HALL ALGEBRAS OF QUIVER REPRESENTATIONS

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Motivation

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Such a Lie algebra gives rise to an integral square matrix $C(\mathfrak{g})$, the so-called *Cartan matrix* of $\mathfrak{g}$. A Theorem by Serre provides a presentation of $\mathfrak{g}$ by generators and relations that only depend on $C(\mathfrak{g})$, the so-called *Serre relations*. This implies that the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ can be described by the same generators and relations that are derived from those for the Lie algebra. More generally, such presentations can be associated to so-called generalized Cartan matrices, which then define certain infinite-dimensional Lie algebras, that are known as Kac-Moody Lie algebras. Deforming the relations yield quantum Serre relations which lead to quantized enveloping algebras $U_q(\mathfrak{g})$.

Integral matrices can also be used to define quivers. In this talk, which is based on lecture notes by Andrew Hubery, I will introduce some basic concepts on quivers and their representations and illustrate how Hall algebras of quivers give rise to similar relations. This is thus the first step towards defining a homomorphism

$$U_q(\mathfrak{g}^+) \longrightarrow \mathcal{H}$$

between half quantum groups and Hall algebras.

Our relations will arise in the following fashion: Let $x, y \in \mathfrak{g}$. In $U(\mathfrak{g})$ we have $\mathrm{ad}\, x = \ell_x - r_x$, the difference between the left and right multiplications effected by x. Hence a relation $(\mathrm{ad}\, x)^n(y) = 0$ in $\mathfrak{g}$ implies

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} x^i y x^{n-i} = 0$$

in $U(\mathfrak{g})$. For the purposes of this talk, "deforming" relations means replacing binomial coefficients by Gaussian binomial coefficients.

1. Quiver representations

Let k be a field, R be a k-algebra. We will work in the category $\operatorname{mod} R$ of finite-dimensional R-modules. Let $M \in \operatorname{mod} R$.

- $\mathrm{Rad}(M) := \bigcap_{U \subsetneq M \text{max.}} U$ is the *radical* of M. We put $\mathrm{Rad}^{n+1}(M) := \mathrm{Rad}(\mathrm{Rad}^n(M))$.
- $\ell\ell(M) := \min\{n \in \mathbb{N}_0 \; ; \; \mathrm{Rad}^n(M) = (0)\}$ is the *Loewy length* of M.
- $\mathrm{Soc}(M) = \sum_{S \subseteq M \text{simple}} S$ is the *socle* of M. We put $\mathrm{Soc}_{n+1}(M) := \{m \in M \; ; \; m + \mathrm{Soc}_n(M) \in \mathrm{Soc}(M/\mathrm{Soc}_n(M))\}$.
- Let $M = \mathrm{Soc}(M)$ be semisimple, S be a simple R-module. Then

$$M_S := \sum_{V \subseteq N; V \cong S} V$$

  is the *S-isotypic component* of M. Thus, if $M = \bigoplus_{i=1}^{\ell} S_i^{d_i}$, then $M_{S_i} = S_i^{d_i}$.
- We say that M is uniserial, if $(\mathrm{Rad}^n(M))_{n \geq 0}$ is a composition series of M. In that case $\mathrm{Rad}^n(M)$ is the unique submodule of M of length $\ell(M) - n$.

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A *quiver* $Q = (Q_0, Q_1)$ consists of a finite set Q_0 of vertices and finite a set Q_1 of arrows between vertices. We postulate that there are no loops, that is, there are no arrows $\alpha : i \to i$.

For every vertex $i \in Q_0$, we pick a path e_i of length 0, which starts and ends at i.

Definition. The path algebra kQ has underlying vector space with basis the set of oriented paths and product given by concatenation or zero. (Arrows are composed like maps).

Examples. (1) Let K_2 be the 2-Kronecker quiver:

$$1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$$

Then

$$kQ = ke_1 \oplus ke_2 \oplus k\alpha \oplus k\beta,$$

and $\alpha\beta = 0 = \beta\alpha$ while $e_i^2 = e_i$ and $e_i e_{3-i} = 0$.

(2) Let $\tilde{A}_1$:

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

be the cyclic quiver. Then

$$k\tilde{A}_1 = ke_1 \oplus ke_2 \oplus k\alpha \oplus k\beta \oplus k\beta\alpha \bigoplus_{0 \leq i+j \leq 2, n \geq 1} k\beta^i(\alpha\beta)^n\alpha^j.$$

Let $Q = (Q_0, Q_1)$ be a quiver. A *representation* $V = ((V_i)_{i \in Q_0}, (V_\alpha)_{\alpha \in Q_1})$ consists of finite-dimensional vector spaces V_i and linear maps $V_\alpha : V_i \longrightarrow V_j$ for every arrow $\alpha : i \to j$. The element $\underline{\dim} V := (\dim_k V_i)_{i \in Q_0} \in \mathbb{N}_0^{Q_0}$ is the *dimension vector* of V.

