Summer School on
Gentle Algebras

Participants' Talks

BIREP

14–18 August 2017

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Contents

1 Introduction to Gentle, String, Biserial and Special Biserial Algebras

Monday 14^{th} 13:00 – Mariusz Kaniecki (Toruń, Poland)

References.

(1) A. Skowroński, J. Waschbüsch. *Representation finite biserial algebras*, 1983.

(2) J. Külshammer's website. *"Biserial algebras"*.

(3) J. Schröer. *Biserial / special biserial / string / gentle algebras*, 2016

(4) A. Skowroński. *The finite-dimensional algebras in the mathematical nature* (Polish).

Notation.

- k a field
- A a finite-dimensional k-algebra

Definition 1.1. *A is* biserial *if it satisfies the following two properties:*

(a) The radical $\mathrm{rad}(P)$ *of each indecomposable projective right A-module P is the sum of at most two uniserial submodules U_1 and U_2 with $\ell(U_1 \cap U_2) \leq 1$.*

(b) The radical $\mathrm{rad}(P)$ *of each indecomposable projective left A-module P is the sum of at most two uniserial submodules U_1 and U_2 with $\ell(U_1 \cap U_2) \leq 1$.*

Definition 1.2. *A is* special biserial *if $A \cong kQ/I$ for an admissible ideal I such that:*

(SB1) $|\{a \in Q_1 \,|\, s(a) = i\}| \leq 2$ *and* $|\{a \in Q_1 \,|\, t(a) = i\}| \leq 2$ *for each* $i \in Q_0$.

(SB2) For arrows $a, b, c \in Q_1$, $a \neq b$, $t(a) = t(b) = s(c)$, it is $ca \in I$ or $cb \in I$.

(SB3) For arrows $a, b, c \in Q_1$, $a \neq b$, $s(a) = s(b) = t(c)$, it is $ac \in I$ or $bc \in I$.

Lemma 1.3 (Skowroński–Waschbüsch)**.** *Any special biserial algebra is a biserial algebra.*

Proof. Let $A = kQ/I$ and $j \overset{a}{\leftarrow} i \in Q_1$. Let $w = a_s \cdots a_2 a_1$ be maximal in the set of all paths starting with a and not belonging to I.

Now $A(a + I) \subseteq \mathrm{rad}(Ae_i)$ is a uniserial module.

Suppose that we have two parallel paths $u = a_n \cdots a_2 a_1$ and $v = b_m \cdots b_2 b_1$ starting in i with $a_1 \neq b_1$ but $A(u + I) = A(v + I) \neq 0$.

By (SB2) $a_n \neq b_m$, so $A(u + I) = K(u + I) \subseteq \mathrm{soc}(Ae_i)$. Assume $c \in Q_1$ and $cu \notin I$. Then $t(c)$ gives the second upper (if any) factor of $A(u + I) = A(v + I)$ leading to the contradiction $cv \notin I$, $ca_n \notin I$, $cb_m \notin I$. □

Example 1.4. Let $A = kQ/I$ for the quiver

$$Q = \begin{array}{ccccccc} 1 & \xrightarrow{a} & 2 & \xrightarrow{b} & 4 & \xrightarrow{e} & 5 \\ & & & {\scriptstyle c}\searrow & & \nearrow{\scriptstyle d} & \\ & & & & 3 & & \end{array}$$

and $I = \langle eb, ba - dca \rangle$. Then Ae_2 and e_4A look as follows

$$\begin{array}{ccc} & e_2 & \\ \swarrow & & \searrow \\ b + I & & c + I \\ & & \downarrow \\ & & dc + I \\ & & \downarrow \\ & & edc + I \end{array} \qquad \begin{array}{ccc} & e_4 & \\ \swarrow & & \searrow \\ (b - dc) + I & & d + I \\ & & \downarrow \\ & & dc + I \\ & & \downarrow \\ & & dca + I \end{array}$$

Here, A is biserial but not special biserial.

Definition 1.5. *A special biserial algebra $A = kQ/I$ is a* string algebra *if additionally to (SB1)–(SB3) the following condition holds:*

(SB4) The ideal I can be generated by zero relations.

Example 1.6.

(a) $A = k[T]/(T^n)$ where Q is the quiver $\bullet \circlearrowleft T$.

(b) Any Nakayama algebra is a string algebra. Recall that A is a *Nakayama algebra* if for any indecomposable projective or indecomposable injective A-module M there is a filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_s = M$ such that all M_j/M_{j-1} are simple.

Definition 1.7. *A string algebra $A = kQ/I$ is a* gentle algebra *if additionally to the conditions (SB1)–(SB4) the following hold:*

(SB5) For arrows $a, b, c \in Q_1$, $a \neq b$, $t(a) = t(b) = s(c)$, it is $ca \notin I$ or $cb \notin I$.

(SB6) For arrows $a, b, c \in Q_1$, $a \neq b$, $s(a) = s(b) = t(c)$, it is $ac \notin I$ or $bc \notin I$.

(SB7) The ideal I can be generated by a set of paths of length 2.

2 The Representation Theory of the Lorentz Group

Monday 14th 14:15 – Philipp Lampe (Durham, United Kingdom)

(after Gel'fand and Ponomarev)

Notes: http://maths.dur.ac.uk/users/philipp.b.lampe/LorentzBadDriburg.pdf

(a) *Minkowski space*: $\mathbb{R}^{1,3} = (\mathbb{R}^4, \eta)$ with the bilinear form $\eta : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ defined as $\eta(x, y) = x_0y_0 + x_1y_1 + x_2y_2 - x_3y_3$.

(b) *Lorentz group*: $\mathrm{O}(\mathbb{R}^{1,3}) = \{f \in \mathrm{GL}(\mathbb{R}^4) \,|\, \eta(x, y) = \eta(f(x), f(y))\, \forall\, x, y \in \mathbb{R}^4\}$.
In matrix form with $G = \mathrm{diag}(1, -1, -1, -1)$:

$$\mathrm{O}(1, 3) = \{\Lambda \in \mathrm{GL}(4, \mathbb{R}) \,|\, \Lambda^T G\Lambda = G\}$$

$$\mathrm{SO}(1, 3) = \{\Lambda \in \mathrm{O}(1, 3) \,|\, \det(\Lambda) = 1\}$$

(c) *One-parameter subgroups*:

$$A_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(t) & -\sin(t) \\ 0 & 0 & \sin(t) & \cos(t) \end{pmatrix} : t \in \mathbb{R} \right\} \qquad \text{“space rotations” (similarly: } A_2,\ A_3)$$

$$B_1 = \left\{ \begin{pmatrix} \cosh(t) & \sinh(t) & 0 & 0 \\ \sinh(t) & \cosh(t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\} \qquad \text{“Lorentz boosts” (similarly: } B_2,\ B_3)$$

(d) *Lie algebra* $\mathfrak{so}(1, 3)$:

Proposition 2.1. *The complexified Lie algebra of* $\mathrm{SO}(1, 3)$ *is isomorphic to*

$$\langle a_i, b_i \,|\, i = 1, 2, 3\rangle_{\mathbb{R}\,\mathbb{C}}$$

with $[a_k, a_{k+1}] = a_{k+2} = -[b_k, b_{k+1}]$, $[a_k, b_{k+1}] = b_{k+2} = [b_k, a_{k+1}]$, $[a_k, b_k] = 0$.
The Lie algebra $\mathfrak{so}(1, 3)_{\mathbb{C}}$ *contains the Lie subalgebra* $\mathfrak{so}(3)_{\mathbb{C}}$ *of simple type* $\mathbb{A}_1$.

(e) *Classification of finite-dimensional irreducible* $\mathfrak{so}(3)_{\mathbb{C}}$*-modules*: The Lie algebra $\mathfrak{so}(3)_{\mathbb{C}}$ has a basis $h_+ = ia_1 - a_2$, $h_- = ia_1 + a_2$, $h_3 = a_3$ with relations

$$[h_+, h_3] = -h_+\,, \qquad [h_-, h_3] = h_-\,, \qquad [h_+, h_-] = 2h_3\,.$$

Theorem 2.2. *Every irreducible finite-dimensional representation of $\mathfrak{so}(3)_\mathbb{C}$ is isomorphic to R_ℓ for some $\ell \in \frac{1}{2}\mathbb{N}_0$ where*

$$R_\ell = \langle e_m \,|\, m = -\ell, -\ell+1, \ldots, \ell \rangle_\mathbb{C}$$

with

$$\begin{aligned} h_+ e_m &= \sqrt{(\ell+m+1)(\ell-m)}\, e_{m+1}\,, \\ h_- e_m &= \sqrt{(\ell-m+1)(\ell+m)}\, e_{m-1}\,, \\ h_3 e_m &= m e_m\,. \end{aligned}$$

(f) *Harish-Chandra module*: A module M over $\mathfrak{so}(1,3)_\mathbb{C}$ is *HC* if restricted to $\mathfrak{so}(3)_\mathbb{C}$ it is isomorphic to $\bigoplus_{\ell \in \frac{1}{2}\mathbb{N}_0} R_\ell^{k_\ell}$ with $k_\ell \in \mathbb{N}$. Let $R_{\ell,m} \subseteq R_\ell^{k_\ell}$ be the eigenspace of h_3 for the eigenvalue m. Then (under some finiteness condition?)

$$M = \bigoplus_{\ell,m} R_{\ell,m}\,.$$

(g) *New bases*:

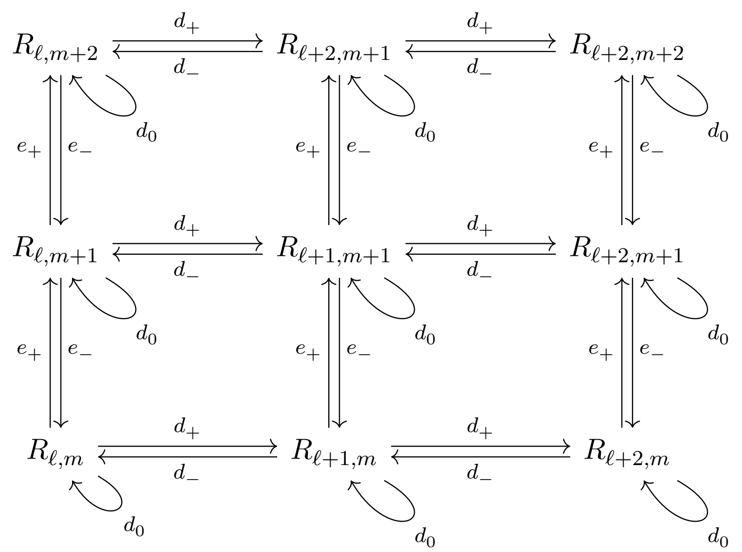
$$\begin{aligned} h_+ &= ia_1 - a_2 \qquad & h_- &= ia_1 + a_2 \qquad & h_3 &= a_3 \\ f_+ &= ib_1 - b_2 \qquad & f_- &= ib_1 + b_2 \qquad & f_3 &= b_3 \end{aligned}$$

$$e_+(x) = \begin{cases} 0 & x \in R_{\ell,m} \text{ with } m = \ell \\ \frac{1}{(\ell+m+1)(\ell-m)}\, h_+(x) & x \in R_{\ell,m} \text{ with } m \neq \ell \end{cases} \qquad (e_-(x) \text{ similarly})$$

(h) *Action on HC modules*: Suppose $d_+, d_-, d_0 : M \to M$ such that

$$\begin{aligned} d_+(R_{\ell,m}) &\subseteq R_{\ell+1,m} \\ d_-(R_{\ell,m}) &\subseteq R_{\ell-1,m} \\ d_0(R_{\ell,m}) &\subseteq R_{\ell,m} \end{aligned}$$

Then we get



such that the diagrams commute and $d_+ d_0 = d_0 d_+$ etc.

