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HOMOLOGICAL CONJECTURES

1. The homological conjectures

We follow the unpublished notes of Happel [Hap90] and the literature from Jan Schröer's short summaries (on his webpage).

Let Λ be a finite-dimensional algebra over an algebraically closed field k. We denote by Λ-mod the category of finite-dimensional left Λ-modules.

(findim) Finitistic dimension conjecture.

The finitistic dimension, defined as follows

$$\operatorname{findim}(\Lambda) = \sup\{\operatorname{pd} X \mid X \in \Lambda\text{-mod}, \operatorname{pd} X < \infty\}$$

is finite.

(VC) Vanishing Conjecture.

$$\mathrm{K}^{\mathrm{b}}(\operatorname{add}(\mathrm{D}\,\Lambda))^{\perp} = \{0\}$$

(NuC) Nunke Condition (also called *strong Nakayama conjecture*).

For every nonzero X in Λ-mod there is an $i \geq 0$ such that $\operatorname{Ext}^i_\Lambda(\mathrm{D}\,\Lambda, X) \neq 0$.

(CC) Complement conjecture.

The number of complements of an almost complete tilting module is finite.

(GNC) Generalized Nakayama conjecture.

Let $0 \to {}_\Lambda\Lambda \to I^0 \to I^1 \to \cdots$ be a minimal injective coresolution in Λ-mod. Then all indecomposable injectives appear as a summand for an I^n for some n.

(KMC) Kvamme-Marczinzik conjecture.

Let $\cdots P_1 \to P_0 \to \mathrm{D}\,\Lambda \to 0$ be a minimal projective resolution. If $\operatorname{id} P_n \leq n+1$ for all $n \geq 0$, then Λ is Co-Goremstein, i.e. $\Omega^\infty(\Lambda\text{-mod}) \subset \ker \operatorname{Ext}^1(-, \Lambda)$.

(NC) Nakayama conjecture.

Let $0 \to {}_\Lambda\Lambda \to I^0 \to I^1 \to \cdots$ be a minimal injective coresolution in Λ-mod. If all I^n are projective, then Λ is self-injective, or shortly:

$$\operatorname{domdim}({}_\Lambda\Lambda) = \infty \Longrightarrow \operatorname{id}{}_\Lambda\Lambda = 0$$

(CDC) Cartan determinant conjecture.

Let $P_1, \ldots, P_n$ be the indecomposable projective Λ-modules (up to isomorphism) and $C_\Lambda = (\underline{\dim} P_j)_j$ the Cartan matrix.

$$\operatorname{gldim}\Lambda < \infty \Longrightarrow \det C_\Lambda = 1$$

(WTC) Wakamatsu tilting conjecture.

If T is exceptional and $\Lambda \in \operatorname{cogen}^\infty(T)$, then $\operatorname{pd} T < \infty$ (i.e. T is tlting)

(GSC) Gorenstein symmetry conjecture.

$$\operatorname{id}{}_\Lambda\Lambda < \infty \Longleftrightarrow \operatorname{id}\Lambda_\Lambda < \infty$$

2. Relative homological conjectures

RHA was introduced by Auslander and Solberg in [AS93a], [AS93b], [AS93c].

(Rfindim) Relative Finitistic dimension conjecture.

For every $F = F_G$, the relative finitistic dimension defined as follows

$$\operatorname{findim}_F(\Lambda) = \sup\{\operatorname{pd}_F X \mid X \in \Lambda\text{-mod}, \operatorname{pd}_F X < \infty\}$$

is finite.

(Gfindim) Global Finitistic dimension conjecture. We can also ask for a global version (i.e. taking the supremum over all possible choices of generators G)

The following supremum

$$\operatorname{Gfindim}(\Lambda) := \sup_{\mathrm{F}}(\operatorname{findim}_{\mathrm{F}}(\Lambda))$$

is finite.

(RVC) Relative Vanishing Conjecture.

For every $F = F^H$ we have

$$\mathrm{K^b(add(H))^{\perp}} := \{\mathrm{X} \in \mathrm{D^b_F}(\Lambda\text{-mod}) \mid \mathrm{Hom(I, X)} = 0 \ \forall \mathrm{I} \in \mathrm{K^b(add(H))}\} = 0$$

(RNuC) Relative Nunke Condition.

For every $F = F^H$ and every X in Λ-mod there is an $i \geq 0$ such that $\operatorname{Ext}^i_F(H, X) \neq 0$.

(RCC) Relative Complement conjecture.

The number of complements of an almost complete relative tilting module is finite

(RGNC) Relative Generalized Nakayama conjecture.

Let $F = F_G = F^H$ and $0 \to G \to H^0 \to H^1 \to \cdots$ a minimal F-exact coresolution by F-injectives. Then every indecomposable summand of H appears as a summand of an H^n for some n.

(RKMC) Relative Kvamme-Marczinzik conjecture.

Let $F = F_G = F^H$ and $\cdots P_1 \to P_0 \to H \to 0$ be a minimal F-exact F-projective resolution. If $\operatorname{id}_F P_n \leq n + 1$ for all $n \geq 0$, then Λ is F-Co-Goremstein, i.e. $\Omega^{\infty}_F(\Lambda\text{-mod}) \subset \ker \operatorname{Ext}^1_F(-, G)$.

