CHASE'S LEMMA AND ITS CONTEXT

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Abstract. Chase's lemma provides a powerful tool for translating properties of (co)products in abelian categories into chain conditions. This note discusses the context in which the lemma is used, making explicit what is often neglected in the literature because of its technical nature.

1. The context of Chase's lemma

In this note we discuss a technical lemma due to Chase [3, 4] which provides a relation between direct products and direct sums of modules. This lemma has several interesting consequences. Chase used this for the study of products of projective modules, but it was then noticed by Gruson and Jensen [9] that it also applies to the study of Σ-pure-injective modules. Recall that a module X is Σ-*pure-injective* if any coproduct of copies of X is pure-injective.

Theorem 1 (Gruson–Jensen). *For a module X over a ring the following conditions are equivalent:*

(1) *The module X is Σ-pure-injective.*
(2) *There exists a cardinal κ such that every product of copies of X is a pure submodule of a coproduct of modules of cardinality at most κ.*
(3) *There exists a module Y such that every product of copies of X is a pure submodule of a coproduct of copies of Y.*

The equivalence (1) $\Leftrightarrow$ (2) is stated in [9] and the proof is rather short; it says (2) $\Rightarrow$ (1) *s'obtiennent par extension des méthodes de* [3]. Further equivalent conditions are formulated in [9] and also studied in independent work by Zimmermann [18] and Zimmermann-Huisgen [19]. There are many references to this result in the literature, but it took more than 20 years until a full proof was published by Huisgen-Zimmermann [10, Theorem 10] in a special volume devoted to infinite length modules [13], using Chase's lemma.

Condition (3) is actually useful in other categorical settings where no obvious notion of cardinality is available. Clearly, (2) and (3) are equivalent, because the isomorphism classes of modules of cardinality bounded by κ form a set and we can take the coproduct of a set of representatives.

Replacing elements of modules with morphisms, Chase's lemma can be formulated more generally for abelian categories, cf. Lemma 4. Then one obtains as a consequence a characterisation of locally noetherian Grothendieck categories which is due to Roos [16]. In particular, we see that *properties of (co)products translate into chain conditions*; this seems to be the real essence of Chase's lemma.

Theorem 2 (Roos). *A locally finitely generated Grothendieck category is locally noetherian (so has a generating set of noetherian objects) if and only if there is an object E such that every object is a subobject of a coproduct of copies of E.*

For such a cogenerating object E we have that every product of copies of E is a subobject of a coproduct of copies of E. Also, we may assume that E is injective,

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because one may replace E with its injective envelope. Then E satisfies condition (3) in Theorem 1, and this yields a first connection between the two theorems.

Getting back to work of Gruson and Jensen [8] one knows that for any ring Λ the fully faithful *transfer functor*

$$T\colon \operatorname{Mod}\Lambda \longrightarrow \operatorname{Add}(\operatorname{mod}(\Lambda^{\mathrm{op}}),\mathrm{Ab}), \qquad X \mapsto X\otimes_\Lambda -$$

identifies pure-injective Λ-modules with injective objects in the category of additive functors $\operatorname{mod}(\Lambda^{\mathrm{op}}) \to \mathrm{Ab}$. Here, $\operatorname{Mod}\Lambda$ denotes the category of right Λ-modules and $\operatorname{mod}\Lambda$ the full subcategory of finitely presented modules. In particular, a Λ-module X is Σ-pure-injective if and only if any coproduct of copies of $T(X)$ is injective.

In order to explain the relevance of Chase's lemma and the close connection between the two theorems we adopt a more general approach, following Crawley-Boevey [5]. We fix a locally finitely presented additive category $\mathcal{A}$ and have a fully faithful functor

$$T\colon \mathcal{A} \longrightarrow \mathbf{P}(\mathcal{A})$$

into its *purity category* $\mathbf{P}(\mathcal{A})$ which is a locally finitely presented Grothendieck category.[1] The functor T preserves all (co)products and identifies pure-injective objects in $\mathcal{A}$ with injective objects in $\mathbf{P}(\mathcal{A})$ [5, §3]. For example, we can take $\mathcal{A} = \operatorname{Mod}\Lambda$ for a ring Λ and then the functor $\mathcal{A} \to \mathbf{P}(\mathcal{A})$ identifies with the above functor $X \mapsto X\otimes_\Lambda -$.

