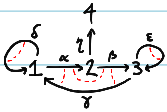
Some descriptions of Σ-pure-injectivity

Henning's talk (last week) included a discussion about equivalent characterisations of Σ-pure-injective objects. Today we will see some uses of these characterisations.



In my previous talk (two weeks ago) we looked at:

$\beta^{-1}\varepsilon^{-1}\gamma^{-1}\delta\gamma\varepsilon^{-1}$

$b_0 \xleftarrow{\zeta} b_1 \xrightarrow{\beta} b_2 \xrightarrow{\varepsilon} b_3 \xrightarrow{\gamma} b_4$

$m_0 \xrightarrow{\beta} m_1 \xrightarrow{\varepsilon} m_2 \xrightarrow{\gamma} m_3$

$d_\delta^{-1}\delta\gamma\varepsilon^{-1}\gamma^{-1}\delta^{-1}\alpha^{-1}d_\alpha$

$\vdots$

- a gentle (or, more generally, string) algebra $\Lambda = KQ/I$
- words w built from the arrows in Q (and their inverses)
- string modules $S(w)$, defined for each w
- a pp-definable subgroup wM for each Λ-module M
- homotopy words C built from paths $\gamma \notin I$ and d_α (arrow α)
- string complexes $S^\bullet(C)$ of projective Λ-modules
- a subspace CM of $M = \bigoplus_{i\in\mathbb{Z}} M^i$ for each complex $M^\bullet$ of proj.'s
- a morphism $a_c : S^\bullet(B) \longrightarrow S^\bullet(D)$ such that, for each $M^\bullet$,

$$CM = \{ f(a_c(e_v)) \mid f : S^\bullet(D) \to M^\bullet \}$$

compact objects of $K(\Lambda\text{-Proj})$ depends only on depends on C

Band modules $B(w,V)$ and band complexes $B^\bullet(C,V)$ are defined in a similar way to string modules and string complexes.

Some significant differences: letters indexed by $\mathbb{Z}$, of the form $\ldots xx\ldots$

- the word w or homotopy word C must be (infinite, and) 'periodic'
- type $\mathbb{A}$ graph depicting construction replaced by type $\tilde{\mathbb{A}}$
- parametised by a $K[T,T^{-1}]$-module V

$$B(w,V) = S(w) \otimes_{K[T,T^{-1}]} V,\quad B^\bullet(C,V) = S^\bullet(C) \otimes_{K[T,T^{-1}]} V$$

Aim for today: explain something about the following results.

possibly infinite dimensional!

Theorem 1 [B-T, Crawley-Boevey '18] Let Λ be a string algebra. Any Σ-pure-injective object in Λ-Mod is a coproduct of modules of the form $S(w)$ and $B(w,V)$ where V is Σ-pure-injective.

must be finite dimensional

Theorem 2 [B-T, 20'] Let Λ be a gentle algebra. Any Σ-pure-injective object in $K(\Lambda\text{-Proj})$ is a coproduct of shifts of complexes of the form $S^{\cdot}(C)$ and $B^{\cdot}(C,V)$ where V is Σ-pure-injective.

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§1: Reminder about purity

In what follows let

$$\mathcal{L} = \begin{cases} \mathcal{L}_{\Lambda\text{-Mod}} = \{0, +, a- \mid a \in \Lambda\} \\ \text{or} \\ \mathcal{L}_{K(\Lambda\text{-Proj})} = \{0_G, +_G, \alpha\circ \mid G, H \in \mathrm{skel}(K^c(\Lambda\text{-Proj})),\ \alpha : G \to H\} \end{cases}$$

... where S = skel$(K^c(\Lambda\text{-Proj}))$ (subcategory of compact objects)

S-sorted language...

Some rough shorthand phrases for this talk → Refer to these as 'the module case' and 'the homotopy case' respectively. Implicitly Λ is a string algebra in the module case, and Λ is a gentle algebra (and f.d) in the homotopy case. Theorem 1 concerns the module case. Theorem 2 concerns the homotopy case.

Objects M in $\mathcal{A}$ define $\mathcal{L}$-structures

Let

$$\mathcal{A} = \begin{cases} \Lambda\text{-Mod (module)} & [M, 0_M, +: M^2 \to M, a- : M \to M] \\ K(\Lambda\text{-Proj}) \text{ (homotopy)} & [\bigsqcup_{G \text{ compact}} \mathrm{Hom}(G,M), 0_G \in \mathrm{Hom}(G,M), +_G : \mathrm{Hom}(G,M)^2 \to \mathrm{Hom}(G,M), \alpha\circ : \mathrm{Hom}(H,M) \to \mathrm{Hom}(G,M)] \end{cases}$$

'underlying sets'

Each morphism in $\mathcal{A}$ defines a homorphism of $\mathcal{L}$-structures*.

