Summer School on Gentle Algebras

Christof's Talks

BIREP

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Contents

1 Classification of Finite-Dimensional Indecomposable Representations of Gentle Algebras via Bimodule Problems I

Wednesday 16^{th} 11:15 – Christof Geiß (Mexico City, Mexico)

History.

- Gelfand problem, 1974: Classify indecomposables for

$$\begin{tikzcd} \bullet \arrow[r, bend right, "b_1"'] & \bullet \arrow[l, bend right, "a_1"'] \arrow[r, bend left, "a_2"] & \bullet \arrow[l, bend left, "b_2"] \end{tikzcd}$$

  with relation $b_1a_1 - b_2a_2$ (*not* special biserial).

- Kiev school:
  - Nazarova–Roiter, 1974: self-reproducing systems
  - Bondarenko, 1992: bundles of semichains
- Bangming Deng, 1995: Ph.D. thesis under Gabriel
- Crawley-Boevey, 1989: using functorial filtrations (clans)

Recall 1.1 (Bimodule problem)**.** *Let $\mathcal{A}$ be an additive (Krull–Schmidt) k-category. For an $\mathcal{A}$-$\mathcal{A}$* bimodule*, i.e. a k-linear functor*

$$M : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow k\text{-}\mathrm{mod}\,,$$

we have a category $\mathrm{Rep}(M)$ *with:*

- *objects:* (X, m) *with* $X \in \mathcal{A}$ *and* $m \in M(X,X)$
- $\mathrm{Hom}((X,m),(Y,n)) = \{\varphi \in \mathcal{A}(X,Y) \mid M(X,\varphi)\cdot m = M(\varphi,Y)\cdot n \textit{ in } M(X,Y)\}$

Remark 1.2 (Special case)**.** $\underline{M} = (M_1^+, M_1^-, \dots, M_n^+, M_n^-)$ functors $\mathcal{A} \to k\text{-}\mathrm{mod}$
$\rightsquigarrow M = \bigoplus_{i=1}^n \mathrm{Hom}_k(M_i^+, M_i^-)$ is a special kind of $\mathcal{A}$-$\mathcal{A}$-bimodule.
$\rightsquigarrow \mathrm{Rep}(\underline{M})$ has objects $(X,(f_1,\dots,f_n))$ with $X \in \mathcal{A}$ and $f_i \in \mathrm{Hom}_k(M_i^+(X), M_i^-(X))$.
This kind of bimodule problem is called a *tangle*.

Example 1.3. Let $\mathcal{A} = (A\text{-}\mathrm{mod}) \times (k\text{-}\mathrm{mod})$, $M^- = \mathrm{Hom}_A(X,-)$ for some $X \in A\text{-}\mathrm{mod}$ and $M^+ = \mathrm{id}_{k\text{-}\mathrm{mod}}$. $\rightsquigarrow \mathrm{rep}(\underline{M}) \cong A[X]\text{-}\mathrm{mod}$ (one-point extension)

Bundles of Chains

Let S_i^ε $(i = 1, \dots, n, \varepsilon = \pm)$ be finite linearly ordered sets:

$$\begin{array}{c}\bullet\\ \vee\\ \bullet\\ \vdots\\ \bullet\\ \vee\\ \bullet\end{array}$$

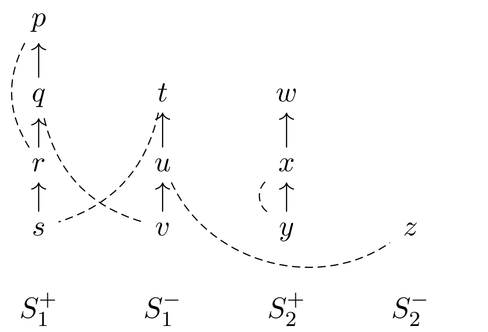
Let $S := \dot{\bigcup}_{i,\varepsilon} S_i^\varepsilon$ with the obvious poset structure.

Let $\sim$ be an equivalence relation on S such that each equivalence class has 1 or 2 elements.