Each object $X \in \mathcal{A}$ gives rise to a localising subcategory $\mathcal{C}_X \subseteq \mathbf{P}(\mathcal{A})$ that is generated by all finitely presented objects $C \in \mathbf{P}(\mathcal{A})$ satisfying $\operatorname{Hom}(C, T(X)) = 0$. We write $\operatorname{Prod} X$ for the full subcategory of products of copies of X and their direct summands.

Lemma 3. *An object $X \in \mathcal{A}$ is Σ-pure-injective if and only if the localised category $\mathbf{P}(\mathcal{A})/\mathcal{C}_X$ is locally noetherian. In this case T induces an equivalence*

$$\operatorname{Prod} X \xrightarrow{\sim} \operatorname{Inj}(\mathbf{P}(\mathcal{A})/\mathcal{C}_X)$$

onto the full subcategory of injective objects in $\mathbf{P}(\mathcal{A})/\mathcal{C}_X$.

This lemma is useful because properties of Σ-pure-injective objects (for example essentially unique decompositions into indecomposable objects) can now be derived from a well developed theory of injective objects in locally noetherian Grothendieck categories [7]. For a proof we refer to [11, §9], which combines the ideas from [5, 8, 17] with the localisation theory for Grothendieck categories [7].

The above approach towards the study of pure-injectivity works equally well for a compactly generated triangulated category $\mathcal{T}$ via the *restricted Yoneda functor*

$$\mathcal{T} \longrightarrow \operatorname{Add}((\mathcal{T}^c)^{\mathrm{op}},\mathrm{Ab}), \qquad X \mapsto \operatorname{Hom}(-,X)$$

where $\mathcal{T}^c$ denotes the full subcategory of compact objects [12].

2. Chase's lemma for additive categories

In the context of modules over a ring, Chase's lemma goes back to an argument in the proof of Theorem 3.1 in [3], though it is not stated explicitly as a lemma. In a subsequent paper [4] Chase formulated this as follows.

[1] $\mathbf{P}(\mathcal{A}) = \operatorname{Lex}(\operatorname{Fp}(\operatorname{fp}\mathcal{A},\mathrm{Ab}),\mathrm{Ab})$, where $\operatorname{fp}\mathcal{A}$ denotes the full subcategory of finitely presented objects in $\mathcal{A}$, $\operatorname{Fp}(\mathcal{C},\mathrm{Ab})$ denotes the abelian category of finitely presented functors $\mathcal{C} \to \mathrm{Ab}$, and $\operatorname{Lex}(\mathcal{D},\mathrm{Ab})$ denotes the category of additive functors $\mathcal{D} \to \mathrm{Ab}$ which send each short exact sequence in $\mathcal{D}$ to a left exact sequence in Ab. The functor T maps $X \in \operatorname{fp}\mathcal{A}$ to $\operatorname{Hom}(\operatorname{Hom}(X,-),-)$ and preserves filtered colimits.

Theorem 1.2. *Let $A^{(1)}, A^{(2)}, \cdots$ be a sequence of left modules over a ring R, and set $A = \prod_{i=1}^{\infty} A^{(i)}$, $A_n = \prod_{i=n+1}^{\infty} A^{(i)}$. Let $C = \sum_{\alpha} \oplus C_\alpha$, where $\{C_\alpha\}$ is a family of left R-modules and α traces an index set I. Let $f\colon A \to C$ be an R-homomorphism, and denote by $f_\alpha\colon A \to C_\alpha$ the composition of f with the projection of C onto C_α. Finally, let $\mathscr{F}$ be a filter of principal right ideals of R. Then there exists aR in $\mathscr{F}$ and an integer $n > 0$ such that $f_\alpha(aA_n) \subseteqq \bigcap_{b\hat{R}\in\mathscr{F}} bC_\alpha$ for all but a finite number of α in I.*

We continue with a version of Chase's lemma for additive categories which seems to be new. For a sequence of morphisms $\gamma = (C_n \to C_{n+1})_{n\in\mathbb{N}}$ we denote by $\gamma_n\colon C_0 \to C_n$ the composite of the first n morphisms. An object C is called *finitely generated* if any morphism $C \to \coprod_{i\in I} X_i$ factors through $\coprod_{i\in J} X_i$ for some finite subset $J \subseteq I$.

