Summer School on
Gentle Algebras

Christof’s Talks

BIREP

14–18 August 2017

To edit these notes please go to:

https://www.overleaf.com/10715945xrvjfwffgtj

Please feel free to correct mistakes and to add/modify whatever seems reasonable.

Contents

1  Classification of Indecomposable Representations via Bimodule Problems I . .  2
2  Classification of Indecomposable Representations via Bimodule Problems II . .  6
3  Strategy of Proof . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 9
1 Classification of Finite-Dimensional Indecomposable Representations of Gentle Algebras via Bimodule Problems I

Wednesday 16\textsuperscript{th} 11:15 – Christof Geiß (Mexico City, Mexico)

History.

• Gelfand problem, 1974: Classify indecomposables for

\[
\begin{array}{ccc}
\bullet & \overset{a_1}{\longrightarrow} & \bullet \\
\downarrow b_1 & & \downarrow b_2 \\
\bullet & \overset{a_2}{\longrightarrow} & \bullet
\end{array}
\]

with relation \(b_1a_1 - b_2a_2\) (\textit{not} special biserial).

• Kiev school:

  • Nazarova–Roiter, 1974: self-reproducing systems
  • Bondarenko, 1992: bundles of semichains

• Bangming Deng, 1995: Ph.D. thesis under Gabriel

• Crawley-Boevey, 1989: using functorial filtrations (clans)

Recall 1.1 (Bimodule problem). \textit{Let \(A\) be an additive (Krull–Schmidt) \(k\)-category. For an \(A\)-\(A\) bimodule, i.e. a \(k\)-linear functor}

\[
M : A^\text{op} \times A \longrightarrow k\text{-mod},
\]

\textit{we have a category \(\text{Rep}(M)\) with:}

\begin{itemize}
  \item \textit{objects:} \((X, m)\) with \(X \in A\) and \(m \in M(X, X)\)
  \item \textit{Hom}((\(X, m\), (\(Y, n\))) = \{\varphi \in A(X, Y) \mid M(X, \varphi) \cdot m = M(\varphi, Y) \cdot n \text{ in } M(X, Y)\}
\end{itemize}

Remark 1.2 (Special case). \(M = (M_1^+, M_1^-, \ldots, M_n^+, M_n^-)\) functors \(A \to k\text{-mod}\)

\(\rightsquigarrow M = \bigoplus_{i=1}^{n} \text{Hom}_k(M_i^+, M_i^-)\) is a special kind of \(A\)-\(A\)-bimodule.

\(\rightsquigarrow \text{Rep}(M)\) has objects \((X, (f_1, \ldots, f_n))\) with \(X \in A\) and \(f_i \in \text{Hom}_k(M_i^+(X), M_i^-(X))\).

This kind of bimodule problem is called a \textit{tangle}.

Example 1.3. Let \(A = (A\text{-mod}) \times (k\text{-mod})\), \(M^- = \text{Hom}_A(X, -)\) for some \(X \in A\text{-mod}\) and \(M^+ = \text{id}_{k\text{-mod}}\). \(\rightsquigarrow \text{rep}(M) \cong A[X]\text{-mod}\) (one-point extension)
Bundles of Chains

Let \( S^\varepsilon_i (i = 1, \ldots, n, \varepsilon = \pm) \) be finite linearly ordered sets:

\[
\begin{align*}
\bullet \\
\lor \\
\bullet \\
\vdots \\
\bullet \\
\lor \\
\bullet
\end{align*}
\]

Let \( S := \bigcup_{i, \varepsilon} S^\varepsilon_i \) with the obvious poset structure.

Let \( \sim \) be an equivalence relation on \( S \) such that each equivalence class has 1 or 2 elements.

Let \( \mathcal{A} = \text{add} S \) be the \( k \)-category where \( S \) has

- as objects the equivalence classes \( S/\sim \),
- \( \text{rad} S(a, b) = \bigoplus_{X \in a, Y \in b, x > y} k(x|y) \),
- and obvious compositions.

Example 1.4.

\[
\begin{array}{cccc}
p & q & t & w \\
\uparrow & \uparrow & \uparrow & \\
r & u & x & z \\
\uparrow & \uparrow & \uparrow & \\
s & v & y & \\
\end{array}
\]

where \( p \sim r, q \sim v, s \sim t, u \sim z, x \sim y \). Morphisms in the radical of \( S \)

\[
\begin{array}{cccc}
\{p, r\} & \rightarrow & \{s, t\} & \rightarrow & \{w\} \\
\downarrow a & & \downarrow d & & \downarrow f \\
\{q, v\} & \leftarrow & \{u, z\} & \leftarrow & \{x, y\}
\end{array}
\]

with relations \( ab, dc, be, g^2 \).
Each chain $S^ε_i$ gives rise to a (uniserial) module $M^i_ε$ with
\[ M^i_ε(a) = \bigoplus_{x \in a \cap S^ε_i} kx. \]
We are interested in
\[ \text{Rep} \left( \bigoplus_{i=1}^n \text{Hom}_k(M^+_i, M^-_i) \right). \]