Let $\mathcal{A} = \operatorname{add} \mathcal{S}$ be the k-category where $\mathcal{S}$ has

- as objects the equivalence classes $S/\sim$,
- $\operatorname{rad} \mathcal{S}(a, b) = \bigoplus_{X \in a, Y \in b, x > y} k(x|y)$,
- and obvious compositions.

Example 1.4.



where $p \sim r$, $q \sim v$, $s \sim t$, $u \sim z$, $x \sim y$. Morphisms in the radical of $\mathcal{S}$

$$\begin{tikzcd}
\{p,r\} \arrow[r, "c"] \arrow[d, shift left, "a"] & \{s,t\} \arrow[d, "d"] & \{w\} \arrow[d, "f"] \\
\{q,v\} \arrow[u, bend left, "b"] & \{u,z\} \arrow[l, "e"] & \{x,y\} \arrow[loop right, "g"]
\end{tikzcd}$$

with relations ab, dc, be, g^2.

Each chain S_i^ε gives rise to a (uniserial) module M_i^ε with

$$M_i^\varepsilon(a) = \bigoplus_{x \in a \cap S_i^\varepsilon} k\underline{x}\,.$$

We are interested in

$$\operatorname{Rep}\left(\bigoplus_{i=1}^{n} \operatorname{Hom}_k(M_i^+, M_i^-)\right)\,.$$

Example 1.5. In Example 1.4 this can be visualized as a matrix problem:

	s	r	q	p
t				
u				
v				

	y	x	w
z			

Coupling between: $t \leftrightarrow s$, $r \leftrightarrow p$, $q \leftrightarrow v$, $u \leftrightarrow z$, $y \leftrightarrow x$

The aim of the Kiev school was to produce normal forms for this kind of problems.

Let $kQ/\langle\rho\rangle$ be a gentle algebra with ρ a set of paths of length 2.
We need polarizations $\sigma, \tau : Q_1, \to \{+, -\}$ with:

- $s(a) = s(b)$ and $a \neq b \Rightarrow \sigma(a) \neq \sigma(b)$
- $t(a) = t(b)$ and $a \neq b \Rightarrow \tau(a) \neq \tau(b)$
- $cb \in \rho \Rightarrow \tau(c) = \sigma(b)$

Now:

- $Q_0 = \{1, 2, \ldots, n\}$
- $S = \{i^\varepsilon \mid i = 1, \ldots, n, \varepsilon = +, -\} \cup \{a^\varepsilon \mid a \in Q_1, \varepsilon = +, -\}$
- $S_i^\varepsilon = \{i^\varepsilon\} \cup \{a^+ \mid a \in Q_1, s(a) = i, \sigma(a) = \varepsilon\} \cup \{b^- \mid b \in Q_1, t(b) = i, \tau(b) = \varepsilon\}$

$\Rightarrow |S_i^\varepsilon| \in \{1, 2, 3\}$

$$\begin{array}{c} a^+ \\ \vee \\ i^\varepsilon \\ \vee \\ b^- \end{array}$$

With equivalence relation: $a^+ \sim a^-$

Example 1.6.

$$\begin{tikzcd} 1 \arrow[loop left, "e"] \arrow[r, bend right, "b"'] & 2 \arrow[l, bend right, "a"'] \end{tikzcd}$$

with relations e^2 and ba. Then S looks as follows:

$$\begin{tikzcd}[row sep=small]
e^+ & b^+ \arrow[ddrr, dashed, no head, bend left=15] & a^+ & \\
1^+ \arrow[u] & 1^- \arrow[u] & 2^+ \arrow[u] & 2^- \\
e^- \arrow[u] \arrow[uu, dashed, no head, bend left=40] & a^- \arrow[u] \arrow[uur, dashed, no head] & & b^- \arrow[u]
\end{tikzcd}$$

2 Classification of Finite-Dimensional Indecomposable Representations of Gentle Algebras via Bimodule Problems II

Thursday 17^{th} 11:15 – Christof Geiß (Mexico City, Mexico)

Recall 2.1.