Proposition 2.3.

$$\begin{pmatrix} f_3(x) \\ f_+(x) \\ f_-(x) \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \begin{pmatrix} d_- x \\ d_0 e_+ x \\ d_+ e_- x \end{pmatrix}$$

Then the b_1, b_2, b_3 given by f_3, f_+, f_- satisfy commutator relations for a_i, b_i if and only if for every $x \in R_{\ell,m}$:

$$\begin{aligned} &\ell d_+ d_0(x) - (\ell+2) d_0 d_+(x) = 0 \\ &(\ell+1) d_- d_0(x) - (\ell-1) d_0 d_-(x) = 0 \\ &(2\ell-1) d_+ d_-(x) - (2\ell-3) d_- d_+(x) = -d_0^2(x) + x \end{aligned}$$

(i) *Harish-Chandra modules from quiver representations*: Let $\ell_0, \ell_1 \in \frac{1}{2}\mathbb{N}_0$ with $\ell_0 \equiv \ell_1$ mod 1. Let $P \in \mathrm{mod}(\mathbb{C}Q/I)$. Then we have

$$\phi_{\ell_0,\ell_1} : \mathrm{mod}(\mathbb{C}Q/I) \to \mathrm{HC}(\mathfrak{so}(1,3)_{\mathbb{C}})$$

with Q sketched here:

$$0 \cdots \rightleftarrows 0 \rightleftarrows P_1 \overset{\mathrm{id}}{\rightleftarrows} P_1 \overset{\mathrm{id}}{\rightleftarrows} \cdots \overset{\mathrm{id}}{\rightleftarrows} P_1 \underset{P_{\delta_-}}{\overset{P_{\delta_+}}{\rightleftarrows}} P_2 \overset{\mathrm{id}}{\rightleftarrows} P_2 \overset{\mathrm{id}}{\rightleftarrows} P_2$$

Theorem 2.4 (Gel'fand–Ponomarev)**.** $\phi_{\ell_0,\ell_1} : \mathrm{mod}(\mathbb{C}Q/I) \to C_s(\lambda_1, \lambda_2)$ *is an equivalence of categories.*

(The right-hand side is the "singular block" of HC modules where the "Laplace operators" have eigenvalues $\lambda_1 = -i\ell_0\ell_1$ and $\lambda_2 = -1 + \ell_0^2 + \ell_1^2$.)

3 Classification of Indecomposable Modules over Special Biserial and String Algebras

Monday 14^{th} 15:45 – Apolonia Gottwald (Bielefeld, Germany)

3.1 Indecomposable Modules

Notation.

- Λ a special biserial algebra, $\Lambda \cong kQ/I$

Lemma 3.1. *For studying indecomposable non-projective modules we can assume that Λ is a string algebra.*

Proof. Write $\Lambda = P_1 \oplus P_2$ where P_1 is the direct sum of the indecomposable non-uniserial projective-injective modules. Then $\Lambda/\operatorname{soc}(P_1)$ is a string algebra. □

Definition 3.2.

(a) For all arrows b let b^{-1} be its "formal inverse" with $s(b^{-1}) = t(b)$ and $t(b^{-1}) = s(b)$.

(b) Consider words over the alphabet of arrows and inverse arrows.

(c) For $u \in Q_u$ let 1_u with $s(1_u) = u = t(1_u)$.

(d) Strings*: $w = 1_u$ or $w = w_1 w_2 \cdots w_n$ such that*

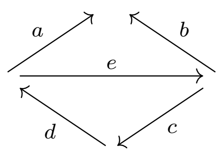
- *$s(w_i) = t(w_{i+1})$ for all $1 \leq i < n$,*
- *there is no $w_i w_{i+1} \cdots w_j \in I$ and no $(w_i w_{i+1} \cdots w_j)^{-1} \in I$,*
- *there is no $w_{i+1} = w_i^{-1}$ for all $1 \leq i < n$.*

(e) Concatenation*: $w_1 \cdots w_m w_{m+1} \cdots w_n$ of $w_1 \cdots w_m$ and $w_{m+1} \cdots w_n$ is said to be* defined *iff it is a string.*

Definition 3.3. *Let $\sim$ be the equivalence relation on strings induced by $w \sim w^{-1}$.*

Let $\mathcal{S}t$ be a complete set of representatives of strings under $\sim$.

Example 3.4.



with relations $ed = 0$ and $ce = 0$. Then $dcb^{-1}a$ and $b^{-1}a \sim a^{-1}b$ are strings.

Definition 3.5. *A string $w = w_1 \cdots w_n$ is a* band *if*

- *all* rotations *$w_i w_{i+1} \cdots w_n w_1 \cdots w_{i-1}$ exist,*
- *all powers exist,*

- *it is not a power itself.*

Definition 3.6. *Let $\sim_r$ be the equivalence relation on bands induced by $w \sim_r w'$ if w' is a rotation of w.*

Example 3.7. In Example 3.4 there are bands $dcb^{-1}a$ and bea^{-1}.

Fact 3.8. *If w is a string $\neq 1_u$ for all $u \in Q_0$ there exists at most one arrow b with wb defined and at most one arrow c with cw defined.*

Definition 3.9. *Let $w = w_1 \cdots w_n$ or $w = 1_u$ be a string.*

Define an algebra C_w and a functor $G_w : C_w\text{-}\mathrm{mod} \to \Lambda\text{-}\mathrm{mod}$.

$\rightsquigarrow$ $C_w(V)$ is the representation over Q_w where $C_w = k$ and Q_w with underlying graph $\mathbb{A}_{n+1}$ with an arrow pointing to the left iff w_i is an arrow.

Example 3.10. $\bullet \underset{b}{\overset{a}{\leftleftarrows}} \bullet$.

For the string $ab^{-1}a$ and $V = k$ we get $G_w(V)$ as a left Λ-module where α and β, respectively, are represented by

$$\begin{pmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&1&0&0 \\ 0&0&0&0 \end{pmatrix} .$$

Draw this as a representation as follows

$$k^2 \underset{\left(\begin{smallmatrix} 0&1 \\ 0&0 \end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix} 1&0 \\ 0&1 \end{smallmatrix}\right)}{\leftleftarrows}} k^2 \quad .$$

Definition 3.11. *Let $w = w_1 \cdots w_n$ be a band (assume w_1 is an arrow).*

Let $C_w = k[x, x^{-1}]$ and Q_w be the quiver that is an oriented cycle with consecutive arrows $w_1, \ldots, w_n$ where w_i is oriented anti-clockwise iff it is an arrow.

$\rightsquigarrow$ $G_w(V)$ is the representation of Q_w where the map at w_1 is x and the maps at w_i for $i \neq 1$ are identities.

$\rightsquigarrow$ There is a band module for all vector spaces V and all linear maps $x : V \to V$.

Example 3.12. $\bullet \underset{b}{\overset{a}{\leftleftarrows}} \bullet$.

There is only one band $w = ba^{-1}$ and $Q_w = \bullet \underset{b}{\overset{a}{\leftleftarrows}} \bullet$.

The total dimension of $G_w(V)$ is $2\dim(V)$. As a Λ-module

$$V \underset{x}{\overset{\mathrm{id}}{\leftleftarrows}} V \quad .$$

For all vector spaces over k and linear maps $x : V \to V$ there is an indecomposable module $M(V, x)$ such that $M(V, x) \cong M(V', x')$ iff $V \cong V'$ and x and x' are similar.

Theorem 3.13. *Let Λ be a string algebra and $I := \mathcal{S}t \,\dot\cup\, \mathcal{B}a$. Then $G_w(V)$ for $w \in I$ form a complete set of representatives of the indecomposable Λ-modules.*

Theorem 3.14. Λ *special biserial* $\Rightarrow$ Λ *tame or of finite type*

3.2 Functorial Filtration

The functors $G_w : C_w\text{-}\operatorname{mod} \to \Lambda\text{-}\operatorname{mod}$ and $F_w : \Lambda\text{-}\operatorname{mod} \to C_w\text{-}\operatorname{mod}$ satisfy:

(1) $F_w G_w \cong \mathrm{id}$, $F_v G_w = 0$ for all $v \neq w$.

(2) $\{F_w : w \in I\}$ is locally finite and reflects isomorphisms.

(3) For all $M \in \Lambda\text{-}\operatorname{mod}$ and $w \in I$ there exists a map $\gamma_{w,M} : G_w F_w(M) \to M$ such that $F_w(\gamma_{w,M})$ is an isomorphism.

(4) For all $M \in \Lambda\text{-}\operatorname{mod}$ the map $\gamma_{w,M} : \bigoplus_{w \in I} G_w F_w(M) \to M$ is an isomorphism.

(5) M indecomposable $\Rightarrow$ a) $F_w(M) = 0$ and b) $M \cong G_w F_w(M)$.

4 Irreducible Maps of Strings and Band Modules

Monday 14th 17:00 – Ögmundur Eiriksson (Bielefeld, Germany)

4.1 A Reminder on AR-theory

- k a field
- A a finite-dimensional k-algebra
- A- mod the category of finite-dimensional A-modules

Notation.

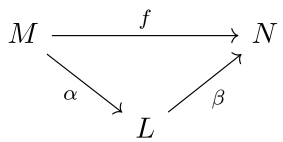
Definition 4.1. *Let $f : M \to N$ be a map in A-* mod*.*

We say f is left almost split *if f is not a split mono and any non-split mono $g : M \to L$ factors through f.* Right almost split *is defined dually.*

Definition 4.2. *We say an exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is an* almost split sequence *(or* AR-sequence*) if f is left almost split and g is right almost split.*

For each non-projective finitely generated indecomposable A-module M there is a unique AR-sequence $0 \to \tau(M) \to N \to M \to 0$. This determines the AR-translate *$\tau(M)$ of M.*

Definition 4.3. *A map $f : M \to N$ with indecomposable A-modules M, N is* irreducible *if for any factorization*



either α is a split mono or β is a split epi.

Example 4.4. Let $Q = 1 \to 2 \to 3$. Then we have an almost split sequence

$$0 \to P(2) \to P(1) \oplus S(2) \to I(2) \to 0\,.$$

4.2 Irreducible Maps for Band Modules

Let Λ be a string algebra over $k = \overline{k}$. Define $C = k[x, x^{-1}]$.

Observe that a finite-dimensional C-module "is the same" as a finite-dimensional vector space with an automorphism. Let $J_n(\lambda)$ be the $n \times n$ Jordan block with eigenvalue λ.

Then we have a $1:1$ correspondence:

$$\text{ind. f.d. } C\text{-modules up to iso} \xleftrightarrow{\;\cong\;} \{J_n(\lambda) \mid \lambda \in k, n \in \mathbb{N}_+\}$$
$$V_n(\lambda) \longleftrightarrow J_n(\lambda)$$

Lemma 4.5. *For $n \geq 1$ and $\lambda \in k^\times$ there is an AR-sequence*

$$0 \to V_n(\lambda) \to V_{n-1}(\lambda) \oplus V_{n+1}(\lambda) \to V_n(\lambda) \to 0\,.$$

In particular, $\tau(V_n(\lambda)) = V_n(\lambda)$ and its AR-component is a tube of rank 1.

Sketch of proof. Fix a basis $w_n, (x-\lambda)w_n, \dots (x-\lambda)^{n-1}w_n$ for $V_n(\lambda)$ where w_n is a generator. Then the matrix for x has Jordan form with respect to this basis. We then put $g(w_{n-1}, 0) := (x-\lambda)w_n$ and $g(0, w_{n+1}) = w_n$. Then g is non-split, surjective, and has kernel $\langle -w_{n-1}, (x-\lambda)w_{n+1}\rangle \cong V_n(\lambda)$. It is enough to check maps to/from $M = V_m(\lambda)$. □

Let w be a band (or an equivalence class of a band) and let $G_w : C\text{-}\mathrm{mod} \to \Lambda\text{-}\mathrm{mod}$ be the functor from the last talk.

Fact 4.6. *G_w sends irreducible maps to irreducible maps.*

Proposition 4.7. *Write $V = V_n(\lambda)$. The sequence*

$$0 \to G_w(V) \xrightarrow{G_w f} G_w(V_{n-1}(\lambda) \oplus V_{n+1}(\lambda)) \xrightarrow{G_w g} G_w(V) \to 0$$

is an AR-sequence. The component of $G_w(V)$ consists of all such $G_w(V_m(\lambda))$ for $m \in \mathbb{N}$.