**Example 1.5.** In Example 1.4 this can be visualized as a matrix problem:

<table>
<thead>
<tr>
<th></th>
<th>s</th>
<th>r</th>
<th>q</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>t</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>u</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>v</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>y</th>
<th>x</th>
<th>w</th>
</tr>
</thead>
<tbody>
<tr>
<td>z</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Coupling between: $t \leftrightarrow s$, $r \leftrightarrow p$, $q \leftrightarrow v$, $u \leftrightarrow z$, $y \leftrightarrow x$

The aim of the Kiev school was to produce normal forms for this kind of problems.

Let $kQ/\langle \rho \rangle$ be a gentle algebra with $\rho$ a set of paths of length 2.

We need polarizations $\sigma, \tau : Q_1 \to \{+,-\}$ with:

- $s(a) = s(b)$ and $a \neq b \Rightarrow \sigma(a) \neq \sigma(b)$
- $t(a) = t(b)$ and $a \neq b \Rightarrow \tau(a) \neq \tau(b)$
- $cb \in \rho \Rightarrow \tau(c) = \sigma(b)$

Now:

- $Q_0 = \{1, 2, \ldots, n\}$
- $S = \{i^ε | i = 1, \ldots, n, \epsilon = +, -\} \cup \{a^ε | a \in Q_1, \epsilon = +, -\}$
- $S^c_i = \{i^ε\} \cup \{a^+ | a \in Q_1, s(a) = i, \sigma(a) = \epsilon\} \cup \{b^- | b \in Q_1, t(b) = i, \tau(b) = \epsilon\}$

$\Rightarrow |S^c_i| \in \{1, 2, 3\}$

\[ a^+ \]
\[ \vee \]
\[ i^ε \]
\[ \vee \]
\[ b^- \]

With equivalence relation: $a^+ \sim a^-$
Example 1.6.

\[ e \Leftrightarrow 1 \quad \xrightarrow{a} \quad 2 \]

with relations \( e^2 \) and \( ba \). Then \( S \) looks as follows:

\[ \begin{array}{cccc}
  e^+ & b^+ & a^+ \\
  \uparrow & \uparrow & \uparrow \\
  1^+ & 1^- & 2^+ & 2^- \\
  \uparrow & \uparrow & \uparrow \\
  e^- & a^- & b^- \\
\end{array} \]
2 Classification of Finite-Dimensional Indecomposable Representations of Gentle Algebras via Bimodule Problems II

Thursday 17th 11:15 – Christof Geiß (Mexico City, Mexico)

Recall 2.1.

- bundles of chains: disjoint linearly ordered sets \(S_1^+, S_1^-, \ldots, S_n^+, S_n^-\)
- equivalence relation \(\sim\) on \(S = \bigcup_{i, \varepsilon} S_i^\varepsilon\) with each class containing at most 2 elements
- define a \(k\)-category \(S\) such that
  - \(\text{obj } S = S/\sim\)
  - \(\text{rad } S = \text{rad } kS\)
  - \(A = \text{add } S\)
- objects in \(A\): \(X = \bigoplus_{a \in S} a \otimes_k V_a\)
- \(M_i^\varepsilon: A \to k\text{-mod} \) functors coming from the \(S_i^\varepsilon\)
- \(M_i^\varepsilon(X) = \bigoplus_{x \in S_i^\varepsilon} V_{\tilde{x}}\) where \(\tilde{x}\) is the equivalence class of \(x\)
- \(\text{rep}(M(S, \sim))\): \((X, f_1, \ldots, f_n)\) with \(f_i \in \text{Hom}_k(M_i^+(X), M_i^-(X))\)
- \(kQ/\langle \rho \rangle\) gentle algebra where \(\rho\) contains only paths of length 2:
  - polarization \(\sigma, \tau: Q_1 \to \{+,-\}\)
  - \(S_i^\varepsilon = \{i^\varepsilon\} \cup \{a^+ | a \in Q_1, s(a) = i, \sigma(a) = \varepsilon\} \cup \{b^- | b \in Q_1, t(b) = i, \tau(b) = \varepsilon\}\)
  - each \(S_i^\varepsilon\) is a chain of length \(\leq 3\)
  - length-3 chains: \(a^+ \succ i^\varepsilon \succ a^-\)

**Proposition 2.2.** With \((S, \sim)\) as just defined for a gentle algebra \(A = kQ/\langle \rho \rangle\) we have an equivalence
\[
F: \text{rep}_b(M(S, \sim)) \xrightarrow{\sim} A\text{-mod}
\]
where \(b\) means:

- \((X, f_i) \in \text{rep}(M(S, \sim))\) such that \(f_i\) is bijective for all \(i\).