* Map between underlying sets which respect predicate/functions in $\mathcal{L}$.

Model theory... Translation into algebra

A homomorphism of $\mathcal{L}$-structures is an $\mathcal{L}$-pure-embedding if solutions to (sorted) pp-formulas are reflected. $L \to M$ is pure provided $M \models \varphi$ implies $L \models \varphi$

Definition: In the module case, an exact sequence $0 \to L \to M \to N \to 0$ in $\mathcal{A}$ is pure-exact if any finite system of linear equations

$$l_i = \sum_{j=1}^{c} a_{ij} x_j \quad (i=1,\dots,r,\ a_{ij} \in \Lambda,\ l_i \in L)$$

which has a solution in M^c must also have a solution in L^c.

Fact: in the module case, $0 \to L \to M \to N \to 0$ is pure exact if and only if, for any finitely presented module G,

$$0 \to \mathrm{Hom}(G,L) \to \mathrm{Hom}(G,M) \to \mathrm{Hom}(G,N) \to 0 \text{ is exact.}$$

Lots of nice theory & results here: for example Ziegler spectrum, ...

Definition [Krause, 00'] In the homotopy (more generally, compactly generated Δ-ated) case, a triangle $L \to M \to N \to L[1]$ is pure-exact provided, for any compact object G in $\mathcal{A}$, the sequence

$$0 \to \mathrm{Hom}(G,L) \to \mathrm{Hom}(G,M) \to \mathrm{Hom}(G,N) \to 0 \text{ is exact.}$$

In both cases, $L \to M$ is then called a pure monomorphism*.

*In the homotopy case, not monic in the categorical sense...

Fact (exercise): In both cases $L \to M$ is pure if and only if the induced homomorphism of $\mathcal{L}$-structures is a pure-embedding.

Due to [Krause, 00'] in homotopy (more general) case

Definition: An object L of $\mathcal{A}$ is called pure-injective if any pure morphism $L \to M$ in $\mathcal{A}$ is a section.

Definition: An object M of $\mathcal{A}$ is called Σ-pure-injective if, for any set I, the coproduct $\coprod_{i \in I} M$ is pure-injective.

We now note some ways to characterise Σ-pure-injectives in the module case and the homotopy case (seperately).

Theorem [Gruson, Jensen 76'], [Zimmerman, 77'], [Huisgen-Zimmerman, 79']*
Consider the module case. TFAE for an object M in $\mathcal{A}$.

Generalised by [Crawley-Boevey, 94'] to locally finitely presented additive categories

(i) M is Σ-pure-injective
(ii) Any descending chain of pp-definable subgroups of M stabilises.
(iii) For any set I the product $\prod_{i \in I} M$ is a coproduct of indecomposable objects with local endomorphism rings.
(iv) There exists a cardinal κ such that, for any set I, the product $\prod_{i \in I} M$ is a coproduct of objects of cardinality at most κ.
(v) ... ← More could be written, but for today these suffice.

Theorem [Krause, 00'], [Krause, Reichenbach 00'], *, [Garcia, Dung 94'], [BT, 20']
Consider the homotopy case. TFAE for an object M in $\mathcal{A}$.

(i) M is Σ-pure-injective
(ii) For any compact object G in $\mathcal{A}$ any descending chain of pp-definable subgroups of M of sort G must stabilise.
(iii) For any set I the product $\prod_{i \in I} M$ is a coproduct of indecomposable objects with local endomorphism rings.
(iv) ... ← As in the module case, more may be written here, superfluous for today.

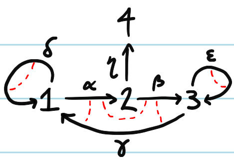
§2: Linear relations

For both the proof of Theorem 1 and the proof of Theorem 2 one can use the so-called functorial filtrations method. To use this method one considers functors built from linear relations. In order to describe objects which are Σ-pure-injective using this method, these linear relations are viewed as representations of the Kronecker quiver. We now explain this.