Lemma 4 (Chase). *Let $(X_n)_{n\in\mathbb{N}}$ and $(Y_i)_{i\in I}$ be families of objects in an additive category and*

$$\phi\colon \prod_{n\in\mathbb{N}} X_n \longrightarrow \coprod_{i\in I} Y_i$$

a morphism. If $\gamma = (C_n \to C_{n+1})_{n\in\mathbb{N}}$ is a sequence of morphisms and $C = C_0$ is finitely generated, then there exists $m \in \mathbb{N}$ such that for almost all $j \in I$ each composite

$$C \xrightarrow{\gamma_m} C_m \xrightarrow{\theta} \prod_{n\in\mathbb{N}} X_n \xrightarrow{\phi} \coprod_{i\in I} Y_i \twoheadrightarrow Y_j$$

with $\theta_n = 0$ for $n < m$ factors through $\gamma_n\colon C \to C_n$ for all $n \in \mathbb{N}$.

It is convenient to introduce further notation. For a morphism $\gamma\colon C \to D$ and an object X we denote by X_γ the image of the map

$$\operatorname{Hom}(D, X) \xrightarrow{-\circ\gamma} \operatorname{Hom}(C, X).$$

Then a sequence of morphisms $\gamma = (C_n \to C_{n+1})_{n\in\mathbb{N}}$ yields a descending chain

$$\cdots \subseteq X_{\gamma_2} \subseteq X_{\gamma_1} \subseteq X_{\gamma_0} = \operatorname{Hom}(C_0, X).$$

We can now rephrase the statement of the lemma as follows. *There exists $m \in \mathbb{N}$ such that*

$$\phi_i\big((\prod_{n\geq m} X_n)_{\gamma_m}\big) \subseteq \bigcap_{n\geq 0} (Y_i)_{\gamma_n}$$

for almost all $i \in I$, where

$$\phi_i\colon \operatorname{Hom}(C, X) \xrightarrow{\phi\circ -} \operatorname{Hom}(C, Y) \longrightarrow \operatorname{Hom}(C, Y_i).$$

Proof. We follow closely the proof of Theorem 1.2 in [4]. Assume the conclusion to be false. We set $X = \prod_{n\in\mathbb{N}} X_n$ and construct inductively sequences of elements $n_j \in \mathbb{N}$, $i_j \in I$, and $\theta_j \in \operatorname{Hom}(C, X)$ with $j \in \mathbb{N}$ and satisfying

(1) $n_{j+1} > n_j$,
(2) $\theta_j \in (\prod_{n\geq n_j} X_n)_{\gamma_{n_j}}$,
(3) $\phi_{i_j}(\theta_j) \notin (Y_{i_j})_{\gamma_{n_{j+1}}}$,
(4) $\phi_{i_j}(\theta_k) = 0$ for $k < j$.

We proceed as follows. Set $n_0 = 0$. Then there exists $i_0 \in I$ such that

$$\phi_{i_0}(X_{\gamma_0}) \not\subseteq \bigcap_{n\geq 0} (Y_{i_0})_{\gamma_n},$$

and hence we may select $\theta_0 \in X_{\gamma_0}$ and $n_1 > 0$ such that $\phi_{i_0}(\theta_0) \not\in (Y_{i_0})_{\gamma_{n_1}}$. Thus conditions (1)–(4) are satisfied for $j = 0$.