**Proof (sketch).** Let \((X, f_\bullet) \in \text{rep}_b(M(S, \sim))\), then \(F(X, f_\bullet)_i := M_i^+(X)\) for all \(i \in Q_0\).

Recall that for \(a \in Q_1\) we have \(a^+ \sim a^-\) in \(S\). If \(s(a) = i, t(a) = j, a^+ \in S_i^\varepsilon, a^- \in S_j^\eta\), then we have a canonical isomorphism
\[
(M_i^\varepsilon / \text{rad } M_i^\varepsilon)(X) \xrightarrow{\xi_X} \text{soc } M_j^\eta
\]
coming from \(a^+ \sim a^-\).

Now we have to distinguish four cases to define \(F(X, f_\bullet)(a)\):
\begin{itemize}
  \item \((\varepsilon, \eta) = (+, +)\):
    \[ M_i^+(X) \to (M_i^+/\text{rad } M_i^+)(X) \xrightarrow{\xi_i^k} \text{soc } M_j^+(X) \to M_j^+(X) \]
  \item \((\varepsilon, \eta) = (+, -)\):
    \[ M_i^+(X) \to (M_i^+/\text{rad } M_i^+)(X) \xrightarrow{\xi_i^k} \text{soc } M_j^-(X) \xrightarrow{f_{j,i}} M_j^+(X) \]
  \item \((\varepsilon, \eta) = (-, -)\) and \((\varepsilon, \eta) = (-, +)\): similar
\end{itemize}

\textbf{Strings and Bands for \((S, \sim)\) Bundle of Chains}

We may assume that \((S, \sim)\) is complete, i.e. each equivalence class contains exactly two elements.

We can define an involution \((-\sim)\) on \(S\) such that for each \(x \in S\) we have \(\overline{x} = \{x, x^{-}\}\).

\[ \hat{S} = \bigcup_{i=1}^{n} (S_i^+ \times S_i^-) \cup \bigcup_{i=1}^{n} (S_i^- \times S_i^+) \subseteq S \times S \]

\textbf{Definition 2.3.} A string for \((S, \sim)\) is a sequence of elements \(\xi = (s_1, s_2, \ldots, s_\ell)\) of \(S\) such that \((s_i^{-}, s_{i+1}) \in \hat{S}\) for \(i = 1, 2, \ldots, \ell - 1\). A band is an infinite periodic string.

\textbf{Remark 2.4.} If \(\xi\) is a string, then so is \(\xi^{-} = (s_\ell^{-}, s_{\ell-1}^{-}, \ldots, s_1^{-})\).

If \(b\) is a band, then so are \(b^{-}\) and \(b^{[1]}\).

\textbf{Definition 2.5.} Define for each string \(\xi = (s_1, \ldots, s_\ell)\) a representation \((X_\xi, f_{\xi,1}, \ldots, f_{\xi,n})\) of \(M = M(S, \sim)\)

\[ X_\xi := \bigoplus_{i=1}^{\ell} \{s_i, s_i^{-}\} \in \mathcal{A} \]

where \(\{s_i, s_i^{-}\} \in \text{obj } S\). Observe

\[ M_j^+(X_\xi) = \bigoplus_{s_i \in S_j^+} k s_i \oplus \bigoplus_{s_i^{-} \in S_j^-} k s_i^{-} \]

with structure maps

\[ f_{\xi,i} = \sum_{s_i^{-} \in S_i^+ \cup S_i^-} f_{\xi,i,r} \]

where

\[ f_{\xi,i,r} : M_i^+(X) \to M_i^-(X) \]

is defined as follows:
(1) If $s_i^r \in S_i^+$, we have a direct summand $ks_i^r$ of $M_i^+(X_\varnothing)$ and a direct summand $ks_{r+1}$ of $M_i^-(X_\varnothing)$. Compose projection and inclusion:

$$f_{s,i,r} : M_i^+(X) \rightarrow ks_i^r \rightarrow ks_{r+1} \hookrightarrow M_i^-(X).$$

(2) If $s_i^r \in S_i^-$, then $s_{r+1} \in S_i^+$ and we can define:

$$f_{s,i,r} : M_i^+(X) \rightarrow ks_{r+1}^r \rightarrow ks_i^r \hookrightarrow M_i^-(X).$$

Example 2.6. $\varnothing = ptuyzq$:

```
  -   +
   p
  t ← r  f_{s,1,1}
 u ← s  f_{s,1,2}
 u ← t  f_{s,2,3}
 u ← q  f_{s,2,4}
 v
```

Theorem 2.7 (well-known?). Let $S$ be a set of strings such that for each string $s$ we have that $|\{s, s^\sim\} \cap S| = 1$. Let $B$ be a system of representatives of the bands. Then the

$$(X_s, f_s, \bullet) \quad \text{and} \quad (Y_{b,n}, f_{b,n,p,\bullet})$$

with $b \in B$, $n \in \mathbb{N}_+$, $p \in \mathcal{P} = \text{“monic irreducible polynomials in } k[X] \setminus \{X\}$” give a complete list of the indecomposable representations of $M = M(S, \sim)$. 

