CLASSICAL AUSLANDER CORRESPONDENCE

Auslander correspondence (Auslander, 1971, [2]). The assignment of (iso-classes) of pairs

$$\Phi\colon [\Lambda, {}_\Lambda M] \mapsto [\Gamma = \mathrm{End}_\Lambda(M), {}_\Gamma M]$$

gives a self-inverse bijection between

(1) Finite-dim. algebras Λ of finite representation-type (and module M with $\mathrm{add}(M) = \Lambda-\mathrm{mod}$)
(2) Finite dim. algebras Γ (with $\Gamma - \mathrm{projinj} = \mathrm{add}(M)$) such that $\mathrm{gldim}\,\Gamma \leq 2 \leq \mathrm{domdim}\Gamma$

Auslander's formula, (Auslander, 1965, [1]). Let $\mathcal{C}$ be a small abelian category, then

$$\mathrm{mod}_1\,\mathcal{C}/\mathrm{eff}(\mathcal{C}) \cong \mathcal{C}$$

where $\mathrm{mod}_1\,\mathcal{C}$ are (contravariant) finitely presented functors and $\mathrm{eff}(\mathcal{C})$ the subcategory represented by epimorphisms in $\mathcal{C}$ (this is a Serre subcategory).

Auslander Corresp. for general rings, (Tachikawa, 1974,[11]). Let Γ be any ring and $\Gamma\,\mathrm{Proj}$ the full subcategory of projective modules in $\Gamma\,\mathrm{Mod}$. The following are equivalent

(1) $\Gamma\,\mathrm{Proj}$ is Grothendieck abelian.
(2) Γ is a semi-primary QF-3 ring with $\mathrm{gldim}\,\Gamma \leq 2 \leq \mathrm{domdim}\,\Gamma$

HIGHER CLASSICAL AUSLANDER CORRESPONDENCE

Higher Auslander correspondence (Iyama, 2007, [8]). Let $n \geq 1$. The assignment Φ gives a self-inverse bijection between

(1) Finite-dim. algebras Λ with M an n-cluster tilting object
(2) Finite dim. algebras Γ with $\Gamma - \mathrm{projinj} = \mathrm{add}(M)$ such that $\mathrm{gldim}\,\Gamma \leq n+1 \leq \mathrm{domdim}\Gamma$

Let R be a com. artinian ring, an R-linear Hom-finite, additive Krull-Schmidt cat. $\mathcal{A}$ is a *dualizing R-variety* if the duality $D = \mathrm{Hom}_R(-, E(R/\mathrm{rad}(R)))$ restricts to a duality $D\colon \mathrm{mod}_1\,\mathcal{A} \to \mathrm{mod}_1\,\mathcal{A}^{op}$.

For dualizing R-varieties, (Iyama-Jasso, 2017, [9]). Then, the following are equivalent for a dualizing R-variety $\mathcal{A}$

(1) $\mathcal{A}$ is n-abelian
(2) There is a fully faithful functor $F\colon \mathcal{A} \to \mathcal{B}$ with $\mathcal{B}$ a small abelian category which is a dualizing R-variety such that $F(\mathcal{A})$ is n-cluster tilting in $\mathcal{B}$
(3) $\mathrm{gldim}\,\mathrm{mod}_1\,\mathcal{A} \leq n+1 \leq \mathrm{domdim}\,\mathrm{mod}_1\,\mathcal{A}$

Higher Auslander formula (HAF) for n-abelian categories, (Ebrahimi–Nasr-Isfahani, 2022, [6]). Let $\mathcal{M}$ be a small n-abelian category, then $\mathrm{eff}(\mathcal{M}) = \mathrm{mod}_{epi}\,\mathcal{M} \subseteq \mathrm{mod}_1\,\mathcal{M}$ is a Serre subcategory and

$$F\colon \mathcal{M} \to \mathrm{mod}_1\,\mathcal{M}/\mathrm{eff}(\mathcal{M}) =: \mathcal{B}, \quad X \mapsto \mathrm{Hom}_{\mathcal{M}}(-, X)$$

is a fully faithful embedding inducing an equivalence of n-exact categories to an n-cluster-tilting subcategory in $\mathcal{B}$.