Proceeding by induction on j, assume that elements $n_{k+1} \in \mathbb{N}$, $i_k \in I$ and $\theta_k \in \operatorname{Hom}(C, X)$ have been constructed for $k < j$ such that conditions (1)–(4) are satisfied. Using that C is finitely generated, there exists a finite subset $I' \subseteq I$ such that for $i \in I \setminus I'$ we have $\phi_i(\theta_k) = 0$ for $k < j$. We may then select $i_j \in I \setminus I'$ such that

$$\phi_{i_j}\big((\prod_{n \geq n_j} X_n)_{\gamma_{n_j}}\big) \not\subseteq \bigcap_{n \geq 0} (Y_{i_j})_{\gamma_n},$$

because otherwise the lemma would be true. Thus there exists $\theta_j \in (\prod_{n \geq n_j} X_n)_{\gamma_{n_j}}$ and $n_{j+1} > n_j$ such that $\phi_{i_j}(\theta_j) \not\in (Y_{i_j})_{\gamma_{n_{j+1}}}$. It is then clear that the elements $n_{k+1} \in \mathbb{N}$, $i_k \in I$, and $\theta_k \in \operatorname{Hom}(C, X)$ for $k \leq j$ satisfy the conditions (1)–(4).

Now let $\theta = \sum_{j \in \mathbb{N}} \theta_j \in \operatorname{Hom}(C, X)$, which is well-defined since the sum for each component $C \to X_n$ is finite. For each $j \in \mathbb{N}$ we have $\phi_{i_j}(\theta) = \phi_{i_j}(\theta_j) + \phi_{i_j}(\sum_{k > j} \theta_k) \neq 0$, since the second summand lies in $(Y_{i_j})_{\gamma_{n_{j+1}}}$, whereas the first does not. On the other hand, the morphism $\phi\theta$ factors through a finite sum $\coprod_{i \in J} Y_i$ for some $J \subseteq I$, since C is finitely generated. This contradiction finishes the proof. □

We include the application from [3] about products of projective modules. Note that the descending chain condition on principal right ideals characterises rings that are left perfect [1].

> **THEOREM 3.1.** *Let R be a ring, and J be an infinite set of cardinality ζ, where $\zeta \geqq \operatorname{card}(R)$. Set $A = \prod_{\alpha \in J} R^{(\alpha)}$, where $R^{(\alpha)} \approx R$ is a left R-module. Suppose that A is a pure submodule of a left R-module of the form $C = \sum_\beta \oplus C_\beta$, where each C_β is generated by a subset of cardinality less than or equal to ζ. Then R must satisfy the descending chain condition on principal right ideals.*

3. Coproducts of injective objects

A motivation for Chase's study of products of projective modules in [3] was the fact that coproducts of injective modules are again injective over a noetherian ring. In fact, this property for right modules characterises right noetherian rings [14, 15]. There are similar results for Grothendieck categories, and this brings us back to Theorem 2. Roos stated this theorem in [16], but again the proof is short: *La démonstration du théorème 1 est analogue à celle du théorème B de* [6]. For this reason it seems appropriate to include a complete proof which is based on Chase's lemma; it is different from that in [6], though the authors do refer to the work of Chase [3].

Proof of Theorem 2. Let $\mathcal{A}$ be a Grothendieck category and fix a generator G. When $\mathcal{A}$ is locally noetherian, then every injective object decomposes into a coproduct of indecomposable objects [7, IV.2]. Each indecomposable injective object arises as injective envelope $E(G/U)$ for some subobject $U \subseteq G$. The subobjects of any object in a Grothendieck form a set. Thus $E = \coprod_{U \subseteq G} E(G/U)$ has the property that every object of $\mathcal{A}$ is a subobject of a coproduct of copies of E, since $\mathcal{A}$ admits injective envelopes.