A *morphism* $f : V \longrightarrow W$ is a family $f_i : V_i \longrightarrow W_i$ of linear maps such that, for each arrow $\alpha : i \to j$, the diagram

$$\begin{array}{ccc} V_i & \xrightarrow{f_i} & W_i \\ {\scriptstyle V_\alpha}\downarrow & & \downarrow{\scriptstyle W_\alpha} \\ V_j & \xrightarrow{f_j} & W_j \end{array}$$

commutes.

Kernels, Images and Cokernels are defined canonically, and we thus have an abelian category $\mathrm{rep}(Q)$. In fact, $\mathrm{rep}(Q)$ is equivalent to $\mathrm{mod}\, kQ$.

Special features:

- Let $i \in Q_0$ $S_i := ((\delta_{ij}k)_j, 0)$ is a simple module, and the S_i $(i \in Q_0)$ exhaust all simple kQ-modules.
- We have $\dim_k \mathrm{Ext}^1(S_i, S_j) = |\alpha : i \to j|$.
- In particular, S_i is projective iff i is a sink and S_i is injective iff i is a source.

2. Hall numbers

Let R be an algebra over a finite field $k = \mathbb{F}_q$.

- Given $M, X, Y \in \operatorname{mod} R$, the Hall number is defined by
$$F^X_{M,N} := |\{U \subseteq X \ ; \ U \cong N \text{ and } X/U \cong M\}|.$$
- Recall that $\mathcal{H}(R) := \bigoplus_{[M]} \mathbb{Z}u_M$ is the Hall algebra, with product
$$u_M u_N = \sum_{[X]} F^X_{M,N} u_X.$$

We let $\operatorname{Gr}_d(k^n)$ be the Grassmannian of d-planes in n-space and recall that
$$|\operatorname{Gr}_d(k^n)| = \binom{n}{d}_q$$
is the Gaussian binomial coefficient. We write $(n)_q := \binom{n}{1}_q$.

By way of example, we prove the following:

Lemma 2.1. *Let Q be a quiver, S be a simple kQ-module,*
$$(0) \longrightarrow S^d \longrightarrow X \xrightarrow{\pi} M \longrightarrow (0)$$
be an exact sequence of kQ-modules.

(1) *If S is injective or X is semisimple, then have*
$$F^X_{M,S^d} = \binom{\dim_k \operatorname{Soc}(X)_S}{d}_q$$
(2) *If S is injective and $M \not\cong S$ is indecomposable, then $F^X_{M,S^d} = 1$ and $u_M u_{S^d} = u_{M \oplus S^d}$.*
(3) *We have $u_{S^r} u_{S^s} = \binom{r+s}{s}_q u_{S^{r+s}}$. In particular, $u^n_S = (n)_q! u_{S^n}$ for all $n \geq 1$.*

Proof. (1) We write $X = ((X_i), (X_\alpha))$ and $S = S_{i_0}$, so that
$$(\operatorname{Soc}(X)_S)_i = \begin{cases} \bigcap_{\alpha: i_0 \to j} \ker X_\alpha & i = i_0 \\ (0) & \text{else.} \end{cases}$$
If S is injective, then i_0 is a source, whence $\sum_{\alpha: i \to i_0} \operatorname{im} X_\alpha = (0)$. This also follows in case X is semisimple.

We write $X_{i_0} = (\operatorname{Soc}(X)_S)_{i_0} \oplus Y_{i_0}$ as a sum of k-spaces. Let $U \subseteq X$ be a submodule, $\varphi : U \longrightarrow S^d$ be an isomorphism. Then $U \subseteq \operatorname{Soc}(X)_S$, so that $U_{i_0} \subseteq \bigcap_{\alpha: i_0 \to j} \ker X_\alpha$. Hence there is a linear map $f_{i_0} \in \operatorname{GL}(X_{i_0})$ such that

(a) $f_{i_0}(\operatorname{Soc}(X)_S) = \operatorname{Soc}(X)_S$, and
(b) $f_{i_0}|_{U_{i_0}} = \varphi_{i_0}$, and
(c) $f_{i_0}|_{Y_{i_0}} = \operatorname{id}_{Y_{i_0}}$.

