamma, \delta\}$ are injective. Furthermore, if $M, N \in \mathrm{rep}_k(Q, \mathbf{d})$ are two indecomposable representations with $M_\varphi(k) = N_\varphi(k)$ for all $\varphi \in \{\alpha, \beta, \gamma, \delta\}$, then M and N are isomorphic. Hence any indecomposable representation $M \in \mathrm{rep}_K(Q, \mathbf{d})$ is determined up to isomorphism by the images $a, b, c, d \in \mathbb{P}^1_k$ of k under $M_\alpha, M_\beta, M_\gamma, M_\delta$. In this case we write $M = M(a, b, c, d)$.

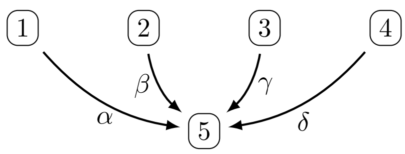

Figure 4: The four subspace quiver

Note that the group of automorphisms of the projective line $\mathbb{P}^1_k$ is isomorphic to $\mathrm{Aut}(\mathbb{P}^1_k) \cong \mathrm{PGL}_k(2)$. Let a, b, c, d and a', b', c', d' be points in $\mathbb{P}^1_k$. Then $M(a, b, c, d)$ and $M(a', b', c', d')$ are isomorphic if and only if there is an automorphism $\Phi \in \mathrm{Aut}(\mathbb{P}^1_k)$ such that $a' = \Phi(a)$, $b' = \Phi(b)$, $c' = \Phi(c)$ and $d' = \Phi(d)$. The group $\mathrm{Aut}(\mathbb{P}^1_k)$ acts 3-transitively on $\mathbb{P}^1_k$. In particular, for every three pairwise different points $a, b, c \in \mathbb{P}^1_k$

there is a unique projective transformation $\Phi \in \mathrm{Aut}(\mathbb{P}^1_k)$ such that $\Phi(a) = 0$, $\Phi(b) = 1$ and $\Phi(c) = \infty$ (where we view k as a subset of $\mathbb{P}^1_k$ under the embedding $z \mapsto (1, z)$).

Definition 2.1. Suppose that $a, b, c, d \in \mathbb{P}^1_k$ are pairwise different. The *cross ratio* of the quadruple (a, b, c, d) is $\Phi(d) \in k$ where Φ is the unique projective automorphism such that $\Phi(a) = 0$, $\Phi(b) = 1$ and $\Phi(c) = \infty$.

It follows from the definition that the cross ratio is invariant under projective transformations. Moreover, two representations $M(a, b, c, d)$ and $M(a', b', c', d')$, where a, b, c, d and a', b', c', d' are four pairwise different points, respectively, are isomorphic if and only if the cross ratios of (a, b, c, d) and (a', b', c', d') coincide. In particular, the set $\mathrm{rep}_k(Q, \mathbf{d})$ contains infinitely many pairwise non-isomorphic representations.

Remark 2.2. The name cross ratio comes from the following geometric construction due to Pappus of Alexandria. Suppose that $k = \mathbb{R}$. Choose a line $l \subseteq \mathbb{R}^2$ with $0 \notin l$ which is not parallel to any of the lines $a, b, c, d \subseteq \mathbb{R}^2$. It meets a, b, c, d in points $A, B, C, D \in \mathbb{R}^2$. Then the cross ratio of (a, b, c, d) is equal to the ratio $(\overline{AC} \cdot \overline{BD})/(\overline{AD} \cdot \overline{BC})$ for every choice of l.

2.2 Thick subcategories of type $\tilde{D}_4$

Let k be an infinite field. Suppose that $\tilde{Q}$ is a connected quiver such that the underlying undirected diagram is not Dynkin. We want to prove that $\tilde{Q}$ is representation infinite.

Lemma 2.3 (Folklore)**.** The quiver $\tilde{Q}$ contains a full subquiver whose underlying undirected diagram is an extended Dynkin diagram of type $\tilde{A}_n$ (with $n \neq 1$), $\tilde{D}_n$ (with $n \geq 4$) or $\tilde{E}_n$ (with $n \in \{6, 7, 8\}$).

Using the Jordan canonical form one can show that $\mathrm{rep}_k(\tilde{Q})$ is representation infinite if the underlying diagram of $\tilde{Q}$ contains an extended Dynkin diagram of type $\tilde{A}$. It is easy to see that $\mathrm{rep}_k(\tilde{Q})$ contains the module category of a quiver of type $\tilde{D}_4$ as a thick subcategory (i. e. an exact abelian subcategory closed under extensions) if $\tilde{Q}$ is an orientation of a Dynkin diagram of type $\tilde{D}$. Without loss of generality we may therefore assume that the underlying undirected diagram of $\tilde{Q}$ has type $\tilde{E}$ and that all arrows are oriented towards the central vertex. Note that the extending vertex x of $\tilde{Q}$ is adjacent to the special vertex of y of the corresponding Dynkin quiver Q.