- *bundles of chains: disjoint linearly ordered sets* $S_1^+, S_1^-, \ldots, S_n^+, S_n^-$
- *equivalence relation* $\sim$ *on* $S = \bigcup_{i,\varepsilon} S_i^\varepsilon$ *with each class containing at most* 2 *elements*
- *define a* k*-category* $\mathcal{S}$ *such that*
  - $\operatorname{obj} \mathcal{S} = S/\sim$
  - $\operatorname{rad} \mathcal{S} = \operatorname{rad} kS$
  - $\mathcal{A} = \operatorname{add} \mathcal{S}$
- *objects in* $\mathcal{A}$*:* $X = \bigoplus_{a \in \mathcal{S}} a \otimes_k V_a$
- $M_i^\varepsilon : \mathcal{A} \to k\text{-}\mathrm{mod}$ *functors coming from the* S_i^ε
- $M_i^\varepsilon(X) = \bigoplus_{x \in S_i^\varepsilon} V_{\tilde{x}}$ *where* $\tilde{x}$ *is the equivalence class of* x
- $\operatorname{rep}(M(S,\sim))$*:* $(X, f_1, \ldots, f_n)$ *with* $f_i \in \operatorname{Hom}_k(M_i^+(X), M_i^-(X))$
- $kQ/\langle\rho\rangle$ *gentle algebra where* ρ *contains only paths of length* 2*:*
  - *polarization* $\sigma, \tau : Q_1 \to \{+,-\}$
  - $S_i^\varepsilon = \{i^\varepsilon\} \cup \{a^+ \mid a \in Q_1, s(a) = i, \sigma(a) = \varepsilon\} \cup \{b^- \mid b \in Q_1, t(b) = i, \tau(b) = \varepsilon\}$
  - *each* S_i^ε *is a chain of length* ≤ 3
  - *length-*3 *chains:* $a^+ > i^\varepsilon > a^-$

Proposition 2.2. *With* $(S,\sim)$ *as just defined for a gentle algebra* $A = kQ/\langle\rho\rangle$ *we have an equivalence*

$$F : \operatorname{rep}_b(M(S,\sim)) \xrightarrow{\simeq} A\text{-}\mathrm{mod}$$

where b *means:*

- $(X, f_\bullet) \in \operatorname{rep}(M(S,\sim))$ *such that* f_i *is bijective for all* i*.*

Proof (sketch). Let $(X, f_\bullet) \in \operatorname{rep}_b(M(S,\sim))$, then $F(X, f_\bullet)_i := M_i^+(X)$ for all $i \in Q_0$.

Recall that for $a \in Q_1$ we have $a^+ \sim a^-$ in S. If $s(a) = i$, $t(a) = j$, $a^+ \in S_i^\varepsilon$, $a^- \in S_j^\eta$, then we have a canonical isomorphism

$$(M_i^\varepsilon / \operatorname{rad} M_i^\varepsilon)(X) \xrightarrow{\xi_X^a} \operatorname{soc} M_j^\eta$$

coming from $a^+ \sim a^-$.

Now we have to distinguish four cases to define $F(X, f_\bullet)(a)$:

- $(\varepsilon, \eta) = (+,+)$:

$$M_i^+(X) \to (M_i^+/\operatorname{rad} M_i^+)(X) \xrightarrow{\xi_X^a} \operatorname{soc} M_j^+(X) \hookrightarrow M_j^+(X)$$

- $(\varepsilon, \eta) = (+,-)$:

$$M_i^+(X) \to (M_i^+/\operatorname{rad} M_i^+)(X) \xrightarrow{\xi_X^a} \operatorname{soc} M_j^-(X) \xrightarrow{f_j} M_j^+(X)$$

- $(\varepsilon, \eta) = (-,-)$ and $(\varepsilon, \eta) = (-,+)$: similar

$\square$

Strings and Bands for $(S, \sim)$ Bundle of Chains

We may assume that $(S, \sim)$ is complete, i.e. each equivalence class contains exactly two elements.