Sketch of proof. The maps occurring after projecting (resp. restricting) $G_w f$ (resp. $G_w g$) to direct summands are irreducible by our fact. It is possible to see that this shows that we have an AR-sequence. Also by uniqueness of AR-sequences, $\tau(G_w(V)) = G_w(V)$. Since we have found AR-sequences for all $G_w(V)$, we obtain the whole component. □

4.3 Irreducible Maps for String Modules

Let Λ still be a string algebra over $k = \overline{k}$. Let C be a string and let $G_C : k\text{-}\mathrm{mod} \to \Lambda\text{-}\mathrm{mod}$ be the functor from the last talk. We write $M(C) := G_C(k)$.

Definition 4.8. *We say C*

(i) starts (resp. ends) on a peak *if there is no arrow b such that $Cb(b^{-1}C)$ is a string,*

(ii) starts (resp. ends) in a deep *if there is no arrow b such that $Cb^{-1}(bC)$ is a string.*

We say $C = c_1 \cdots c_n$ is directed (resp. inverse) *if all the c_i (resp. c_i^{-1}) are arrows.*

If C, D are strings and b is an arrow such that CbD is a string, then there is a canonical exact sequence

$$0 \to M(C) \to M(CbD) \to M(D) \to 0\,.$$

Similarly, if $Db^{-1}C$ is a string, there is a canonical exact sequence

$$0 \to M(C) \to M(Db^{-1}C) \to M(D) \to 0\,.$$

Hooks and Co-Hooks

Definition 4.9. *If C does not start (resp. end) on a peak, so $Cb(b^{-1}C)$ is a string, there is a unique directed D such that $C_h := CbD^{-1}$ starts (resp. ${}_hC := Db^{-1}C$ ends) in a deep.*

Here, C_h (resp. ${}_hC$) is called a hook.

If C does not start (resp. end) on a deep, so $Cb^{-1}(bC)$ is a string, there is a unique directed D such that $C_c := Cb^{-1}D$ starts (resp. ${}_cC := D^{-1}bC$ ends) on a peak.

Here, C_c (resp. ${}_cC$) is called a co-hook.

Proposition 4.10. *The canonical maps $M(C) \to M(C_h)$ and $M(C) \to M({}_hC)$ and the canonical maps $M(C_c) \to M(C)$ and $M({}_cC) \to M(C)$ are irreducible.*

Irreducible Maps Ending at Projectives (resp. Beginning at Injectives)

For a vertex u, the projective $P(u)$ is a string module: Let C_1, C_2 be the maximal directed paths beginning in u. Then $P(u) \cong M(C_1C_2^{-1})$. If both have length zero, then $P(u)$ is simple. Assume $C_1 = \overline{C}_1 b$ has length ≥ 1. Then there is an irreducible map

$$M(\overline{C}_1) \to M((\overline{C}_1)_h) \cong P(u)\,.$$

Similarly,

$$M(\overline{C}_2) \to M({}_h(\overline{C}_2)) \cong P(u)\,.$$

4.3.1 AR-Sequences

Now there are five families of AR-sequences:

(1) For any b there are C, D maximal directed such that $C^{-1}bD^{-1}$ is a string and starts in a deep and ends on a peak. Note that $\Lambda e_u/\Lambda b \cong M(D^{-1})$. We have an AR-sequence

$$0 \to M(C^{-1}) \to M(C^{-1}bD^{-1}) \to M(D^{-1}) \to 0\,.$$

(2) If C neither starts nor ends on a peak, we have an AR-sequence

$$0 \to M(C) \to M({}_hC) \oplus M(C_h) \to M({}_hC_h) \to 0\,.$$

(3) If C does not start on a peak but ends on a peak, we have with $C = {}_cD$ an AR-sequence

$$0 \to M({}_cD) \to M(D) \oplus M({}_cD_h) \to M(D_h) \to 0\,.$$

(4) If C starts on a peak but does not end on a peak, we have with $C = D_c$ an AR-sequence

$$0 \to M(D_c) \to M(D) \oplus M({}_hD_c) \to M({}_hD) \to 0\,.$$

(5) If C starts and ends on a peak, we have with $C = {}_cD_c$ an AR-sequence

$$0 \to M({}_cD_c) \to M(D_c) \oplus M({}_cD) \to M(D) \to 0\,.$$

5 The Structure of Biserial Algebras

Tuesday 15^{th} 8:30 – Manuel Flores Galicia (Bielefeld, Germany)

$$\text{gentle} \Longrightarrow \text{string} \Longrightarrow \text{special biserial} \underset{?}{\overset{\Longrightarrow}{\Longleftarrow}} \text{biserial}$$

Notation.

- $k = \overline{k}$ a field
- Λ an associative k-algebra with 1, finite-dimensional over k
- $Q = (Q_0, Q_1, s, t)$ a quiver with a trivial path ε_u for each $u \in Q_0$

5.1 Description of Basic Biserial Algebras

Recall 5.1. *Λ is basic if there exists a complete set of primitive orthogonal idempotents e_i (c.s.p.o.i) such that $\Lambda e_i \ncong \Lambda e_j$ for all $i \neq j$.*

Definition 5.2. *Λ is* biserial *if every indecomposable projective left or right Λ-module P contains uniserial submodules U and V such that $U + V = \operatorname{rad}(P)$ and $U \cap V$ is either zero or simple.*

Example 5.3. Nakayama algebras and algebras whose Auslander-Reiten sequences have at most two non-projective summands in their middle term are biserial.

Definition 5.4. *Let Q be a finite quiver.*

(a) A bisection *of Q is a pair (σ, τ) of functions $Q_1 \to \{\pm 1\}$ such that if $a \neq b$ are arrows starting (resp. ending) at the same vertex, then $\sigma(a) \neq \sigma(b)$ (resp. $\tau(a) \neq \tau(b)$.*

(b) The quiver Q is biserial *if for every vertex u, there are at most two arrows starting at u and at most two arrows ending at u.*

Observation 5.5. *Q has a bisection $\Leftrightarrow$ Q is biserial*

Definition 5.6. *Let Q be a quiver and (σ, τ) a bisection of Q. We say a path $a_r \cdots a_1$ is a* good path *or (σ, τ)-*good *if $\sigma(a_i) = \tau(a_{i-1})$ for all $1 < i \leq r$. Otherwise, we say it is a* bad path*. The trivial paths ε_u are* good*.*

Definition 5.7. *By a* bisected presentation *(Q, σ, τ, p, q) of an algebra Λ we mean that Q is a biserial quiver with bisection (σ, τ) and $p, q : kQ \to \Lambda$ are surjective algebra homomorphisms with $p(\varepsilon_u) = q(\varepsilon_u)$ for all $u \in Q_0$ and $p(a), q(a) \in \operatorname{rad}(\Lambda)$ for all $a \in Q_1$ and $q(a)p(x) = 0$ whenever $a, x \in Q_1$ such that ax is a bad path.*

Theorem 5.8 (Vila-Freyer)**.** *Every basic biserial algebra Λ has a bisected presentation (Q, σ, τ, p, q) in which Q is the ordinary quiver of Λ.*

Conversely, any algebra with a bisected presentation is basic and biserial.

Let kQ^+ be the arrow ideal of kQ.

Theorem 5.9 (Vila-Freyer). *Let Q be a quiver with bisection (σ, τ). For each bad path ax of length 2 let d_{ax} be elements in kQ^+ such that*

(1) $d_{ax} = 0$ or $d_{ax} = wb_t \cdots b_1$, $w \in k^\times$, $t \geq 1$, and $b_t \cdots b_1 x$ a good path with $t(b_t) = t(a)$ and $b_t \neq a$,

(2) if $d_{ax} = \phi b$ and $d_{by} = \psi a$ with $\phi, \psi \in k^\times$, then $\phi\psi \neq 1$.

If I is admissible in kQ containing all the elements $(a - d_{ax})x$, then kQ/I is a basic biserial algebra. Conversely, every basic biserial algebra is isomorphic to a quotient of this form.

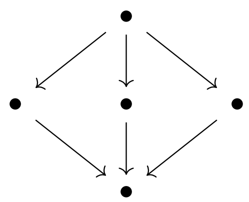
5.2 Distributive Algebras

Let $\mathcal{S}(\Lambda)$ be the *lattice of (left) ideals* of Λ.

Remark 5.10. In general, the distributive law $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ in a lattice does not hold.

Definition 5.11. Λ *is* distributive *iff $\mathcal{S}(\Lambda)$ is distributive.*

Fact 5.12. Λ *is distributive* $\overset{\text{Thrall}}{\Leftrightarrow}$ *the Hasse diagram of $\mathcal{S}(\Lambda)$ does not contain*

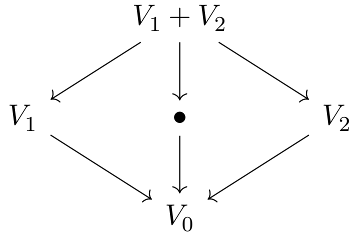


Theorem 5.13 (Jans). *If V is a module over Λ, then the lattice of Λ-submodules of V is finite iff it is distributive.*

Corollary 5.14. *The lattice of left (right, two-sided) ideals of a finite-dimensional algebra over k is finite iff it is distributive.*

Theorem 5.15 (Jans). *If Λ is of finite-representation type, then Λ has a finite ideal lattice. Therefore it is distributive.*

Sketch of the proof of Theorem 5.13. "$\Leftarrow$": Suppose the lattice is not distributive, so there is a diagram



It is enough to show that the submodule lattice of V/V_0 is infinite. So assume $V_0 \neq 0$. Then V_1 and V_2 are distinct direct summands of $V_1 + V_2$. Moreover, $V_1 \oplus U \cong V_2 \oplus U$. Hence, $V_1 \overset{\varphi}{\cong} V_2$. Let $\{v_i\}_{i=1}^r$ be a k-basis of V_1. Then $\{\varphi(v_i)\}_{i=1}^r$ is a k-basis of V_2. One verifies that the set $\{v_i + \kappa\varphi(v_i)\}_{i=1}^r$ for a fixed $\kappa \in k$ is a basis for a Λ-submodule V_κ and that $V_{\kappa_1} \neq V_{\kappa_2}$ for $\kappa_1 \neq \kappa_2$. Since $k = \overline{k}$ is infinite, we have proved "$\Leftarrow$". □

5.3 Representation-Finite Biserial Algebras Are Special Biserial

Recall 5.16. Λ *is* special biserial *if it is Morita-equivalent to a bound quiver algebra kQ/I where (Q, I) satisfies:*

(1) Q is biserial.

(2) For every arrow $a \in Q_1$ there is at most one arrow $b \in Q_1$ and at most one arrow $c \in Q_1$ such that ba and ac are not in I.

Theorem 5.17 (Skowroński-Waschbüsch)**.** *Any distributive biserial algebra is special biserial.*

Corollary 5.18. *Representation-finite biserial algebras are special biserial.*

6 Repetitive Algebras of Gentle Algebras

Tuesday 15^{th} 10:00 – Jordan McMahon (Graz, Austria)

Recall 6.1. *kQ/I is* special biserial *if the following hold:*

(SB1) *Each vertex $i \in Q_0$ has at most 2 arrows starting (resp. ending) at i.*

(SB2) *For each arrow $b \in Q_1$ there is at most one $a \in Q_1$ with $ab \notin I$.*

(SB2') *For each arrow $b \in Q_1$ there is at most one $c \in Q_1$ with $bc \notin I$.*

(G1) *I is generated by paths of length 2.*

(G2) *For each arrow $b \in Q_1$ there is at most one $a \in Q_1$ with $ab \in I$.*

(G3) *For each arrow $b \in Q_1$ there is at most one $c \in Q_1$ with $bc \in I$.*

Definition 6.2. *A path $p \in kQ/I$ is* maximal *if for each $b \in Q_1$ we have $bp = pb = 0$.*

Assume $A = kQ/I$ is locally bounded (i.e. each arrow is contained in a maximal path) and I generated by zero relations and commutativity relations.

Let $DA = \mathrm{Hom}_k(A, k)$ and for each path p let $\varphi_p \in DA$ be the dual path.