<u>Definition</u>: For a field K the category K-Rel of linear relations over K is defined as follows:

- the objects of K-Rel are pairs (V, R) where V is a vector space over K and R is a K-subspace of $V \oplus V$
- the morphisms $(f): (V, R) \to (W, S)$ are given by a K-linear map $f: V \to W$ such that $\{(f(x), f(y)) \mid (x,y) \in R\} \subseteq S$.

<u>Examples</u>: recall the gentle algebra $\Lambda = KQ/I$ given by



<u>Example 1</u>: consider the word $w = \gamma\varepsilon\gamma^{-1}\delta$. For any Λ-module M recall

$$wM = \{m_0 \in e_1 M \mid \exists m_1, m_2, m_3, m_4 : m_0 \xleftarrow{\gamma-} m_1 \xleftarrow{\varepsilon-} m_2 \xrightarrow{\gamma-} m_3 \xleftarrow{\delta-} m_4\}$$

Consider instead

$$\underline{w}_M = \{(m_0, m_4) \in e_1 M \oplus e_1 M \mid \exists m_1, m_2, m_3 : \text{———''———}\}$$

Note that $(e_2 M, \underline{w}_M)$ is an object of K-Rel. Furthermore if $\theta: M \to N$ is a homomorphism of Λ-modules then $(\theta): (e_2 M, \underline{w}_M) \to (e_2 N, \underline{w}_N)$ is a morphism in K-Rel.

<u>Example 2</u>: for the homotopy word $C = \gamma^{-1} d_\gamma d_\delta^{-1} \delta\gamma\varepsilon$ we can analogously consider, for any complex $M^\bullet$ in $K_{min}(\Lambda\text{-Proj})$, the object $(e_3 M^0, \underline{C}^0_M)$ of K-Rel given by

$$\underline{C}^0_M = \{(m_0, m_2) \in e_3 M^0 \oplus e_3 M^0 \mid \exists\, m_2 : m_0 \overset{\gamma-}{\mapsto} \gamma m_0 \overset{d_{\gamma,M}}{\longleftarrow\!\shortmid} m_1 \overset{d_{\delta,M}}{\longmapsto} \delta\gamma\varepsilon m_2 \overset{\delta\gamma\varepsilon-}{\longleftarrow\!\shortmid} m_2\}$$

As in Example 1, if $\theta^\bullet : M^\bullet \to N^\bullet$ is a morphism of complexes then we have that $(\theta^0) : (e_3 M^0, \underline{C}^0_M) \to (e_3 N^0, \underline{C}^0_N)$ is a morphism in K-Rel.

<u>Remark</u>: any object (V, R) of K-Rel defines a representation

$$(x,y) \in R \underset{q}{\overset{p}{\rightrightarrows}} V \ni x, y$$

of the Kronecker quiver $\bullet \rightrightarrows \bullet$

In this way one can show K-Rel is equivalent to the full subcategory of the category $K\{\bullet \rightrightarrows \bullet\}$-Mod consisting of representations

$$A \underset{s}{\overset{r}{\rightrightarrows}} B \quad \text{where the map } \binom{r}{s} : A \to B^2 \text{ is injective.}$$

$$a \longmapsto (r(a), s(a))$$

Hence definitions and results from §1 (module case) are inherited, from the category $K\{\bullet \rightrightarrows \bullet\}$-Mod, by the category K-Rel. For example, define (V, R) to be Σ-pure-injective provided the $K\{\bullet \rightrightarrows \bullet\}$-module $R \underset{q}{\overset{p}{\rightrightarrows}} V$ is Σ-pure-injective.

With a view toward Theorem 1 and Theorem 2, we want to associate to each object (V, R) of K-Rel a $K[T, T^{-1}]$-module: in such a way so that this $K[T, T^{-1}]$-module is Σ-pure-injective whenever (V, R) is.

This will be our main focus for the remainder of §2. For each object (V,R) of K-Rel consider the sets

$$R^{\#}=\{x\in V \mid \exists (x_i)_{i\in\mathbb{Z}}\in V^{\mathbb{Z}} : x=x_0,\ (x_i,x_{i+1})\in R\ \forall i\}$$

$$\cdots \leftarrow x_{-2}\leftarrow x_{-1}\leftarrow x_0 \xleftarrow{R} x_1\leftarrow x_2\leftarrow x_3\leftarrow\cdots$$

$(x_0,x_1)\in R$

$$R^{b}=\{y+z \mid y,z\in R^{\#},\ y_i=0 \text{ for } i\gg 0,\ z_j=0 \text{ for } j\ll 0\}$$

$$\cdots 0\leftarrow 0 \leftarrow z_j \leftarrow \cdots \leftarrow z_0 \quad y+z \quad y_0\leftarrow\cdots\leftarrow y_i=0\leftarrow 0\leftarrow 0$$

It is straightforward to check $R^{b}\subseteq R^{\#}\subseteq V$ are subspaces.