To prove the converse we need to assume that the Grothendieck category is *locally finitely generated*, so it has a set of finitely generated generators. Let $C \in \mathcal{A}$ be a finitely generated object. We wish to show that C is noetherian. To this end fix a chain of finitely generated subobjects $0 = B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$ and set

$C_n = C/B_n$. This yields a sequence of epimorphisms $\gamma = (C_n \twoheadrightarrow C_{n+1})_{n\in\mathbb{N}}$. For $X \in \mathcal{A}$ we set $X_{\bar{\gamma}_n} = \operatorname{Hom}(B_{n+1}/B_n, X)$ and obtain an exact sequence

$$0 \longrightarrow X_{\gamma_{n+1}} \longrightarrow X_{\gamma_n} \longrightarrow X_{\bar{\gamma}_n} \longrightarrow 0$$

provided that X is injective or a coproduct of injective objects.

Now consider a cogenerator E such that each object of $\mathcal{A}$ embeds into a coproduct of copies of E. We may assume that E is injective by replacing E with its injective envelope. Let $\kappa = \max(\aleph_0, \operatorname{card}\operatorname{Hom}(C, E))$ and choose a monomorphism

$$\phi\colon \prod_{n\in\mathbb{N}} E^{\kappa} \longrightarrow \coprod_{i\in I} E.$$

For each $m \in \mathbb{N}$ we apply $\operatorname{Hom}(C_m, -)$ and obtain a monomorphism

$$\phi_m\colon \prod_{n\in\mathbb{N}} (E_{\gamma_m})^{\kappa} \longrightarrow \coprod_{i\in I} E_{\gamma_m}$$

since $X \mapsto X_{\gamma_m}$ preserves products and coproducts. Then it follows from Lemma 4 that for some $m \in \mathbb{N}$ the map ϕ_m restricts to an embedding

$$\prod_{n\geq m} (E_{\gamma_m})^{\kappa} \longrightarrow (\coprod_{i\in J} E_{\gamma_\infty}) \amalg (\coprod_{\text{finite}} E_{\gamma_m})$$

for some cofinite subset $J \subseteq I$, where $E_{\gamma_\infty} = \bigcap_{n\geq 0} E_{\gamma_n}$. Comparing this with ϕ_{m+1} and passing to the quotient yields a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod_{n\geq m} (E_{\gamma_{m+1}})^{\kappa} & \longrightarrow & \prod_{n\geq m} (E_{\gamma_m})^{\kappa} & \longrightarrow & \prod_{n\geq m} (E_{\bar{\gamma}_m})^{\kappa} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\coprod_{i\in J} E_{\gamma_\infty}) \amalg (\coprod_{\text{finite}} E_{\gamma_{m+1}}) & \longrightarrow & (\coprod_{i\in J} E_{\gamma_\infty}) \amalg (\coprod_{\text{finite}} E_{\gamma_m}) & \longrightarrow & \coprod_{\text{finite}} E_{\bar{\gamma}_m} & \longrightarrow & 0
\end{array}$$

where we use the fact that E is injective. The vertical map on the right is a monomorphism because it is a restriction of $\operatorname{Hom}(B_{m+1}/B_m, \phi)$. From the choice of κ it follows that $E_{\bar{\gamma}_m} = 0$, cf. Lemma 5 below. Thus $C_m = C_{m+1}$ since E cogenerates $\mathcal{A}$. We conclude that C is noetherian. □

Lemma 5. *Let A be an abelian group with $\alpha = \operatorname{card} A$ and let $\kappa \geq \max(\aleph_0, \alpha)$. If there is a monomorphism $A^{\kappa} \to A^n$ for some $n \in \mathbb{N}$, then $A = 0$.*

Proof. Suppose $A \neq 0$. Then we have

$$\operatorname{card}(A^{\kappa}) = \alpha^{\kappa} \geq 2^{\kappa} > \kappa = \kappa^n \geq \alpha^n = \operatorname{card}(A^n).$$

This contradicts the fact that there is an injective map $A^{\kappa} \to A^n$. □

Remark 6. The paper of Roos [16] formulates Theorem 2 for Grothendieck categories satisfying Grothendieck's condition (AB6).

We end this note with some further references. Huisgen-Zimmermann provides in [10] a detailed survey about pure-injective modules, emphasising the role of Chase's lemma. For a more recent treatment of Chase's lemma and its generalisations we refer to work of Bergman [2].

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