Setting $f_i = \operatorname{id}_{X_i}$ for $i \neq i_0$, one checks that $f = (f_i) \in \operatorname{Aut}(X)$, while $f|_U = \varphi$. Thus, $\ker(\pi \circ f) = f^{-1}(S^d) = U$, and we have $X/U \cong M$. Consequently, F^X_{M,S^d} counts the d-dimensional subspaces of $\operatorname{Soc}(X)_S$.

(2) Since S is injective, the sequence splits and $X \cong M \oplus S^d$. By the same token, S is not isomorphic to a submodule of M, whence $\operatorname{Soc}(X)_S \cong \operatorname{Soc}(M)_S \oplus S^d = S^d$.

(3) Since the quiver has no loops, we have $\operatorname{Ext}^1(S, S) = (0)$, whence $\operatorname{Ext}^1(S^r, S^s) = (0)$. Thus, every exact sequence
$$(0) \longrightarrow S^s \longrightarrow X \longrightarrow S^r \longrightarrow (0)$$

splits and $X \cong S^{r+s}$ is semisimple. Consequently, part (1) yields

$$u_{S^r}u_{S^s} = \binom{r+s}{s}_q u_{S^{r+s}}.$$

The second assertion now follows by induction. □

3. The n-Kronecker quiver

Let $k = \mathbb{F}_q$. The quiver K_n is given by

$$1 \overset{n}{\to} 2.$$

Consequently,

- $\underline{\dim}\, M \in \mathbb{N}_0^2$ for every $M \in \mathrm{rep}(K_n)$.
- S_1 is injective and S_2 is projective.

The category $\mathrm{rep}(K_n)$ affords a duality $D : \mathrm{rep}(K_n) \longrightarrow \mathrm{rep}(K_n)$

$$D(M_1, M_2, (\varphi_i)) := D(M_2^*, M_1^*, (\varphi_i^*)),$$

so that $\underline{\dim}\, D(M) = (\dim_k M_2, \dim_k M_1)$.

Let

$$(0) \longrightarrow N \longrightarrow X \longrightarrow M \longrightarrow (0)$$

be an exact sequence. Then

$$(0) \longrightarrow D(M) \longrightarrow D(X) \longrightarrow D(N) \longrightarrow (0)$$

is exact, and we have

$$F^{D(X)}_{D(M),D(N)} = F^X_{N,M}.$$

This implies that the map

$$D : \mathcal{H}(K_n) \longrightarrow \mathcal{H}(K_n) \;\; ; \;\; u_M \mapsto u_{D(M)}$$

is an involution: $D(ab) = D(b)D(a)$; $D^2 = \mathrm{id}$.

Given $\underline{\mathrm{d}} \in \mathbb{N}_0^2$, we put

$$\mathrm{ind}^{\underline{\mathrm{d}}}(K_n) := \{[M] \; ; \; M \in \mathrm{rep}(K_n) \text{ indecomposable}, \underline{\dim}\, M = \underline{\mathrm{d}}\}$$

as well as

$$u_{\underline{\mathrm{d}}} := \sum_{[M]\in \mathrm{ind}^{\underline{\mathrm{d}}}(K_n)} u_M.$$

Lemma 3.1. *The following statements hold:*

(1) *If N is indecomposable with $N \not\cong S_2$, then $u_{S_2^s}u_N = u_{S_2^s \oplus N}$.*
(2) $u_{S_2^r}u_{S_1}u_{S_2^s} = \sum_{a=0}^{s} \binom{r+s-a}{r}_q u_{S_2^{r+s-a}} u_{(1,a)}$.

Proof. (1) Since Since $S_1 = D(S_2)$ is injective and $D(N)$ is indecomposable with $D(N) \not\cong S_1$, Lemma 2.1 yields

$$u_{S_2^s} u_N = D(u_{S_1^s}) D(u_{D(N)}) = D(u_{D(N)} u_{S_1^s}) = D(u_{D(N) \oplus S_1^s}) = u_{S_2^s \oplus N}.$$

(2) If

$$(0) \longrightarrow S_2^s \longrightarrow X \longrightarrow S_1 \longrightarrow (0)$$

is exact, then $\mathrm{Soc}(X)_{S_2} \cong S_2^s$, so that every $U \subseteq X$ with $U \cong S_2^s$ equals $\mathrm{Soc}(X)_{S_2}$. Consequently, $F^X_{S_1, S_2^s} = 1$, and we obtain

$$u_{S_1} u_{S_2^s} = \sum_{\underline{\dim}\, X = (1,s)} F^X_{S_1, S_2^s} u_X = \sum_{\underline{\dim}\, X = (1,s)} u_X,$$

so that

$$u_{S_2^r} u_{S_1} u_{S_2^s} = \sum_{\underline{\dim}\, X = (1,s)} u_{S_2^r} u_X.$$