6.1 Repetitive Algebra $\widehat{A}$ of A

As k-vector space we have

$$\widehat{A} = \bigoplus_{z\in\mathbb{Z}} A[z] \oplus \bigoplus_{z\in\mathbb{Z}} DA[z]$$

with multiplication

$$\begin{aligned}(a[z], \varphi[z])(b[z], \psi[z]) &= (a[z]b[z], a[z]\psi[z] + \varphi[z]b[z-1]) \\ &= (ab[z], (a\psi)[z], (\varphi b)[z]) .\end{aligned}$$

Define a quiver $\widehat{Q} = (\widehat{Q}_0, \widehat{Q}_1)$ where

$$\begin{aligned}\widehat{Q}_0 &= Q_0 \times \mathbb{Z} , \\ \widehat{Q}_1 &= \{a[z] : u[z] \to v[z] \,|\, a : u \to v \in Q_1\} \\ &\cup \ \{\widehat{p}[z] : v[z] \to u[z] \,|\, p \text{ max. path } u \to \cdots \to v\} ,\end{aligned}$$

and an ideal

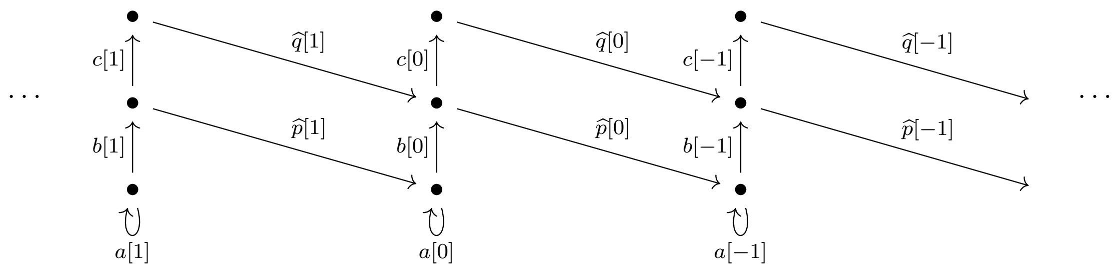
$$\begin{aligned}\widehat{I} &= \{p[z] \,|\, p \in I\} \cup \{p_1[z] - p_2[z] \,|\, p_1 - p_2 \in I\} \\ &\cup \ \{p \in k\widehat{Q} \,|\, p \text{ contains a connecting arrow and is not a subpath of a full path}\} \\ &\cup \ \{p_2[z]\widehat{p}[z]p_1[z-1] - q_2[z]\widehat{q}[z]q_1[z-1] \,|\, p = p_1xp_2,\ q = q_1xq_2 \text{ max. paths}\} ,\end{aligned}$$

where a *full path* is any of the form $p_2[z]\widehat{p}[z]p_1[z-1]$ where $p = p_1p_2$ is a maximal path.

Example 6.3. Consider $A = kQ/I$ where

$$Q = \; a \circlearrowleft 1 \xrightarrow{\;b\;} 2 \xrightarrow{\;c\;} 3$$

and $I = \langle a^2bc \rangle$. The maximal paths are $\{p = ab, q = c\}$.



Then $\widehat{I} = \langle a[z]a[z], b[z]c[z], c[z]\widehat{q}[z] - \widehat{p}[z]a[z-1]b[z-1], \widehat{q}[z]\widehat{p}[z-1], \widehat{p}[z]b[z-1] \rangle$.

Theorem 6.4 (Schröer; see also: Asashiba, Hille, Roggenkamp)**.** *$\widehat{A} = k\widehat{Q}/\widehat{I}$ where the ideal $\widehat{I}$ is generated by relations $p[z]q[z] = pq[z]$, $\varphi_p[z](p[z]) = \varphi_1(z)$, $\varphi_1[z]\varphi_1[z-1] = 0$.*

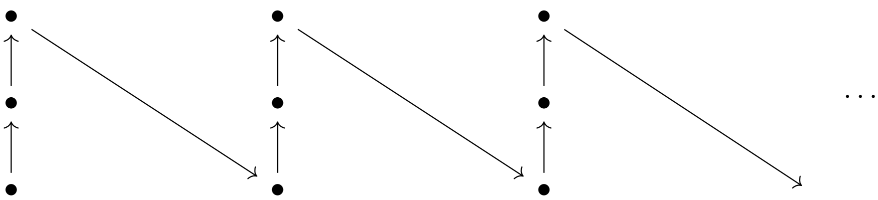
Sketch of proof. Draw a picture.
If $q = q_1 p$, then $\varphi_{q_1}[z] = p[z]\varphi_q[z]$.
If $q = pq_2$, then $\varphi_{q_2}[z] = \varphi_q[z]p[z]$. □

6.2 Interlude: Trivial Extensions

Let $T(A)$ be the trivial extension of A with the "same" multiplication as in the repetitive algebra. So $\mathrm{mod}_{\mathbb{Z}}(T(A)) = \mathrm{mod}(\widehat{A})$.

Example 6.5. For $A = \bullet \to \bullet \to \bullet$ we have

$\widehat{A} = \cdots$



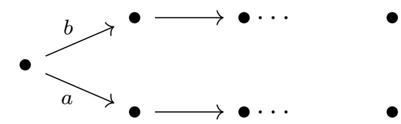
and

$T(A) =$

Theorem 6.6 (Schröer; see also: Assem, Ringel, Pogorzały, Skowroński)**.** *A is gentle if and only if $\widehat{A}$ is special biserial.*

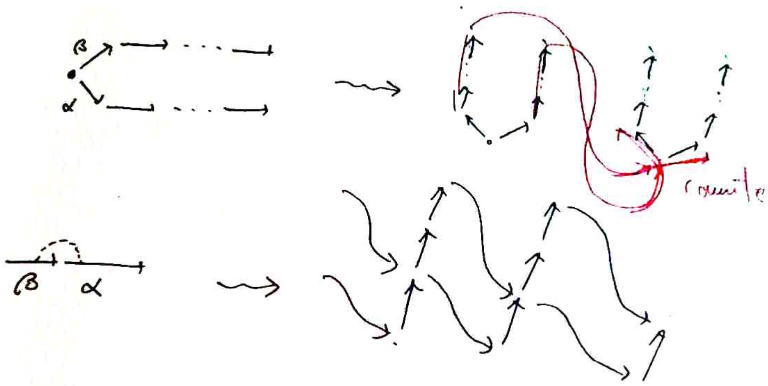
Proof. Assume A is gentle. We need only to check endpoints of maximal paths.

Case 1.

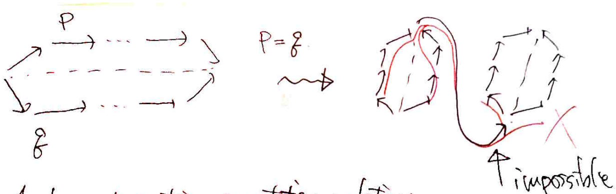


Case 2. $\bullet \xrightarrow{b} \bullet \xrightarrow{a} \bullet$ with $ba = 0$.

Draw some nice pictures in both cases ...

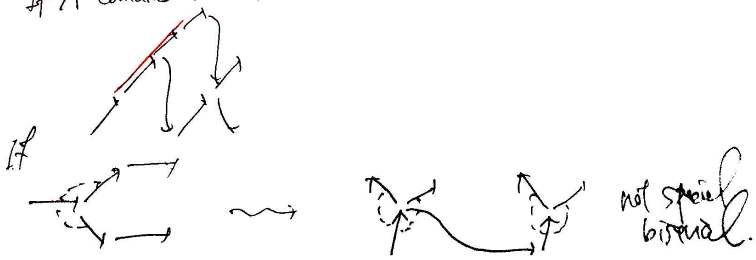


Conversely, assume now that $\widehat{A}$ is special biserial. Then A is special biserial. Distinguish again a few cases and draw some more pictures ...



So A does not contain commutative relation.

If A contains a relation more than 3, then



□

7 Brauer Graph Algebras (BGA) = Symmetric Special Biserial Algebras (SSB)

Tuesday 15th 11:15 – Wassilij Gnedin (Bochum, Germany)

7.1 Origins of BGA

(a) G a group, $\mathrm{char}(k) = p \mid \#G < \infty$, $B = kG \Rightarrow B$ is SSB

(b) A gentle $\overset{\text{last talk}}{\Rightarrow}$ $\widehat{A}$ SSB $\Rightarrow$ $B = T(A)$ is SSB and $B \twoheadrightarrow A$

Remark: $D^b(A) \xrightarrow{\sim} D^b(A') \;\Rightarrow\; D^b(B) \xrightarrow{\sim} D^b(B')$ where $B' = T(A')$

(c) Γ a "graph on an oriented surface S" (e.g. a triangulation) $\overset{\S\, 7.2}{\rightsquigarrow}$ A_Γ BGA

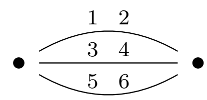
7.2 From BGA to SSB

Definition 7.1. *A Brauer graph $\Gamma = (H, \sigma, \alpha, V, m)$ is given by*

- $H = \{1, \ldots, 2n\}$ *"half-edges",*
- $\sigma : H \xrightarrow{\cong} H \rightsquigarrow \sigma$ *has cycle decomposition* $\sigma = \sigma_1 \cdots \sigma_s$,
- $\alpha : H \xrightarrow{\cong} H$ *such that* $\alpha^2 = \mathrm{id}$ *and* $\alpha(h) \neq h$ *for all* $h \in H$
  $\rightsquigarrow h$ *and* $\alpha(h)$ *form an edge in* Γ,
- $V = \{v_1, \ldots, v_s\} \rightsquigarrow f : H \to V,\ h \mapsto v_j$ *if* h *occurs in* σ_j,
- $m = (m_v)_{v \in V}$ *with* $m_v \in \mathbb{N}_+$.

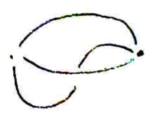
Example 7.2.

Γ =



$$\begin{aligned} \sigma &= (135)(264) \\ \alpha &= (12)(34)(56) \\ m &= (m_1, m_2, m_3) \end{aligned}$$

Γ' =

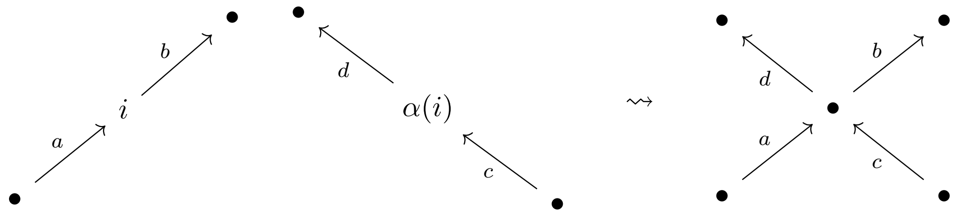


$$\begin{aligned} \sigma' &= (135)(246) \\ \alpha' &= \alpha \\ m' &= m \end{aligned}$$

Definition 7.3. Γ *a Brauer graph. We get its BGA in three steps:*

(S1) Define $\widetilde{Q}$ *by* $\widetilde{Q}_0 = H$ *and* $\exists\, a : i \to j$ *in* $\widetilde{Q}$ *if* $\sigma(i) = j$.

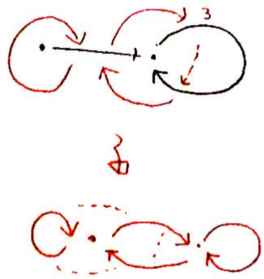
(S2) For each $i \in \widetilde{Q}_0$ glue i and $\alpha(i)$ to obtain (Q, I):



with relations $da = 0 = bc$.

$\rightsquigarrow$ *(Q, I) is "complete gentle (CG)"*

Example 7.4.



Remark 7.5. (Q, I) is CG $\Rightarrow$ For each $a \in Q_1$ there is a unique $c_a \in \mathcal{C} = \{\text{simple cycle}\}$ such that c_a begins with a. $\rightsquigarrow$ $Q_1 \to \mathcal{C} \to V$, $a \mapsto c_a \mapsto v(c_a) =$ "center of the cycle c_a".

Set

$$z_a := c_a^{m_{v(c_a)}}.$$

Notation 7.6. A cyclic path $c = a_n \cdots a_1$ is a *simple cycle* in (Q, I) if $a_i \neq a_j$ for all $i \neq j$ and $c \notin I$ and c has "maximal length".

(S3) Set $A_\Gamma = kQ/(I + J)$ where $J = \langle z_a - z_b \,|\, s(a) = s(b), a \neq b \rangle$.