(RNC) Relative Nakayama conjecture.

Let $F = F_G = F^H$ and $0 \to G \to H^0 \to H^1 \to \cdots$ a minimal F-exact coresolution by F-injectives. If all H^n are F-projective, then Λ is F-self-injective , or shortly:

$$\operatorname{domdim}_F(G) = \infty \Longrightarrow \operatorname{id}_F G = 0$$

(RCDC) Relative Cartan determinant conjecture.

Let $F = F_G$ and $G_1, \ldots, G_n$ be the indecomposable summands of G (up to isomorphism). For X in Λ-mod we define $\underline{\dim}^F X := (\dim_k \operatorname{Hom}_\Lambda(G_i, X))_{1 \leq i \leq n}$ and $C^F_\Lambda := (\underline{\dim}^F G_j)_j$, then

$$\operatorname{gldim}_F \Lambda < \infty \Longrightarrow \det C^F_\Lambda = 1$$

(RWTC) Relative Wakamatsu tilting conjecture.

For every $F = F_G = F^H$ and an F-exceptional module T with $G \in \operatorname{cogen}^{\infty}_F(T)$ we have $\operatorname{pd}_F T < \infty$ (i.e. T is F-tilting).

(RGSC) Relative Gorenstein symmetry conjecture.

For every $F = F_G = F^H$ one has

$$\operatorname{id}_F G < \infty \Longleftrightarrow \operatorname{pd}_F H < \infty$$

3. History, hierarchy, variants and related results, known cases

3.1. **History.**

(findim) *Stated by:* [AB57] for commutative noetherian rings and in [AB58] for arbitrary noetherian rings, also by Bass in 60-ies [Bas60], [Bas62] (survey: [ZH95])
Newest development: Green, Psaroudakis, Solberg reduction techniques.
(VC) *Stated in:* unknown, mentioned in [Hap90].
(NuC) *Stated in:* [Nun55]
(CC) *Stated by:* [BS98], independently by [HU98].
(GNC) *Stated in:* [AR75]
Here are Happel and Unger's [HU98] and Buan and Solberg's [BS98] reformulations in terms of complements of tilting modules the most interesting development.
(KMC) *Stated in:* Kvamme-Marczinzik [KM18]
(NC) *Stated in:* [Nak58] later [Mue68]
There has been a manuscript of Rene Marczinzik with an attempted proof of the Nakayama conjecture. It was using results of Beligianis which are not completely verifiable, so one has to assume that the conjecture is not proven. A talk of Sondre Kvamme (in the Birep seminar) gave an explanation of the gap. It seems to have lead to the joint work [KM18] and an intermediate conjecture between (GNC) and (NC).
(CDC) *Stated in:* [FZH86]
(WTC) *Stated in:* Wakamatsu [Wak88]
(GSC) *Stated in:* [AR91]

3.2. **Hierarchy.**

Happel [Hap90] mentions the following implications with references:

$$(findim) \Rightarrow (VC) \Rightarrow (NuC) \Rightarrow (GNC) \Rightarrow (NC)$$

In [BS98] the first two implications are proven, in Kvamme-Marczinzik the later two

$$(findim) \Rightarrow (CC) \Rightarrow (GNC) \Rightarrow (KMC) \Rightarrow (NC)$$

In Bill's lecture notes the following implication is proven:

$$(findim) \Rightarrow (WTC) \Rightarrow (GSC)$$

What about (CDC)?

3.3. **Variants and related results.**

(findim) **Thm** [Hap90] If $\mathcal{P}(\Lambda) = \{\mathrm{pd} < \infty\}$ is contravariantly finite, then one has $\operatorname{findim}\Lambda < \infty$.
Thm [Hap90] If Λ' is tilted from Λ, then: $\operatorname{findim}\Lambda < \infty \Leftrightarrow \operatorname{findim}\Lambda' < \infty$.
(GNC) **Thm (for the eq. (2) see** [HU98, Cor 2.4]**)** The following are equivalent to the generalized Nakayama conjecture for Λ:
(1) For each simple S there is an integer t such that $\operatorname{Ext}^t(\mathrm{D}\,\Lambda, S) \neq 0$.
(2) Every almost complete tilting module which is projective has only finitely many complements.
(NC) **Thm** [AR75] The conjecture (NC) for Λ is equivalent to:
Every selforthogonal generator-cogenerator is projective.
(CDC) **Thm** [Eil54] If $\operatorname{gldim}\Lambda < \infty$, then one has $\det C_\Lambda \in \{-1, 1\}$.

3.4. **Known cases.**

(findim) Λ (Iwanaga-)Gorenstein [CB17, Lem 2, p.32]
Λ monomial [GKK91], [IZ90] (cited in [Hap90])
Λ with $\operatorname{rad}^3(\Lambda) = 0$ [GZH91]
(GNC) $\operatorname{rad}^{2l+1}(\Lambda) = 0$ and $\Lambda/\operatorname{rad}^{2l}(\Lambda)$ representation finite. ([DH92])

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