Theorem 2.4 (Ringel)**.** Let (N_1, N_2, N_3) be the special antichain triple of Q (with support in Q'). We consider the subcategory $\mathcal{E}$ of $\mathrm{rep}_k(\tilde{Q})$ of all representation of $\tilde{Q}$ that admit a filtration with factors isomorphic to N_1, N_2, N_3, $S(x)$ or $S(y)$. Then $\mathcal{E}$ is a thick subcategory and it is equivalent to the module category of a quiver of type $\tilde{D}_4$.

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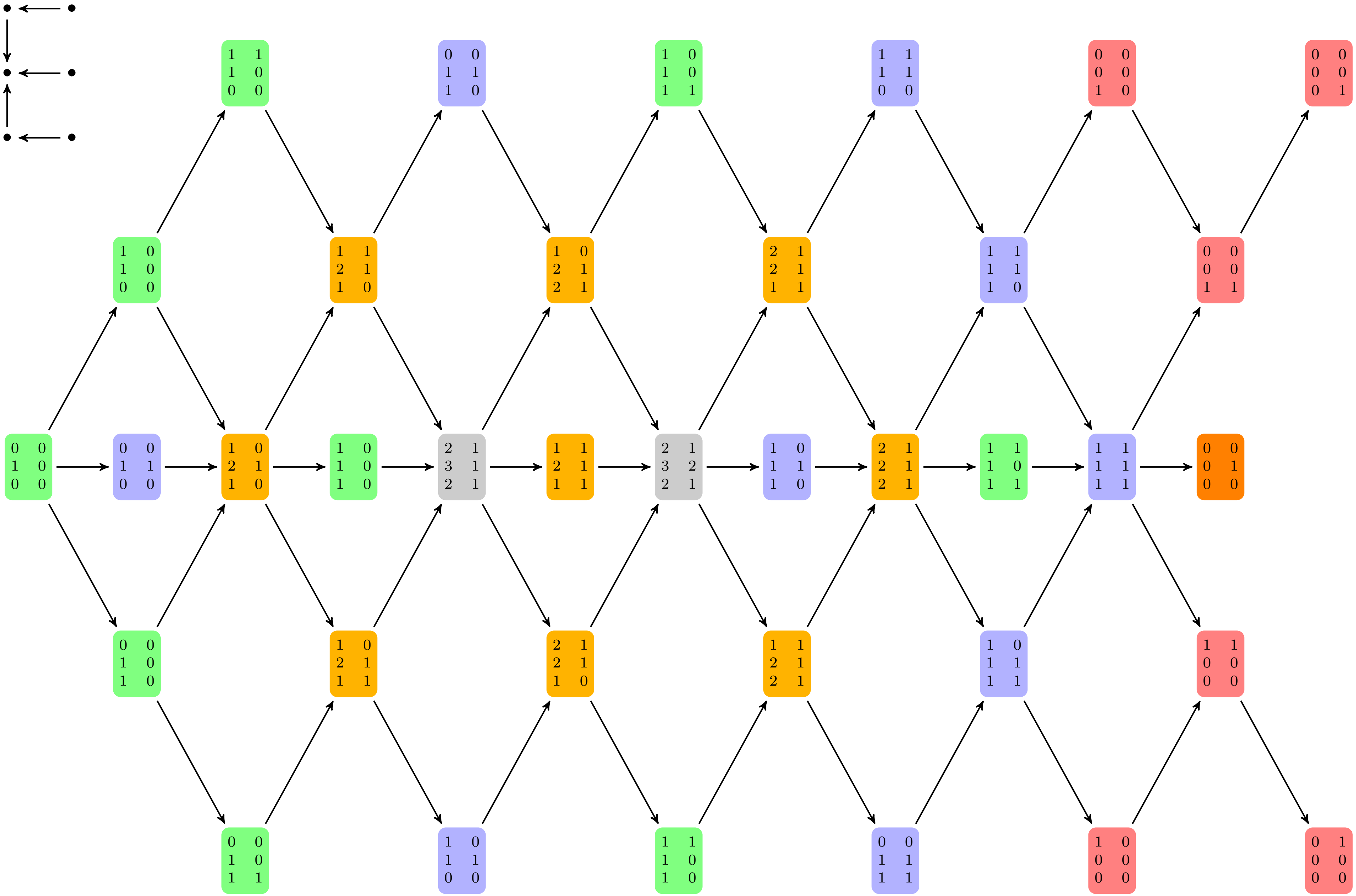

Figure 5: The indecomposable representations of a Dynkin quiver of type E_6