Remark 7.7. J is not admissible. For example, $\Gamma = \; p-1$ ——— 1 gives

$$A_\Gamma = k[x, y]/(xy, x^{p-1} - y) \cong k[x]/(x^p).$$

Remark 7.8. $A_\Gamma \cong k\overline{Q}/R$ with $\overline{Q}_1 = Q_1 \setminus \{\ell \in Q_1 : z_\ell = \ell\}$.

Proposition 7.9. *A_Γ is finite-dimensional, symmetric and special biserial.*

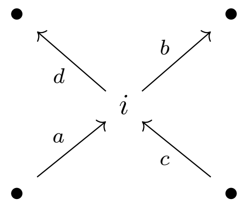
Proof. (i) For all $a \in Q_1$ there is $b \in Q_1$ such that $c_a^{m_a+1} = z_a c_a = z_b c_a$ where $m_a = m_{v(c_a)}$. Then $z_b c_a \in J$ because $bc_a = 0$. Hence, $\dim A_\Gamma < \infty$.

(ii) A_Γ is symmetric iff there exists $\varphi : A_\Gamma \to k$ such that $\varphi(pq) = \varphi(qp)$ and if $\mathfrak{a} \subseteq \ker(\varphi)$ is a left ideal, then $\mathfrak{a} = 0$. Define

$$\varphi(p) = \begin{cases} 1 & \text{if } p = z_a \text{ for some } a \in Q_1, \\ 0 & \text{else.} \end{cases}$$

Let $\mathfrak{a}$ be as above. Assume there exists $p \in \mathfrak{a} \setminus \{0\}$. Then $p = \overline{p}a$ where a is the first arrow in p. $\Rightarrow$ There is $q \in A_\Gamma$ such that $qp = z_a$. $\Rightarrow \varphi(qp) \neq 0 \Rightarrow \varphi(\mathfrak{a}) \neq 0$, a contradiction.

(iii) $A_\Gamma \cong k\overline{Q}/R$. For all $i \in Q_0$ we have



or

$ba = 0 = cd$ $\qquad\qquad$ $ba = 0$

□

7.3 SSB are BGA

Let $k = \overline{k}$ and $B = kQ/I$ a finite-dimensional SSB.

Goal.

Find a Brauer graph Γ_B such that $B \cong A_{\Gamma_B}$.

Main Observation.

B is up to isomorphism uniquely determined by its maximal paths.

Idea.

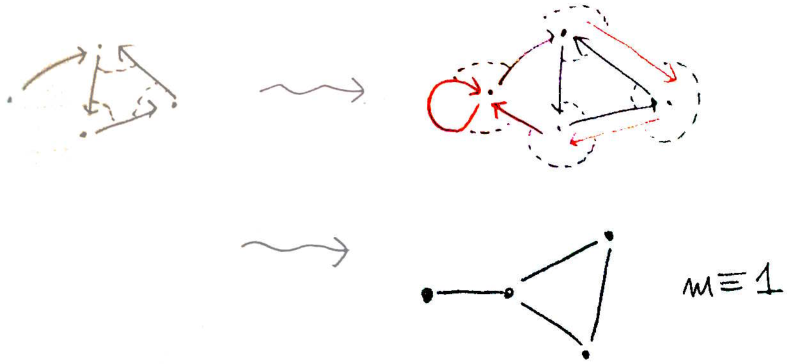
Encode maximal paths in Γ_B.

Theorem 7.10 (Roggenkamp '96, Schroll '15)**.** *Let $B = kQ/I$ be finite-dimensional. Then there exists a Brauer graph Γ such that $B \cong A_\Gamma$ iff B is SSB.*

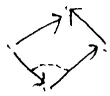
Example 7.11. For gentle A to obtain $B = T(A) \dots$

- complete maximal paths in A to cycles,
- add loops $\dots$,
- set $c_a = c_b$ if $s(a) = s(b)$.

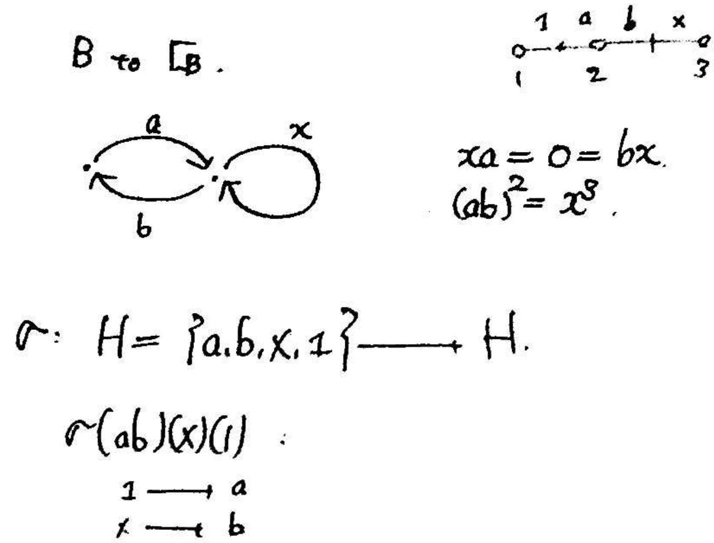
$\rightsquigarrow$ algebra $B \rightsquigarrow \Gamma_B$ with $m_v \equiv 1$



Remark 7.12. If $B \cong T(A')$ for another algebra B, then $D^b(A') \not\cong D^b(A)$.



Example 7.13.



8 Introduction to Triangulated Categories

Tuesday 15th 14:00 – Karin M. Jacobsen (Trondheim, Norway)

(following Happel '88)

Triangulated categories

- were introduced by Verdier in the '60s, published in '77,
- codify "abelian-like" behavior.

Definition 8.1. *Let $\mathcal{T}$ be an additive category with an autoequivalence $\Sigma : \mathcal{T} \to \mathcal{T}$. Triangles are sequences of the form*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X\,. \tag{$\star$}$$

Definition 8.2. *A set Δ of triangles is called a* triangulation *of $\mathcal{T}$ if it fulfills the following axioms*

(TR1) For all morphisms $f : X \to Y$ in $\mathcal{T}$ there exists

$$X \xrightarrow{f} Y \to Z \to \Sigma X \in \Delta\,.$$

For all objects X in $\mathcal{T}$

$$X \xrightarrow{\mathrm{id}_X} X \to 0 \to \Sigma X \in \Delta\,.$$

If $X' \to Y' \to Z' \to \Sigma X'$ is isomorphic to $X \to Y \to Z \to \Sigma X$ then

$$X' \to Y' \to Z' \to \Sigma X' \in \Delta\,.$$

(TR2) If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$, then

$$Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y \in \Delta\,.$$

(TR3) Given a commutative diagram

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\
\downarrow & & \downarrow & & & & \downarrow \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X'
\end{array}$$

there exists $h : Z \to Z'$ making the following diagram commute

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle h} & & \downarrow \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X'\,.
\end{array}$$

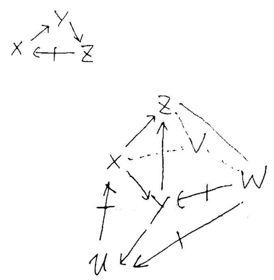
(TR4) Given

$$
\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & U & \longrightarrow & \Sigma X \\
\downarrow & & \downarrow & & \downarrow & & \| \\
X & \xrightarrow{gf} & Z & \longrightarrow & V & \longrightarrow & \Sigma X \\
\downarrow{\scriptstyle f} & & \| & & \downarrow & & \downarrow \\
Y & \xrightarrow{g} & Z & \longrightarrow & W & \longrightarrow & \Sigma Y \\
 & & & & \downarrow & & \downarrow \\
 & & & & \Sigma U & = & \Sigma U
\end{array}
$$

there exists a dashed triangle in Δ as indicated.

In this case $\mathcal{T}$ is called a triangulated category.

Remark 8.3.



Lemma 8.4.

(1) In $(\star)$: $vu = 0$ and $wv = 0$

(2) In (TR3): f, g iso $\Rightarrow h$ iso

(3) $X \xrightarrow{f} Y \to 0 \to \Sigma X \in \Delta \Leftrightarrow f$ iso

(4) In $(\star)$ the following are equivalent:

(i) u split mono

(ii) v split epi

(iii) $w = 0$

Lemma 8.5. *Let $T \in \mathcal{T}$. Then*

$$\mathrm{Hom}_{\mathcal{T}}(T, -) : \mathcal{T} \to \mathrm{mod}(\mathrm{End}\, T)^{\mathrm{op}}$$
$$\mathrm{Hom}_{\mathcal{T}}(-, T) : \mathcal{T} \to \mathrm{mod}(\mathrm{End}\, T)$$

are cohomological functors, i.e. for each triangle as in $(\star) \in \Delta$ the induced sequences

$$\cdots \to \mathrm{Hom}_{\mathcal{T}}(T, X) \to \mathrm{Hom}_{\mathcal{T}}(T, Y) \to \mathrm{Hom}_{\mathcal{T}}(T, Z) \to \mathrm{Hom}_{\mathcal{T}}(T, \Sigma X) \to \mathrm{Hom}_{\mathcal{T}}(T, \Sigma Y) \to \mathrm{Hom}_{\mathcal{T}}(T, \Sigma Z) \to \mathrm{Hom}_{\mathcal{T}}(T, \Sigma^2 X) \to \cdots$$

$$\cdots \to \mathrm{Hom}_{\mathcal{T}}(Z, T) \to \mathrm{Hom}_{\mathcal{T}}(Y, T) \to \mathrm{Hom}_{\mathcal{T}}(X, T) \to \mathrm{Hom}_{\mathcal{T}}(\Sigma^{-1} Z, T) \to \mathrm{Hom}_{\mathcal{T}}(\Sigma^{-1} Y, T) \to \mathrm{Hom}_{\mathcal{T}}(\Sigma^{-1} X, T) \to \mathrm{Hom}_{\mathcal{T}}(\Sigma^{-2} Z, T) \to \cdots$$

are long exact sequences.

Proof. For $\mathrm{Hom}_{\mathcal{T}}(T,-)$, given (TR2), it is enough to check the exactness once:

$$\begin{array}{ccccccc}
T & \xrightarrow{\mathrm{id}} & T & \longrightarrow & 0 & \longrightarrow & X \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow \\
\Sigma X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \longrightarrow & \Sigma X
\end{array}$$

Now:

$$f \in \ker(\mathrm{Hom}_{\mathcal{T}}(T,v)) \Leftrightarrow f = ug \text{ for some } g \Leftrightarrow f \in \mathrm{im}(\mathrm{Hom}_{\mathcal{T}}(T,u))$$

□

Example 8.6. Stable module categories $\underline{\mathrm{mod}}(A) = \mathrm{mod}(A)/\,\mathrm{proj}(A)$ where A is a self-injective locally bounded algebra with $\Sigma = \Omega^{-1}$ the syzygy functor given as

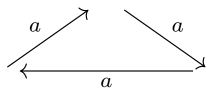
$$X \xrightarrow{\text{inj. env.}} I \longrightarrow \Omega^{-1}X \longrightarrow 0$$

and triangles

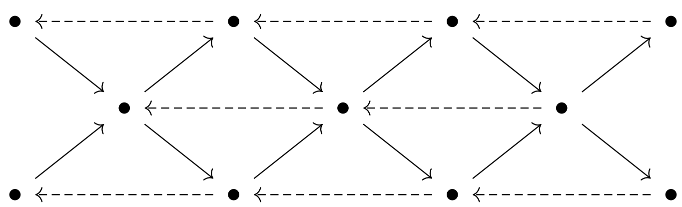
$$\underline{X} \to \underline{E} \to \underline{Y} \to \underline{\Omega^{-1}X} \in \Delta$$

induced by short exact sequences $0 \to X \to E \to Y \to 0$ in $\mathrm{mod}(A)$.

For example take $A = kQ/I$ with

$Q =$  and $I = \langle a^3 \rangle$.

Then $\underline{\mathrm{mod}}(A)$ looks as follows:



Example 8.7. Derived categories:

$\mathcal{A}$ abelian category $\rightsquigarrow C(\mathcal{A})$ category of complexes:

$$\cdots \xrightarrow{d} \bullet \xrightarrow{d} \bullet \xrightarrow{d} \bullet \xrightarrow{d} \bullet \xrightarrow{d} \bullet \xrightarrow{d} \cdots$$

in $\mathcal{A}$ with $d^2 = 0$

$\rightsquigarrow K(\mathcal{A})$ homotopy category (this is triangulated with Σ given by shifting complexes)

$\rightsquigarrow D(\mathcal{A})$ derived category (obtained by localizing at quasi-isomorphisms)

9 A Construction of the Happel Functor

Tuesday 15th 15:15 – Gabriele Bocca (Norwich, United Kingdom)

References.