Higher Auslander algs. and big module categories, (Ding-Keshavarz-Zhou, 2024, [3]).
For an artin algebra Γ, TFAE to be an n-Auslander algebra (cf. (2) HAC, [8])

(a) $\Gamma\ \mathrm{proj}^{\leq(\mathrm{n}-1)}$ abelian, $\Gamma\ \mathrm{proj}^{\leq \mathrm{n}} \cap \mathrm{inj} \subseteq \Gamma\ \mathrm{proj}$, ${}^{\perp_0}\Gamma \subseteq {}^{\perp_{[1,n]}}\Gamma$ in Γ mod
(b) $\Gamma\ \mathrm{Proj}^{\leq(\mathrm{n}-1)}$ abelian, $\Gamma\ \mathrm{Proj}^{\leq \mathrm{n}} \cap \mathrm{Inj} \subseteq \Gamma\ \mathrm{Proj}$, ${}^{\perp_0}\Gamma \subseteq {}^{\perp_{[1,n]}}\Gamma$ in Γ Mod

AUSLANDER CORRESPONDENCE FOR EXACT CATEGORIES

Definition 0.1. A (1-)**Auslander exact category** is an exact category $\mathcal{F}$ with enough projectives $\mathcal{P}$ such that

(AE1) $({}^{\perp_0}\mathcal{P}, \mathrm{copres}(\mathcal{P}))$ is a torsion pair in $\mathcal{F}$
(AE2) Every morphism to an object in ${}^{\perp_0}\mathcal{P}$ is admissible with image in ${}^{\perp_0}\mathcal{P}$
(AE3) $\mathrm{Ext}^1({}^{\perp_0}\mathcal{P}, \mathcal{P}) = 0$
(AE4) $\mathrm{gldim}\, \mathcal{F} \leq 2$

For every small exact category $\mathcal{E}$, we can define $\mathrm{AE}(\mathcal{E}) := \mathrm{mod}_{adm}\, \mathcal{E}$ to be the (contravariant) functors represented by admissible morphisms. This is a fully exact subcategroy of $\mathrm{mod}_1\, \mathcal{E}$. It is Auslander exact with ${}^{\perp_0}\mathcal{P} = \mathrm{eff}(\mathcal{E}) = \mathrm{mod}_{defl}\, \mathcal{E}$ is the category of (contravariant) functors represented by deflations

Auslander's formula for exact categories (Henrard-van Roosm.-Kvamme, 2022, [7]).
$\mathrm{eff}(\mathcal{E})$ is a *percolating* subcategory of $\mathrm{AE}(\mathcal{E})$ and

$$\mathrm{AE}(\mathcal{E})/\mathrm{eff}(\mathcal{E}) \cong \mathcal{E}$$

as exact categories.

Auslander correspondence for exact categories (still [7]).
$\mathcal{E} \mapsto \mathrm{AE}(\mathcal{E})$ is an equivalence of 2-categories.
Alternatively: This assignment together with $\mathcal{F} \mapsto \mathcal{F}/{}^{\perp_0}\mathcal{P}$ gives a one-to-one correspondence between small exact categories and Auslander exact categories (up to equivalence of exact categories)

HIGHER AUSLANDER CORRESPONDENCE (HAC) FOR EXACT CATEGORIES

Definition 0.2. An n-**Auslander exact category** is an exact category $\mathcal{F}$ with enough projectives $\mathcal{P}$ such that

(nAE1) $({}^{\perp_0}\mathcal{P}, \mathrm{copres}(\mathcal{P}))$ is a torsion pair in $\mathcal{F}$
(nAE2) Every morphism to an object in ${}^{\perp_0}\mathcal{P}$ is admissible with image in ${}^{\perp_0}\mathcal{P}$
(nAE3) $\mathrm{Ext}^{1\sim n}({}^{\perp_0}\mathcal{P}, \mathcal{P}) = 0$
(nAE4) $\mathrm{gldim}\, \mathcal{F} \leq n+1$
(nAE5) $\mathcal{P}$ is admissibly covariantly finite