We can define an involution $(-)^\sim$ on S such that for each $x \in S$ we have $\overline{x} = \{x, x^\sim\}$.

$$\widehat{S} \;=\; \bigcup_{i=1}^{n} (S_i^+ \times S_i^-) \;\cup\; \bigcup_{i=1}^{n} (S_i^- \times S_i^+) \;\subseteq\; S \times S$$

Definition 2.3. *A* string *for $(S, \sim)$ is a sequence of elements $\underline{s} = (s_1, s_2, \ldots, s_\ell)$ of S such that $(s_i^\sim, s_{i+1}) \in \widehat{S}$ for $i = 1, 2, \ldots, \ell - 1$. A* band *is an infinite periodic string.*

Remark 2.4. If $\underline{s}$ is a string, then so is $\underline{s}^\sim = (s_\ell^\sim, s_{\ell-1}^\sim, \ldots, s_1^\sim)$.
If $\underline{b}$ is a band, then so are $\underline{b}^\sim$ and $\underline{b}[1]$.

Definition 2.5. *Define for each string $\underline{s} = (s_1, \ldots, s_\ell)$ a representation $(X_{\underline{s}}, f_{\underline{s},1}, \ldots, f_{\underline{s},n})$ of $M = M(S, \sim)$*

$$X_{\underline{s}} \;:=\; \bigoplus_{i=1}^{\ell} \{s_i, s_i^\sim\} \;\in\; \mathcal{A}$$

where $\{s_i, s_i^\sim\} \in \operatorname{obj} \mathcal{S}$. Observe

$$M_j^\varepsilon(X_{\underline{s}}) \;=\; \bigoplus_{\substack{i=1 \\ s_i \in S_j^\varepsilon}}^{\ell} k s_i \oplus \bigoplus_{\substack{i=1 \\ s_i^\sim \in S_j^\varepsilon}}^{\ell} k s_i^\sim \,.$$

with structure maps

$$f_{\underline{s},i} \;=\; \sum_{\substack{r=1 \\ s_r^\sim \in S_i^+ \dot\cup S_i^-}}^{\ell-1} f_{\underline{s},i,r}$$

where

$$f_{\underline{s},i,r} : M_i^+(X) \to M_i^-(X)$$

is defined as follows:

(1) If $s_r^{\sim} \in S_i^{+}$, we have a direct summand $ks_r^{\sim}$ of $M_i^{+}(X_{\underline{s}})$ and a direct summand ks_{r+1} of $M_i^{-}(X_{\underline{s}})$. $\rightsquigarrow$ Compose projection and inclusion:

$$f_{\underline{s},i,r} : M_i^{+}(X) \twoheadrightarrow ks_r^{\sim} \xrightarrow{1} ks_{r+1} \hookrightarrow M_i^{-}(X)\,.$$

(2) If $s_r^{\sim} \in S_i^{-}$, then $s_{r+1} \in S_i^{+}$ and we can define:

$$f_{\underline{s},i,r} : M_i^{+}(X_{\underline{s}}) \twoheadrightarrow ks_{r+1}^{\sim} \xrightarrow{1} ks_r^{\sim} \hookrightarrow M_i^{-}(X_{\underline{s}})\,.$$

Example 2.6. $\underline{s} = ptuyzq$:

$$\begin{tikzcd}[row sep=small]
- & + & \\
 & p \arrow[d, dash, dashed] & \\
t \arrow[dr, dash, dashed] & r \arrow[l] & f_{\underline{s},1,1} \\
u \arrow[d, dash, dashed] & s \arrow[l] & f_{\underline{s},1,2} \\
z & y \arrow[l] \arrow[d, dash, dashed] & f_{\underline{s},2,3} \\
z \arrow[d, dash, dashed] & x \arrow[l] & f_{\underline{s},2,4} \\
u & q \arrow[l] \arrow[dl, dash, dashed] & f_{\underline{s},1,5} \\
v & &
\end{tikzcd}$$