- [Hap] Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, 1988.
- [BM] Barot–Mendoza, *An explicit construction for the Happel functor*, 1991.

Notation.

- k any field
- A a finite-dimensional k-algebra
- $\mathrm{mod}(A)$ the category of finitely generated modules over A
- $\widehat{A}$ the repetitive algebra of A
- $\underline{\mathrm{mod}}(\widehat{A})$ the stable module category over $\widehat{A}$

Remark 9.1. $\mathrm{Ob}(\underline{\mathrm{mod}}(\widehat{A})) = \mathrm{Ob}(\mathrm{mod}(\widehat{A}))$ and $\underline{\mathrm{Hom}}_{\widehat{A}}(X,Y) = \mathrm{Hom}_{\widehat{A}}(X,Y)/I(X,Y)$ where $I(X,Y)$ consists of the morphisms factoring through injectives.

History and Motivation

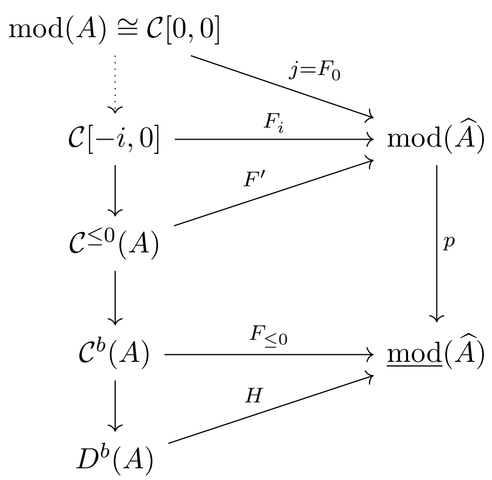
Theorem 9.2 (Happel)**.** *There exists a triangulated, full and faithful functor*

$$H : D^b(\mathrm{mod}(A)) \longrightarrow \underline{\mathrm{mod}}(\widehat{A})\,.$$

If $\mathrm{gl.dim}(A) < \infty$*, then* H *is dense.*

Proof strategy:

$$\mathcal{C}^b(A) \supseteq \mathcal{C}^{\leq 0}(A) \supseteq \mathcal{C}[-i,0] = \{X : \cdots \to 0 \to X^{-i} \to \cdots \to X^0 \to 0 \to \cdots\}$$



Here j is exact, full and faithful.

Theorem 9.3 (Rickard)**.** *Let Λ be a Frobenius k-algebra. Then there exists an equivalence*

$$F : \underline{\mathrm{mod}}(\Lambda) \longrightarrow D^b(\Lambda)/K^b(P_\Lambda)$$

where P_Λ is the full subcategory of $\mathrm{mod}(\Lambda)$ *of projective modules.*

Remark 9.4.

(1) A k-algebra Λ is *Frobenius* if it is locally bounded and the projective and injective modules coincide.

(2) For all $X \in \mathrm{mod}(\Lambda)$ consider $0 \to X \to I(X) \to \Omega^{-1}(X) \to 0$ and then

$$\begin{array}{ccccc} X & \longrightarrow & I(X) & \longrightarrow & \Omega^{-1}(X) \\ \downarrow{\scriptstyle u} & & \downarrow & & \| \\ Y & \longrightarrow & C_u & \longrightarrow & \Omega^{-1}(X) \end{array}$$

where the left square is a pushout. We get

$$X \longrightarrow Y \longrightarrow C_u \longrightarrow \Omega^{-1}(X)\,. \qquad (\star)$$

In $\underline{\mathrm{mod}}(\Lambda)$ let

$$\mathcal{T} \;=\; \{\text{sequences isomorphic to } (\star)\ \}\,.$$

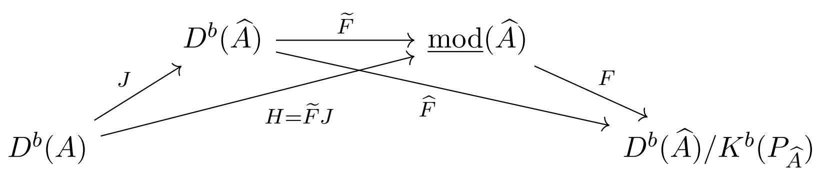
Then $\mathcal{T}$ is a triangulation for $\underline{\mathrm{mod}}(\widehat{A})$ with suspension functor Ω^{-1}.

In particular:

Proposition 9.5 ([Hap, II.2.2)**.** *] Let A be a finite-dimensional k-algebra. Then $\widehat{A}$ is Frobenius and so $\underline{\mathrm{mod}}(\widehat{A})$ is triangulated.*

The construction in [BM] is the following:

- $\mathrm{mod}(A) \xrightarrow{j} \mathrm{mod}(\widehat{A})$ exact, full and faithful:



- **$G : D^b(\Lambda) \to D^b(\Lambda)$, $G \cong_{\text{nat}} \text{id}$:**

  For $X \in \mathcal{C}^b(\Lambda)$ and $n \in \mathbb{Z}$:

$$\begin{array}{ccccccccccc} X & & \cdots \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & X^{n+1} & \longrightarrow & X^{n+2} & \longrightarrow \cdots \\ \downarrow{\scriptstyle \lambda_{n,x}} & & & \| & & \downarrow & & \downarrow & & \| & \\ L_n X & & \cdots \longrightarrow & X^{n-1} & \longrightarrow & I & \longrightarrow & C_n & \longrightarrow & X^{n+2} & \longrightarrow \cdots \end{array}$$

  Dually we can define $R_n X$ and $\rho_{n,x} : R_n X \to X$.

  $\rightsquigarrow$ For every morphism $f : X \to Y$ in $\mathcal{C}^b(X)$ we get $L_n f$ and $R_n f$.

Lemma 9.6.

(a) *For all $n \in \mathbb{Z}$, $X \in \mathcal{C}^b(\Lambda)$ the maps $\lambda_{n,x}$ and $\rho_{n,x}$ are quasi-isomorphisms.*

(b) *"Different choices" for $L_n f$ and $R_n f$ lead to homotopic morphisms.*

For all $X \in \mathcal{C}[s, n]$, $s, n \in \mathbb{Z}$ with $s < 0 < n$,

$$L_{<0} X = L_{-1} L_{-2} \cdots L_s(X)\,,$$
$$R_{>0} X = R_1 R_2 \cdots R_n(X)$$

the maps $\lambda_{<0,x} : X \to L_{<0} X$ and $\rho_{>0,x} : R_{>0} X \to X$ are quasi-isomorphisms.

We have:

$$\begin{array}{ccc} \mathcal{C}^b(\Lambda) & \overset{L_{<0}}{\dashrightarrow} & \mathcal{C}^b(\Lambda) \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\ K^b(\Lambda) & \overset{\overline{L}_{<0}}{\dashrightarrow} & K^b(\Lambda) \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi'} \\ D^b(\Lambda) & \overset{\widetilde{L}_{<0}}{\dashrightarrow} & D^b(\Lambda) \end{array}$$

Then $\widetilde{\lambda}_{<0,x}$ and $\widetilde{\rho}_{>0,x}$ are isomorphisms for all $X \in D^b(\Lambda)$. Moreover, $\widetilde{L}_{<0}$ and $\widetilde{R}_{>0}$ are equivalences naturally isomorphic to $\text{id} : D^b(\Lambda) \to D^b(\Lambda)$*.*

Definition 9.7. $G = \widetilde{R}_{>0} \widetilde{L}_{<0} : D^b(\Lambda) \to D^b(\Lambda)$, $G \cong_{\text{nat}} \text{id}$.

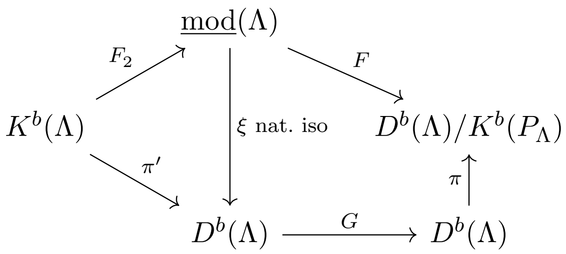
- **$\mathcal{C}^b(\Lambda) \to \underline{\text{mod}}(\Lambda)$:**

  $F_1 X = p(R_{>0} L_{<0}(X))^0$

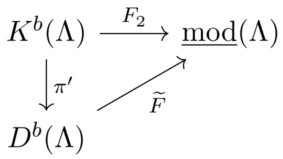
$$\rightsquigarrow \quad \begin{array}{ccccc} \mathcal{C}^b(\Lambda) & \overset{R_{>0}L_{<0}}{\dashrightarrow} & \mathcal{C}^b(\Lambda) & \xrightarrow{X \mapsto X^0} & \text{mod}(\Lambda) \\ \downarrow & & \searrow{\scriptstyle F_1} & & \downarrow \\ K^b(\Lambda) & & \xrightarrow{\quad F_2 \quad} & & \underline{\text{mod}}(\Lambda) \end{array}$$

Properties.

- $FF_2 \cong \pi G \pi'$:



- F_2 factors through π':



  $F_2 = \widetilde{F}\pi' \Rightarrow F\widetilde{F}\pi' = FF_2 \cong \pi G \pi'$

- $F\widetilde{F} \cong_{\mathrm{nat}} \pi G \cong \pi'$

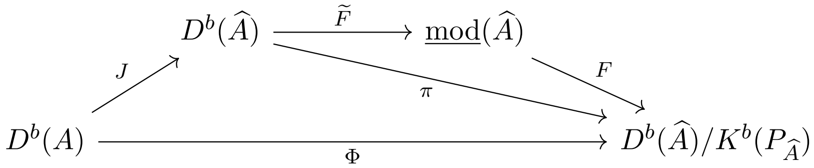
Remark 9.8. $\widetilde{F}$ is triangulated since π is triangulated and F is a triangulated equivalence.

- **Definition of H:**

  $H := \widetilde{F}J : D^b(A) \xrightarrow{J} D^b(\widehat{A}) \xrightarrow{\widetilde{F}} \underline{\mathrm{mod}}(\widehat{A})$

  $(\star H)$:
  - H is triangulated, full and faithful
  - $\mathrm{gl.\,dim}(A) < \infty \Rightarrow H$ dense

  Define $\Phi = \pi J$:



  $(\star\Phi) \Leftrightarrow (\star H)$: $\Phi = \pi J \cong F\widetilde{F}J = FH$

- Φ is triangulated and full, since π and J are.
- Φ is faithful: main idea is to show $X \ncong 0 \Rightarrow \Phi(X) \ncong 0$.

  $\rightsquigarrow$ apply Rickard's argument about F

- By [Hap, II.3.2]: $\mathrm{gl.dim}(A) < \infty \Rightarrow \mathrm{mod}(A)$ generates $\underline{\mathrm{mod}}(\widehat{A})$ as a triangulated category.
- $\mathrm{mod}(A)$ generates $D^b(A)$ as a triangulated category.
- $\Phi(\mathrm{mod}(A)) = \mathrm{mod}(A) \Rightarrow \Phi$ is dense ([Hap, II.3.4]).

10 Classification of Indecomposable Objects in the Bounded Derived Category of a Gentle Algebra

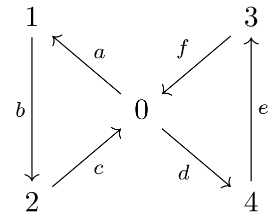
Tuesday 15th 17:00 – Sebastian Opper (Cologne, Germany)

Notation.

- $k = \overline{k}$ a field
- $\Lambda = k\Gamma/I$

Example 10.1. Running example: $\Lambda = k\Gamma/I$ with

$\Gamma =$



and $I = \langle ac, ba, cb, ed, fe, df \rangle$.