8
3 Strategy of Proof of Theorem 2.7

Friday 18^{th} 11:15 – Christof Geiß (Mexico City, Mexico)

For \((X, f_\bullet) \in \text{rep}(M(S, \sim))\) define

\[
\dim(X, f_\bullet) := \sum_{i=1}^{n} (\dim M_i^+(X) + \dim M_i^-(X)).
\]

We show the claim by induction on \(\dim(X, f_\bullet)\) for all bundles of chains simultaneously.

More precisely, given an indecomposable representation \((X, f_\bullet) \in \text{rep}(M(S, \sim))\) we find a subcategory \(\mathcal{M} \subseteq \text{rep}(M(S, \sim))\) containing \((X, f_\bullet)\) such that there is a “reduction” (equivalence) \(\mathcal{M} \xrightarrow{\Phi} \mathcal{N} \subseteq \text{rep}(M(S', \sim'))\) with \(\dim \Phi(X, f_\bullet) < \dim(X, f_\bullet)\).

→ name of “self-reproducing systems”

Reduction Algorithm

Let \((X, f_\bullet) \in \text{rep}(M(S, \sim))\) be indecomposable. We may suppose that some \(f_i \neq 0\).

Then:

\[
f_i = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

For some \(x \in S_i^-\) and \(y \in S_i^+\) we have \((f_i)_{xy} \neq 0\) and this block shape.

Let \(\mathcal{M} \subseteq \text{rep}(M(S, \sim))\) where \(f_i\) has this block shape. In more categorical terms we demand that

\[
R_i(\text{rad}^{\mathcal{F}}(M_i^+(X))) \subseteq \text{rad}^{\mathcal{F}}(M_i^-(X)).
\]

There are two cases:

(1) “\(y \neq x^-\)” We can perform (within column block \(x\) and row block \(y\)) row and column transformations independently. We get the following:
“Somewhere else” (e.g. $y^\sim \in M_j^-$, $x^\sim \in M_j^+$):

\[
f_j' = \begin{array}{c}
x^\sim \\
x_1 \\
\end{array}
\]

This tells us how to define $(S', \sim')$:

- Go to the chain which contains $x^\sim$, substitute $x^\sim$ by $x_1 > x^\sim$.
- Go to the chain which contains $y^\sim$, substitute $y^\sim$ by $y^\sim > y_1$ and set $y_1 \sim' x_1^\sim$.

Recall that $(X, f_{\bullet}) \cong (X, f'_{\bullet})$ and $\Phi(X, f_{\bullet})_j = f_j'$ for $j \neq i$.

Just insert a new division in the blocks of $x^\sim / y^\sim$.

Define

\[
\Phi(X, f_{\bullet})_i = \begin{cases} \gamma & (\ast) \\
\end{cases}
\]

We claim that this defines a functor from $\mathcal{M} \to \mathcal{N}/I$ where $\mathcal{N} \subseteq \text{rep}(\mathcal{M}(S', \sim'))$ and $\mathcal{N}$ consists of the objects $(Y, g_{\bullet})$ that have the block shape in $(\ast)$.

This functor is an equivalence and

strings / bands $\cong$ strings / bands.

**Example 3.1.** $\Phi$ on strings / bands: $x^\sim y \leftrightarrow x_1$
(2) “$y = x^\sim$”: So column transformations in $x$ are conjugate to row transformations in $y$. We can bring $f_{1xy}$ to rational normal form (or Jordan normal form if $k = \overline{k}$). Blocks which are invertible correspond to indecomposable direct summands which correspond to the band $\cdots x x x \cdots$.

So we have to worry only about the nilpotent block of shape

$$
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & 0 \\
\vdots & & & 0 & 1 \\
0 & \cdots & \cdots & \cdots & 0 \\
\end{pmatrix}
$$

Thus we can obtain by admissible transformations:

Now define a new $(S', \sim')$:

- Substitution in $S_1^+$: $x_k > x_{k-1} > \cdots > x_1 > x$
- Substitution in $S_2^-$: $y > x_1 > y_2 > \cdots > x_k$ and set $y_k \sim' x_k$.

Now we can define in a similar way our functor

$$
\Phi : M \to N/I
$$

where $N \subseteq \text{rep}(M(S', \sim'))$.