If $\mathcal{M}$ is an n-cluster tilting subcategory in an exact category $\mathcal{E}$, then $n - \mathrm{AE}(\mathcal{M}) := \mathrm{mod}_{\mathcal{E}-adm}\, \mathcal{M}$ where $\mathcal{E}-adm$ refers to morphisms in $\mathcal{M}$ which are admissible in $\mathcal{E}$. This is a fully exact subcategory of $\mathrm{mod}_1\, \mathcal{M}$. It is n-Auslander exact, the subcategory ${}^{\perp_0}\mathcal{P} = \mathrm{mod}_{\mathcal{E}-defl}(\mathcal{M}) =: \mathrm{eff}(\mathcal{M})$ is the subcategory of functors represented by $\mathcal{E}$-deflations $M \to M'$ with M, M' in $\mathcal{M}$.

Higher Auslander's formula for n-cluster-tilting subcats. (E.-N., 2021 (arxiv), [5]).
Let $\mathcal{M}$ be n-cluster tilting in an exact category $\mathcal{E}$, then $\mathrm{eff}(\mathcal{M})$ is a percolating subcategory of $n - \mathrm{AE}(\mathcal{M})$ and

$$n - \mathrm{AE}(\mathcal{M})/\mathrm{eff}(\mathcal{M}) \cong \mathcal{E}$$

as exact categories.

HAC for exact categories, (also [5]). The assignments $(\mathcal{E}, \mathcal{M}) \mapsto n-\mathrm{AE}(\mathcal{M})$ and $\mathcal{F} \mapsto \mathcal{F}/^{\perp_0}\mathcal{P}$ give mutually inverse one-to-one correspondences between

(1) small exact categories together with an n-cluster tilting subcategory (up to equiv. of exact category preserving the n-ct subcategory) and
(2) n-Auslander exact categories (up to equivalence of exact categories)

In the next result, we put "Auslander's formula" in quotation marks as no *quotient/localization* is established but a new construction which fulfills the wanted properties and is therefore the generalization of HAF in this situation.

Higher "Auslander's formula" for n-exact categories, Kvamme, 2025 (arxiv), [10]. Every weakly idempotent complete n-exact category $\mathcal{M}$ is equivalent as n-exact category to an n-cluster tilting subcategory in a weakly idempotent complete exact category $\mathcal{E}$. Furthermore, the exact category $\mathcal{E}$ is uniquely determined up to exact equivalence.

Remark 0.3. (1) The uniqueness is not a direct consequence of HAF for n-cluster tilting subcategories, as we also need the class of $\mathcal{E}$-admissible morphisms in the n-cluster tilting category to construct the ambient exact category $n-\mathrm{AE}(\mathcal{M})/\mathrm{eff}(\mathcal{M})$.
(2) The construction of $\mathcal{E}$ is using the Garbiel-Quillen embedding of an n-exact category $\mathcal{M} \to \mathcal{L}(\mathcal{M})$, $X \mapsto \mathrm{Hom}(-, X)$ (cf. [4]). The essential image is n-rigid and closed under n-extensions. The ambient exact category $\mathcal{E}$ is constructed as a fully exact category of $\mathcal{L}(\mathcal{M})$ given by first define $\mathcal{C} = \mathrm{Cores}_n^{\mathcal{L}(\mathcal{M})}(\mathcal{M})$ as the subcategory of all object with a n-coresolution in $\mathcal{M}$ and then

$$\mathcal{E} = \{F \in \mathcal{L}(\mathcal{M})\colon \exists 0 \to M_n \to M_{n-1} \to \cdots \to M_1 \to F \to 0 \text{ exact}, \mathrm{coker}(M_{i+1} \to M_i) \in \mathcal{C}, 1 \leq i \leq n-1\}$$

In particular, $\mathcal{E} \subseteq \mathrm{Cores}_n^{\mathcal{L}(\mathcal{M})}(\mathcal{M}) \cap \mathrm{Res}^n_{\mathcal{L}(\mathcal{M})}(\mathcal{M})$ but the definition differs slightly from just taking this (obvious candidate) intersection.
For the uniqueness of $\mathcal{E}$ a new universal property for these ambient categories is established.

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