Fact 10.2. $D^b(\mathrm{mod}(\Lambda))$ *contains* 3 *types of indecomposable objects:*

- *band complexes* $\in K^b(\mathrm{proj}(\Lambda))$
- *string complexes* $\in K^b(\mathrm{proj}(\Lambda))$
- *infinite string complexes*

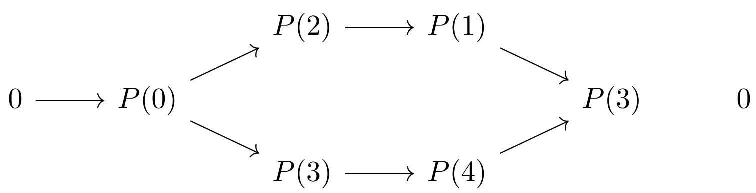
10.1 String Complexes

For $x \in \Gamma_0$ let $P(x)$ be the indecomposable projective module of x.

Example 10.3.

$$\underset{-1}{0} \longrightarrow \underset{0}{P(0)} \xrightarrow{\begin{pmatrix} c \\ f \end{pmatrix}} \underset{1}{P(2) \oplus P(3)} \xrightarrow{\begin{pmatrix} b & 0 \\ 0 & e \end{pmatrix}} \underset{2}{P(1) \oplus P(4)} \xrightarrow{\begin{pmatrix} af & 0 \end{pmatrix}} \underset{3}{P(3)} \longrightarrow \underset{4}{0}$$

Rewrite it as:



This unfolds as:

$$\begin{array}{ccccccccccc} P(4) & \xleftarrow{e} & P(3) & \xleftarrow{f} & P(0) & \xrightarrow{c} & P(2) & \xrightarrow{b} & P(1) & \xrightarrow{af} & P(3) \\ 2 & & 1 & & 0 & & 1 & & 2 & & 3 \end{array}$$

$\rightsquigarrow$ diagram of type $\mathbb{A}_6$ with:

- vertices: pairs (indec. proj. module, integer)
- arrows: admissible (i.e. no subpath in I) paths in Γ

What properties are needed to construct an indecomposable complex from an $\mathbb{A}_n$-diagram via "folding"?

Given

$$\begin{array}{ccccccc} P_n & \overset{w_n}{\text{———}} & P_{n-1} & \overset{w_{n-1}}{\text{———}} & \cdots & \overset{w_1}{\text{———}} & P_0 \\ d_n & & d_{n-1} & & & & d_0 \end{array}$$

with P_i indecomposable projective, w_i admissible path in (Γ, I) and $d_i \in \mathbb{Z}$.

(S1) Degrees increase by 1 along arrows.

(S2) If $\xrightarrow{w_i} P_i \xrightarrow{w_{i-1}}$, then $P(s(w_i)) = P_i = P(t(w_{i-1})$ and $w_i w_{i-1} \in I$.

(S3) If $\xleftarrow{w_i} P_i \xleftarrow{w_{i-1}}$, then $P(t(w_i)) = P_i = P(s(w_{i-1})$ and $w_{i-1} w_i \in I$.

(S4) If $\xrightarrow{w_i} P_i \xleftarrow{w_{i-1}}$, then $P(s(w_i)) = P_i = P(s(w_{i-1})$ and w_{i-1} and w_i do not start with the same arrow.

(S5) If $\xleftarrow{w_i} P_i \xrightarrow{w_{i-1}}$, then $P(t(w_i)) = P_i = P(t(w_{i-1})$ and w_{i-1} and w_i do not end with the same arrow.

Definition 10.4. *An $\mathbb{A}_n$-diagram satisfying (S1)–(S5) is called a* string diagram*.*

string diagram $\overset{\text{fold}}{\rightsquigarrow}$ string complex

Example 10.5.

$$\begin{array}{ccccccccccccc} P(0) & \xrightarrow{c} & P(2) & \xrightarrow{b} & P(1) & \xrightarrow{a} & P(0) & \xleftarrow{d} & P(4) & \xleftarrow{e} & P(3) & \xleftarrow{f} & P(0) \\ 0 & & 1 & & 2 & & 3 & & 2 & & 1 & & 0 \end{array}$$

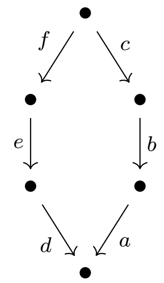
$\overset{\text{fold}}{\rightsquigarrow}$

$$\begin{array}{cccccccc} P(0) & \xrightarrow{f} & P(3) & \xrightarrow{e} & P(4) & \searrow^{d} & & \\ & & & & & & P(0) & \\ P(0) & \xrightarrow[c]{} & P(2) & \xrightarrow[b]{} & P(1) & \nearrow_{a} & & \\ 0 & & 1 & & 2 & & 3 & \end{array}$$

10.2 Band complexes

Example 10.6. Take Example 10.5.

identify outer vertices $\rightsquigarrow$

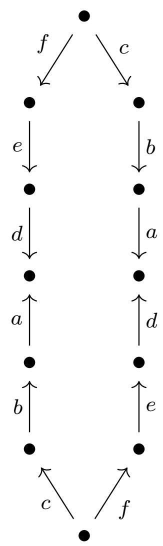


$\rightsquigarrow$ diagram of type $\widetilde{\mathbb{A}}$

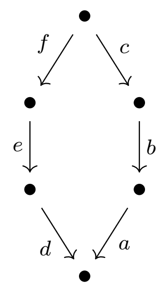
Remark 10.7. Rotating and reflecting gives isomorphic complexes.

Definition 10.8. *A diagram of type* $\widetilde{\mathbb{A}}$ *satisfying (S1)–(S5) and not covering any such diagram of strictly smaller size is called a* band diagram*.*

Example 10.9. Example of a cover:

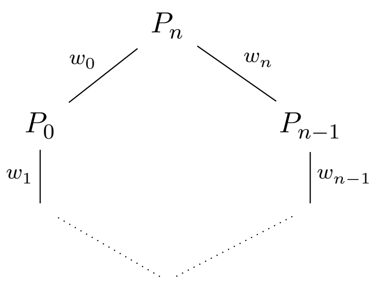


$\rightsquigarrow$

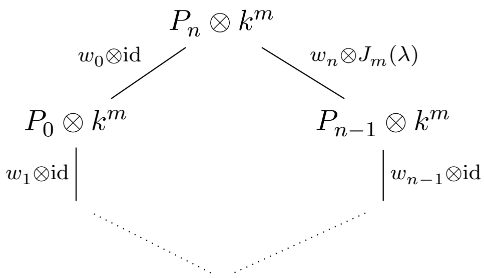


band diagram $\overset{\text{folding}}{\rightsquigarrow}$ $k^\times \times \mathbb{N}_+$ family of pairwise non-isomorphic band complexes

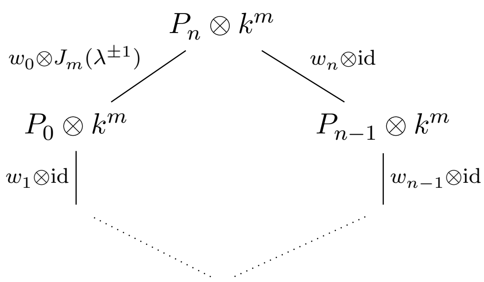
Given $\lambda \in k^\times$, $m \in \mathbb{N}_+$ and a band diagram



we get a band complex



This is isomorphic to



10.3 Infinite String Complexes

Definition 10.10. *A* cycle *is a string diagram of* (Γ, I) *(up to reflection)*

$$P_n \xleftarrow{\alpha_n} \cdots \xleftarrow{\alpha_1} P_0$$

where α_i *are arrows in* Γ *and* $P_n = P_0$*.*

Example 10.11. $\xrightarrow{a}\xrightarrow{c}\xrightarrow{b}$ and $\xrightarrow{d}\xrightarrow{e}\xrightarrow{f}$ are cycles in the running example.

Definition 10.12. *Start with a string diagram*

$$\begin{matrix} P_n \xrightarrow{\;w_n\;} \cdots \xrightarrow{\;w_1\;} P_0 \\ d_n \qquad\qquad\qquad d_0 \end{matrix}$$

It is called ...

- left resolvable *if* $\xrightarrow{w_n}$ *and* $d_n = \min\{d_j\}$ *and there exists a cycle* $P_n \xrightarrow{\alpha_m} \cdots \xrightarrow{\alpha_1} P_n$ *such that* $\xrightarrow{\alpha_1}\xrightarrow{w_n}$ *is a string diagram,*
- right resolvable *if it satifies the dual condition,*
- two-sided resolvable *if it is left and right resolvable.*

Suppose

$$\begin{matrix} P_n & \xrightarrow{w_n} \cdots \xrightarrow{w_1} & P_0 \\ d_n & & d_0 \end{matrix}$$

is left resolvable and $P_n \xrightarrow{\alpha_m} \cdots \xrightarrow{\alpha_1} P_n$*, then*

$$\begin{matrix} \cdots \xrightarrow{\alpha_1} P_n \xrightarrow{\alpha_m} \cdots \xrightarrow{\alpha_1} & P_n & \xrightarrow{w_n} \cdots \xrightarrow{w_1} & P_0 \\ & d_n & & d_0 \end{matrix}$$

is an infinite string diagram. $\overset{fold}{\rightsquigarrow}$ *infinite string complex*

Theorem 10.13 (Bekkert–Merklen, Burban–Drozd, Raphael)**.** *There is a bijection between*

$$\{\textit{isoclasses of indecomposables in } D^b(\mathrm{mod}(\Lambda))\}$$

and

$$\begin{aligned} & \{\textit{string diagrams}\}/\textit{reflection} \\ \dot\cup\ & \{\textit{band diagrams}\}/\textit{reflection and rotation} \\ \dot\cup\ & \{\textit{infinite string diagrams}\}/\textit{reflection}. \end{aligned}$$

11 Derived Equivalences

Wednesday 16^{th} 8:30 – Fajar Yuliawan (Bielefeld, Germany)

References.

(1) Schröer, Zimmermann. *Stable endomorphism algebras of modules over special biserial algebras.*

(2) Schröer. *Modules without self-extensions over gentle algebras.*

(3) Crawley-Boevey. *Maps between representations of zero relation algebras.*

(4) Rickard. *Morita theory for derived categories.*

Definition 11.1. *Let Q be a (not necessarily finite) quiver and ρ a set of relations. Then (Q, ρ) is* special biserial *if (SB1, SB1') and (SB2, SB2') and*

(SB3) Each infinite path in Q contains a subpath in ρ.

Remark 11.2. $A = kQ/(\rho)$ finite-dimensional gentle $\rightsquigarrow$ $(\widehat{Q}, \widehat{\rho})$ special biserial

Definition 11.3. *A k-algebra is called* special biserial *(resp.* gentle*) if it is up to Morita equivalence an algebra $kQ/(\rho)$ with (Q, ρ) special biserial (resp. gentle).*

Theorem 11.4 (Main Theorem)**.** *Let A be a special biserial algebra and M a finite-dimensional A-module with $\operatorname{Ext}^1_A(M, M) = 0$. Then $\underline{\operatorname{End}}_A(M)$ is gentle.*

Corollary 11.5. *Let A be finite-dimensional, $T \in D^b(A)$ and $\operatorname{Hom}_{D^b(A)}(T, T[1]) = 0$. Then $\operatorname{End}_{D^b(A)}(T)$ is gentle.*

In particular, any algebra B which is derived equivalent to A is gentle.