Lemma [CB, 18'] For any (V,R) there is a K-linear automorphism of the quotient $R^{\#}/R^{b}$, given by $x+R^{b}\mapsto x'+R^{b}$ iff

$$x'\in R^{\#}\cap(R^{b}+Rx) \text{ where } Rx=\{y\in V \mid (x,y)\in R\}$$

We say that (V,R) is automorphic if both the map $p:R\to V$ and the map $q:R\to V$ are isomorphisms.

canonical projections

For any subspace $U\subseteq V$ let $R|_U=R\cap(U\oplus U)$. Hence one obtains a new object $(U,R|_U)$ in K-Rel.

Key point for this talk: we want to look at examples of relations (V,R) which are Σ-pure-injective...

Theorem [BT, Crawley-Boevey 18'] For any object (V,R) of K-Rel the inclusions

$$(R^{b},R|_{R^{b}})\hookrightarrow(V,R)\hookleftarrow(R^{\#},R|_{R^{\#}})$$

are pure-monomorphisms (considered as $K\{\bullet\rightrightarrows\bullet\}$-modules).

Consequently, if (V,R) is Σ-pure-injective then there is a subspace $U\subseteq V$ such that $R^{\#}=R^{b}\oplus U$, $(U,R|_U)$ is automorphic and the quotient $R^{\#}/R^{b}$ is a Σ-pure-injective $K[T,T^{-1}]$-module.

§3: Purity of relations from words.

Here we use the charaterisations from §1 with the theory from §2, in order to prove the following results – which are used in the proofs of Theorem 1 and Theorem 2.

<u>Lemma</u> [B-T, Crawley-Boevey 18'] In the module case, if M is a Σ-pure injective Λ-module and if for example, $w = \gamma\varepsilon\gamma^{-1}\delta$; then the object $(e_2 M, \underline{w}_M)$ of K-Rel is Σ-pure-injective, and so $\underline{w}_M^{\#}/\underline{w}_M^{b}$ is a Σ-pure-injective $K[T,T^{-1}]$-module.

<u>Proof</u>: By the earlier characterisations of Σ-pure-injectivity from §1, there exists a cardinal κ such that, for any set I, we have

$$\prod_{i\in I} M \simeq \bigoplus_{j\in J} N_j$$

for some set J and modules N_j with $|N_j| \leq \kappa$ for all j.

We then have, for any set I,

$$\begin{aligned}\prod_{i\in I}(e_2 M, \underline{w}_M) &\simeq (e_2 \textstyle\prod_{i\in I} M, \underline{w}_{\Pi M})\\ &\simeq (e_2 \textstyle\bigoplus_{j\in J} N_j, \underline{w}_{\oplus N_j})\\ &\simeq \textstyle\bigoplus_{j\in J}(e_2 N_j, \underline{w}_{N_j})\end{aligned}$$

Finally, note the $K\{\bullet \rightrightarrows \bullet\}$-module

$$e_2 N^j \oplus e_2 N^j \supseteq \underline{w}_{N_j} \overset{p}{\underset{q}{\rightrightarrows}} e_2 N_j$$

has cardinality at most $\mu = 3\kappa$, as required. ▨

There is an analogue of this result in the homotopy case.

We now explain why, in the first talk, we realised the subspaces wM (in the module case) and $CM^{\cdot}$ (homotopy case) in terms of pp-definable subgroups.

each $C_i = \begin{cases} p^{-1}d_x \\ \text{or} \\ d_x^{-1}p \end{cases}$
where
$p = xq$