Theorem 2.7 (well-known?)**.** *Let $\mathcal{S}$ be a set of strings such that for each string $\underline{s}$ we have that $|\{\underline{s}, \underline{s}^{\sim}\} \cap \mathcal{S}| = 1$. Let $\mathcal{B}$ be a system of representatives of the bands. Then the*

$$\big(X_{\underline{s}}, f_{\underline{s},\bullet}\big) \qquad \textit{and} \qquad \big(Y_{\underline{b},n}, f_{\underline{b},n,p,\bullet}\big)$$

with $\underline{b} \in \mathcal{B}$, $n \in \mathbb{N}_{+}$, $p \in \mathcal{P}$ = "monic irreducible polynomials in $k[X] \setminus \{X\}$" give a complete list of the indecomposable representations of $M = M(S, \sim)$.

3 Strategy of Proof of Theorem 2.7

Friday 18^{th} 11:15 – Christof Geiß (Mexico City, Mexico)

For $(X, f_\bullet) \in \operatorname{rep}(M(S, \sim))$ define

$$\dim(X, f_\bullet) := \sum_{i=1}^{n} (\dim M_i^+(X) + \dim M_i^-(X)) \,.$$

We show the claim by induction on $\dim(X, f_\bullet)$ for all bundles of chains simultaneously.

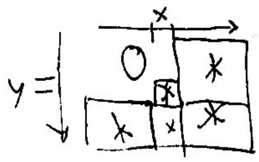
More precisely, given an indecomposable representation $(X, f_\bullet) \in \operatorname{rep}(M(S, \sim))$ we find a subcategory $\mathcal{M} \subseteq \operatorname{rep}(M(S, \sim))$ containing $(X, f_\bullet)$ such that there is a "reduction" (equivalence) $\mathcal{M} \xrightarrow{\Phi} \mathcal{N} \subseteq \operatorname{rep}(M(S', \sim'))$ with $\dim \Phi(X, f_\bullet) < \dim(X, f_\bullet)$.

$\rightsquigarrow$ name of "self-reproducing systems"

Reduction Algorithm

Let $(X, f_\bullet) \in \operatorname{rep}(M(S, \sim))$ be indecomposable. We may suppose that some $f_i \neq 0$.

Then:

f_i = 

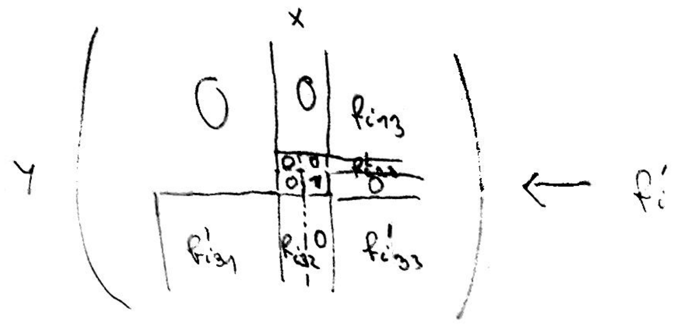
For some $x \in S_i^-$ and $y \in S_i^+$ we have $(f_i)_{xy} \neq 0$ and this block shape.

Let $\mathcal{M} \subseteq \operatorname{rep}(M(S, \sim))$ where f_i has this block shape. In more categorical terms we demand that

$$R_i(\operatorname{rad}^{k_i}(M_i^+(X))) \subseteq \operatorname{rad}^{\ell_i}(M_i^-(X)) \,.$$

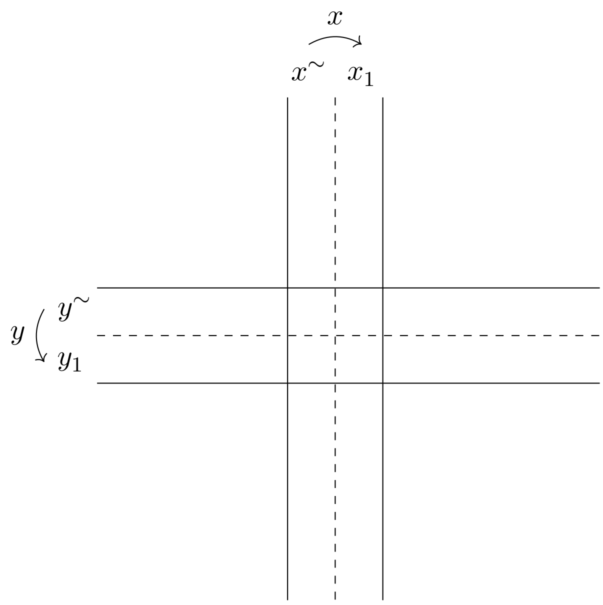
There are two cases:

(1) "$y \neq x^\sim$": We can perform (within column block x and row block y) row and column transformations independently. We get the following:



“Somewhere else” (e.g. $y^{\sim} \in M_j^-$, $x^{\sim} \in M_j^+$):

$f'_j \;=$



This tells us how to define $(S', \sim')$:

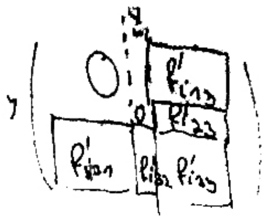
- Go to the chain which contains $x^{\sim}$, substitute $x^{\sim}$ by $x_1 > x^{\sim}$.
- Go to the chain which contains $y^{\sim}$, substitute $y^{\sim}$ by $y^{\sim} > y_1$ and set $y_1 \sim' x_1^{\sim}$.

Recall that $(X, f_\bullet) \cong (X, f'_\bullet)$ and $\Phi(X, f_\bullet)_j = f'_j$ for $j \neq i$.

Just insert a new division in the blocks of $x^{\sim}$ / $y^{\sim}$.

Define

$$\Phi(X, f_\bullet)_i \;=$$



$(\star)$

We claim that this defines a functor from $\mathcal{M} \to \mathcal{N}/I$ where $\mathcal{N} \subseteq \operatorname{rep}(M(S', \sim'))$ and $\mathcal{N}$ consists of the objects $(Y, g_\bullet)$ that have the block shape in $(\star)$.

This functor is an equivalence and

$$\text{strings / bands} \overset{\Phi}{\leftrightarrow} \text{strings / bands}.$$

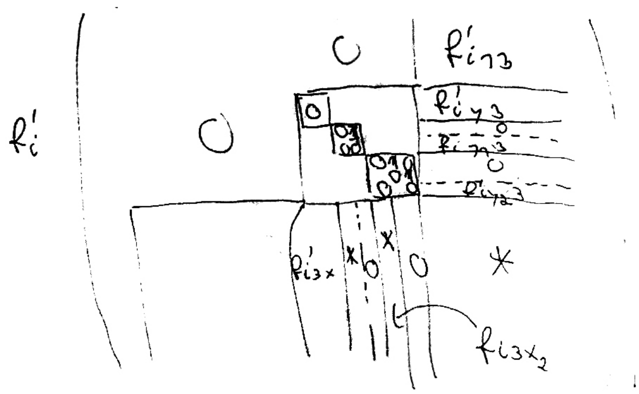
Example 3.1. Φ on strings / bands: $x^{\sim} y \leftrightarrow x_1$

(2) "$y = x^{\sim}$": So column transformations in x are conjugate to row transformations in y. We can bring f_{1xy} to rational normal form (or Jordan normal form if $k = \overline{k}$). Blocks which are invertible correspond to indecomposable direct summands which correspond to the band $\cdots xxx \cdots$.

So we have to worry only about the nilpotent block of shape

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

Thus we can obtain by admissible transformations:



Now define a new $(S', \sim')$:

- Substitution in S_i^+: x by $x_k > x_{k-1} > \cdots > x_1 > x$
- Substitution in S_i^-: y by $y > x_1 > y_2 > \cdots > x_k$ and set $y_k \sim' x_k$.

Now we can define in a similar way our functor

$$\Phi : \mathcal{M} \to \mathcal{N}/I$$

where $\mathcal{N} \subseteq \mathrm{rep}(M(S', \sim'))$.