Given M with $\underline{\dim}\, M = (1, s)$, we have $M \cong N \oplus S_2^{s-a}$, where N is indecomposable and $\underline{\dim}\, N = (1, a)$. Now (1) implies

$$u_{S_2^r} u_{S_1} u_{S_2^s} = \sum_{a=0}^{s} \sum_{N \in \mathrm{ind}^{(1,a)}(K_n)} u_{S_2^r} u_{S_2^{s-a} \oplus N} = \sum_{a=0}^{s} (u_{S_2^r} u_{S_2^{s-a}}) u_{(1,a)} = \sum_{a=0}^{s} \binom{r+s-a}{r}_q u_{(1,a)},$$

where the last equation follows from Lemma 2.1(3). □

Lemma 3.2. *We have:*

(1) $\sum_{r=0}^{n+1} (-1)^r q^{\binom{r}{2}} u_{S_2^r} u_{S_1} u_{S_2^{n+1-r}} = 0.$

(2) $\sum_{r=0}^{n+1} (-1)^{n+1-r} q^{\binom{n+1-r}{2}} u_{S_1^r} u_{S_2} u_{S_1^{n+1-r}} = 0.$

Proof. (1) This follows directly from Lemma 3.1 and the formula

$$\sum_{r=0}^{m} (-1)^r q^{\binom{r}{2}} \binom{m}{r}_q = 0$$

along with $\binom{m}{s}_q = 0$ for $s > m$.

(2) This follows by applying D to (1), while observing $D(S_i) = S_{3-i}$. □

Lemma 2.1(3) now implies that (1) yields

$$\sum_{r=0}^{n+1} (-1)^r q^{\binom{r}{2}} \binom{n+1}{r}_q u_{S_2}^r u_{S_1} u_{S_2}^{n+1-r} = 0.$$

For $n = 2$, this resembles one of the q-Serre relations for affine $\mathfrak{sl}(2)$, but (2) shows that we obtain a second relation that is different.

4. The cyclic quiver $\tilde{A}_1$

We consider the quiver

$$1 \leftrightarrows 2.$$

Since we are interested in finding relations involving only S_1 and S_2 and their iterated extensions, we will be working in the full subcategory of $\operatorname{mod} k\tilde{A}_1$, whose objects have composition series involving S_1 and S_2 only. This is in fact the subcategory $\operatorname{rep}^{\mathrm{nil}}(\tilde{A}_1)$ of nilpotent representations.

Fact:

- Every indecomposable module $M \in \operatorname{rep}^{\mathrm{nil}}(\tilde{A}_1)$ is uniquely determined by its top $\operatorname{Top}(M) := M/\operatorname{Rad}(M)$ and its length $(= \ell\ell(M) = \dim_k M)$. We denote by $S_i(n)$ the indecomposable $\operatorname{rep}^{\mathrm{nil}}(\tilde{A}_1)$-module such that $\operatorname{Top}(S_i(n)) \cong S_i$ and $\dim_k S_i(n) = n$.

Lemma 4.1. *The following statements hold:*

(1) $u_{S_2} u_{S_1^s} = u_{S_2(2) \oplus S_1^{s-1}} + u_{S_1^s \oplus S_2}$.

(2) $u_{S_1^r} u_{S_2} u_{S_1^s} = \binom{r+s-2}{r-1}_q u_{S_1(3) \oplus S_1^{r+s-2}} + \binom{r+s-1}{r}_q u_{S_2(1) \oplus S_1^{r+s-1}} + \binom{r+s-1}{r-1}_q u_{S_1(2) \oplus S_1^{r+s-1}} + \binom{r+s}{r}_q u_{S_1^{r+s} \oplus S_2}$

Proof. (1) Let

$$(0) \longrightarrow S_1^s \longrightarrow X \longrightarrow S_2 \longrightarrow (0)$$

be an exact sequence. Then we have

$$(0) \longrightarrow S_1^s \longrightarrow \operatorname{Soc}(X)_{S_1} \longrightarrow \operatorname{Soc}(S_2)_{S_1}$$

so that $\operatorname{Soc}(X)_{S_1} = S_1^s$. This readily yields $F^X_{S_2 S_1^s} = 1$. If $\ell\ell(X) = 2$, then $X \cong S_2(2) \oplus S^{s-1}$, alternatively $X = S_1^s \oplus S_2$ is semisimple.