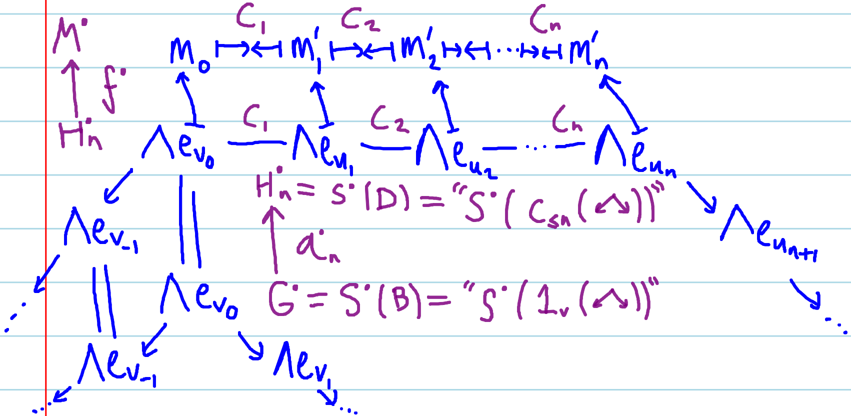
Lemma [B-T, 20'] In the homotopy case, suppose M° is a Σ-pure-injective object of $K_{min}(\Lambda\text{-Proj})$.
Let $C = C_1C_2C_3\ldots$ be an infinite homotopy word.
Define truncations of C, such as
$$C_{\leq i} = C_1\ldots C_i \quad \text{and} \quad C_{>i} = C_{i+1}C_{i+2}\ldots \quad \forall i\geq 1.$$
Then we have
$$\bigcap_{n\geq 1} C_{\leq n}M = \left\{m_0 \,\middle|\, \exists m_i (i\geq 1): \begin{pmatrix} m_i \overset{p^-}{\mapsto} pm_i \overset{d_x}{\leftarrow\!\shortmid} m_{i+1} & \text{if } C_i = p^{-1}d_x \\ \text{or} & \\ m_i \overset{d_x}{\mapsto} pm_{i+1} \overset{p^-}{\leftarrow\!\shortmid} m_{i+1} & \text{if } C_i = d_x^{-1}p \end{pmatrix} \forall i\geq 0\right\}$$
$$m_0 \mapsto\ \leftarrow\!\shortmid m_1 \mapsto\ \leftarrow\!\shortmid m_2 \mapsto\ \leftarrow\!\shortmid m_3 \mapsto\ \leftarrow\!\shortmid \ldots$$

Proof (rough sketch):
Assuming $m_0 \in$ RHS then, $\forall n\geq 1$, the existence of $m_1,\ldots,m_n$ shows that $m_0 \in C_{\leq n}M$. Converseley suppose $m \in$ LHS.

Let $m_0 = m$. For each $n\geq 1$ let $\Delta_n = C_1^{-1}m_0 \cap (C_{>1})_{\leq n}M$, where $C_1^{-1}m_0$ is the set of m_1' such that $m_0 \overset{C_1}{\mapsto\!\leftarrow\!\shortmid} m_1'$.

$CM = \{f(a_c(e_v)) \mid f: S^\circ(D)\to M^\circ\}$
$a_c: S^\circ(B)\to S^\circ(D) \quad G^\circ = S^\circ(B)$

By assumption each Δ_n is non-empty. By the discussion in the first talk, each Δ_n is a coset of a subspace given by a pp-definable subgroup of M° of sort G°



Note that G° is independant of n, although the subgroup that corresponds to Δ_n is given by a map $a_n^\circ : G^\circ \to H_n^\circ$, whose codomain depends on n.

Note that $\Delta_1 \supseteq \Delta_2 \supseteq \Delta_3 \supseteq \dots$, hence any finite intersection of the Δ_n's is non-empty.

Since M° is Σ-pure-injective, by the characterisations in §1, we have that any descending chain of pp-definable subgroups of M° of sort G° must stabilise. Altogether this means $\bigcap_{n\geq 1} \Delta_n \neq \emptyset$. Hence choose $m_1 \in \bigcap_{n\geq 1} \Delta_n$.

Iterating this argument defines the required sequence $m_0, m_1, m_2, \dots$ to ensure $m \in$ RHS. ▨

There is an analogue of this result in the module case.

We conclude by noting a current limitation to both Theorem 1 and Theorem 2: although we have that any Σ-pure-injective is a coproduct of string and band modules/complexes, it is unclear which coproducts of strings and bands are Σ-pure-injective:

- for the module case, a band module $B(w, V)$ is Σ-pure-injective if and only if the $K[T, T^{-1}]$-module V is; and in the thesis of [Harland, '11] a set of conditions on the word w are given which are equivalent to saying $S(w)$ is Σ-pure-injective

- in general a finite coproduct of Σ-pure-injectives is Σ-pure injective, but this is false ~~in general~~ in the infinite case...

- for the homotopy case, less is known; in case Λ is derived-discrete, [Arnesen, Laking, Pauksztello, Prest, '17] showed any string complex is pure-injective...

Thankyou for your attention!