(2) By way of example, we compute the product

$$u_{S_1^r} u_{S_2(2) \oplus S_1^{s-1}} = \sum_{[X]} F^X_{S_1^r, S_2(2) \oplus S_1^{s-1}} u_X.$$

Thus, we have to consider exact sequences

$$(0) \longrightarrow S_2(2) \oplus S_1^{s-1} \longrightarrow X \longrightarrow S_1^r \longrightarrow (0).$$

Then we have $2 \leq \ell\ell(X) \leq 3$ and $\underline{\dim}\, X = (r+s, 1)$. Suppose that $\ell\ell(X) = 3$. Since $\underline{\dim}\, S_2(3) = (1, 2)$, we get

$$X \cong S_1(3) \oplus S_1^{r+s-2}.$$

We write this in the from $X = X^1 \oplus X^2$ and denote the canonical projection by $\pi : X \longrightarrow X^2$.

Let $Y \subseteq X$ be an indecomposable module such that $Y \cong S_2(2)$. Since $\operatorname{Hom}(S_2(2), S_1) = (0)$, we get $\pi(Y) = (0)$, so that $Y \subseteq X^1$. This implies $Y = \operatorname{Soc}_2(X^1)$.

Let $U \subseteq X$ be such that $U \cong S_2(2) \oplus S_1^{s-1}$. Then $\ell\ell(U) = 2$, whence

$$U \subseteq \operatorname{Soc}_2(X) = \operatorname{Soc}_2(X^1) \oplus \operatorname{Soc}_2(X^2) \cong S_2(2) \oplus S_1^{r+s-2}.$$

By the above, we have

$$U = \operatorname{Soc}_2(X^1) \oplus V,$$

where $S^{s-1} \cong V \hookrightarrow X^2 \cong S_1^{r+s-2}$.

Given such a submodule $U \subseteq \operatorname{Soc}_2(X)$, we have

$$(0) \longrightarrow U/\operatorname{Soc}_2(X^1) \longrightarrow X/\operatorname{Soc}_2(X^1) \longrightarrow X/U \longrightarrow (0),$$

where $U/\operatorname{Soc}_2(X^1) \cong S_1^{s-1}$ and $X/\operatorname{Soc}_2(X^1) \cong S_1^{r+s-1}$. Thus, $X/U \cong S^r$, and

$$F^X_{S_1^r, S_2(2)\oplus S_1^{s-1}} = \binom{r+s-2}{s-1}_q.$$

If $\ell\ell(X) = 2$, then we obtain $X \cong S_2(2)\oplus S_1^{r+s-1}$ and the arguments above yield $F^X_{S_1^r, S_2(2)\oplus S_1^{s-1}} = \binom{r+s-1}{s-1}_q$. □

Using this, one can verify the following relation for $\mathcal{H}(\tilde{A}_1)$:

$$qu_{S_1^3}u_{S_2} - u_{S_1^2}u_{S_2}u_{S_1} + u_{S_1}u_{S_2}u_{S_1^2} - qu_{S_2}u_{S_1^3} = 0.$$

By symmetry there is another such relation with the roles of S_1 and S_2 interchanged. These relations markedly differ from those of the quiver K_2 and illustrate the dependence on the orientation of the quiver.

5. The twisted Hall Algebra

Since our categories $\operatorname{rep}(Q)$ are hereditary, we have the Euler form $\langle\, ,\, \rangle : K_0(\operatorname{rep}(Q))^2 \longrightarrow \mathbb{Z}$

$$\langle M, N\rangle = \dim_k \operatorname{Hom}(M,N) - \dim_k \operatorname{Ext}^1(M,N).$$

The simple modules from a basis for $K_0(\operatorname{rep}(Q))$, an the representing matrix relative to this basis is

$$E_Q = I_n - (a_{ij}),$$

where a_{ij} is the number of arrows from i to j and $n := |Q_0|$. Hence we get

$$E_{K_n} := \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} \quad ; \quad E_{\tilde{A}_1} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

We pick $v \in \mathbb{R}$ such that $v^2 = q$ and let $\mathbb{Q}_v \subseteq \mathbb{R}$ be the subfield generated by v. We consider

$$\mathcal{H}_v(Q) := \bigoplus_{[M]} \mathbb{Q}_v u_M$$

and define a new product

$$u_M * u_N := v^{\langle M,N\rangle} u_M u_N.$$

With respect to this new product, the algebras $\mathcal{H}_v(K_2)$ and $\mathcal{H}_v(\tilde{A}_1)$ satisfy the quantum Serre relations.