Proof of Corollary 11.5. A gentle $\overset{\text{Jordan's talk}}{\Rightarrow}$ $\widehat{A}$ special biserial

$\exists\, H : D^b(A) \xrightarrow{\sim} \underline{\operatorname{mod}}(\widehat{A})$ fully faithful and triangulated

Take $M \in \operatorname{mod}(\widehat{A})$ to be $M = H(T)$, then

$$\operatorname{End}_{D^b(A)}(T) \;\cong\; \underline{\operatorname{End}}_{\widehat{A}}(M)$$

and

$$\operatorname{Ext}^1_{\widehat{A}}(M, M) \;\cong\; \underline{\operatorname{Hom}}_{\widehat{A}}(\Omega M, M) \;\cong\; \operatorname{Hom}_{D^b(A)}(T[-1], T) = 0\,.$$

Thus by Theorem 11.4 $\operatorname{End}_{D^b(A)}(T)$ is gentle. □

Lemma 11.6. *Let A, B be finite-dimensional k-algebras and $F : D^b(B) \to D^b(A)$ a fully faithful and triangulated functor. Then $T = F(B)$ satisfies*

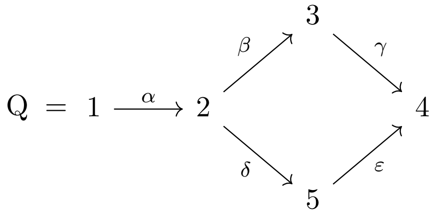
$$B = \operatorname{End}_{D^b(A)}(T) \qquad \textit{and} \qquad \operatorname{Hom}_{D^b(A)}(T, T[1]) = 0\,.$$

Known Facts on Special Biserial Algebras

Let (Q, ρ) be special biserial and $A = kQ/(\rho)$. Assume ρ contains only zero relations and commutativity relations. Define

$$\rho^{+} \;=\; \rho + \text{all paths which are contained in a commutativity relation in } \rho\,.$$

Example 11.7. Let



and $\rho = \{\alpha\beta, \beta\gamma - \delta\varepsilon\}$. Then $\rho^{+} = \{\alpha\beta, \beta\gamma, \delta\varepsilon\}$ and $kQ/(\rho^{+})$ is a string algebra.

Indecomposables in A:

- non-uniserial projective-injectives
- string modules
- band modules

If M_1 is a band module, then $\mathrm{Ext}^1_A(M_1, M_1) \neq 0$.
Let $C = C_1 \cdots C_n$ be a string with $s(C) = s(C_1)$ and $t(C) = t(C_n)$.
$\mathrm{Ext}^1_A(M, M) = 0 \rightsquigarrow M$ does not contain band modules as direct summands
For every vertex i we define two strings of length 0, starting and ending at i:

$$1_{(i,1)} \quad \text{and} \quad 1_{(i,-1)}$$

Concatenation of strings of length 0 depends on chosen "orientation" $\sigma, \varepsilon : \mathcal{S} \to \pm 1$ where

$$\mathcal{S} \;=\; \{\text{all strings for } (Q, \rho^{+})\}\,.$$

Remark 11.8. If C starts at i, then only one of $1_{(i,1)}C$ and $1_{(i,-1)}C$ is defined.

Definition 11.9 (Main definition)**.** *For a string C define*

$$\mathcal{P}(C) \;=\; \{(D, E, F) \mid DEF = C, D, E, F \in \mathcal{S}\}\,.$$

We call (D, E, F) a factor string *of C if*

(1) either $|D| = 0$ or D ends with an inverse arrow,

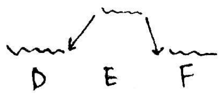
(2) either $|F| = 0$ or F starts with a directed arrow.

A substring *(D, E, F) is defined dually.*
We call a pair $a = ((D_1, E_1, F_1), (D_2, E_2, F_2)) \in \mathrm{fac}(C_1) \times \mathrm{sub}(C_2)$ admissible *where*

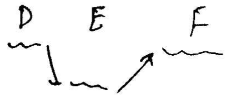
$$E_1 \sim E_2 \;\Rightarrow\; E_1 = E_2 \text{ or } E_1 = E_2^{-}\,.$$

The set of all admissible pairs is denoted $\mathcal{A}(C_1, C_2)$.

Example 11.10. E.g. if $|D| > 0$ and $|F| > 0$ then a factor string has the form



and a substring has the form



For each $a \in \mathcal{A}(C_1, C_2)$ we define

$$f_a : M(C_1) \to M(C_2)$$

and call it a *graph map*.

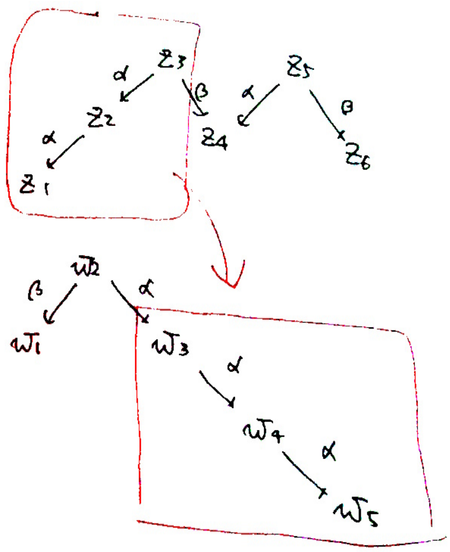
Example 11.11. Let $A = kQ/(\rho)$ with

$$Q = \alpha \circlearrowleft \bullet \circlearrowright \beta$$

and $\rho = \langle \alpha\beta, \beta\alpha, \alpha^4, \beta^3 \rangle$.

Let $C_1 = \alpha^-\alpha^-\beta\alpha^-\beta$ and $C_2 = \beta^-\alpha\alpha\alpha$ and $a = ((1, \alpha^-\alpha^-, \beta\alpha^-\beta), (\beta^-\alpha, \alpha\alpha, 1))$. Then:

- $M(C_1)$ has basis $z_1, \ldots, z_6$,
- $M(C_2)$ has basis $w_1, \ldots, w_5$.



Observe $M(\alpha^-\alpha^-) \cong M(\alpha\alpha)$.

We have:

- (D_1, E_1, F_1) factor string of $C_1 \Rightarrow M(C_1) \twoheadrightarrow M(E_1)$
- (D_2, E_2, F_2) substring of $C_2 \Rightarrow M(E_2) \hookrightarrow M(C_2)$
- admissible $\Rightarrow M(E_1) \xrightarrow{\cong} M(E_2)$

Thus f_a is just

$$M(C_1) \twoheadrightarrow M(E_1) \xrightarrow{\cong} M(E_2) \hookrightarrow M(C_2)\,.$$

Theorem 11.12 (Crawley-Boevey)**.** *The graph maps form a basis of the hom spaces. In particular,* $\dim \operatorname{Hom}_A(M(C_1), M(C_2)) = |\mathcal{A}(C_1, C_2)|$.

Definition 11.13. *Let* $a = ((D_1, E_1, F_1), (D_2, E_2, F_2)) \in \mathcal{A}(C_1, C_2)$. *We call* f_a *...*

- oriented *if* $E_1 = E_2$,
- left (resp. right) sided *if* $|D_1| = |D_2| = 0$ *(resp.* $|F_1| = |F_2| = 0$*),*
- weakly one-sided *if* a *or* $((F_1^-, E_1^-, D_1^-), (D_2, E_2, F_2))$ *is one-sided,*
- two-sided *if it is not weakly one-sided.*

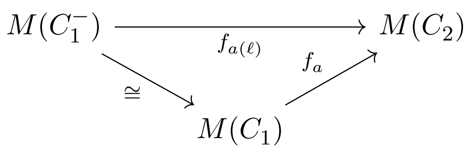
Define

$$a(\ell) = \begin{cases} a & \text{if } a \text{ is oriented} \\ ((F_1^-, E_1^-, D_1^-), (D_2, E_2, F_2)) & \text{otherwise} \end{cases}$$

and $a(r)$ *dually.*

Remark 11.14.

- a is weakly one-sided $\Leftrightarrow a(\ell)$ is one-sided $\Leftrightarrow a(r)$ is one-sided
- a is not oriented $\Rightarrow E_2 = E_1^-$



Proof

Lemma 11.15. *Let* $f_{a_i} : M(C_1) \to M(C_2)$ *with* $1 \le i \le s$ *be pairwise different which are weakly one-sided. If* $\underline{f_{a_i}} \neq 0$*, then the* $\underline{f_{a_i}}$ *are linearly independent in* $\underline{Hom}(M(C_1), M(C_2))$.

Proof. Let $f_a : M(C_1) \to M(C_2)$ be a two-sided graph map and $\operatorname{Ext}^1(M(C_2), M(C_1)) = 0$. Then $\underline{f_a} = 0$. □

Theorem 11.16. *Let* $M \in A\text{-}\mathrm{mod}$ *with* $\operatorname{Ext}_A^1(M, M) = 0$. *Then* $\underline{End}_A(M)$ *is gentle.*

Proof.

- M does not contain band modules

- M does not contain projective indecomposables

- $M_i \ncong M_j$ for all $i \neq j$

$\Rightarrow M = \bigoplus_{i=1}^{n} M_i$ with $M_i = M(C_i)$ and $C_i \not\sim C_j$ for all $i \neq j$

Thus Theorem 11.12 and Lemma 11.15 imply that

$$\underline{\mathcal{B}} = \{ \underline{f_a} \mid f_a \in \mathrm{End}_A(M) \text{ weakly one-sided with } \underline{f_a} \neq 0 \ \}$$

is a basis of $\underline{\mathrm{End}}_A(M)$ which behaves multiplicatively:

$$\underline{f_a f_b} = 0 \qquad \text{or} \qquad \underline{f_a f_b} \in \underline{B}$$

□

$Q_0 = \{ \underline{id} : M(C_i) \to M(C_i) \text{ with } 1 \leq i \leq n \}$
$Q_1 = \underline{B} \setminus (Q_0 \cup \{ \underline{f_a} \in \underline{B} \text{ such that } \underline{f_a} = \underline{f_b f_c} \})$

Lemma 11.17 (Key Lemma 1)**.** *Let $X, Y, Z \in \{ M(C_i) \mid 1 \leq i \leq n \}$ and $f_a : X \to Z$, $f_b : Y \to Z$ be different such that $\underline{f_a}, \underline{f_b} \in Q_1$.*

Then $f_{a(\ell)}$ is left-sided and $f_{b(\ell)}$ is right-sided or vice versa.

Proof. ... □

Lemma 11.18 (Key Lemma 2)**.** *Let $X \xrightarrow{f_a} Y \xrightarrow{f_b} Z$ with $\underline{f_a}, \underline{f_b} \in Q_1$. If $f_{a(\ell)}$ and $f_{b(r)}$ are both left-sided or both right-sided, then $\underline{f_a f_b} \neq 0$. Otherwise, $\underline{f_a f_b} = 0$.*

Proof. ... □

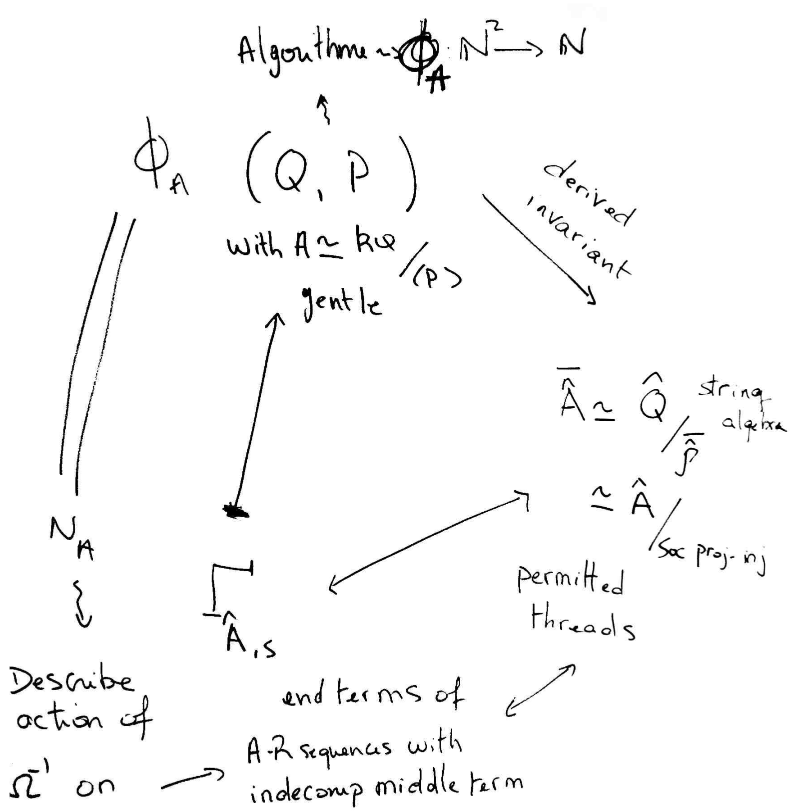
12 Combinatorial Derived Invariants

Wednesday 16^{th} 10:00 – Nicolas Berkouk (Paris, France)

References.

- C. Geiß and Diana Avella-Alaminos.

"Quiver" Plan of the Talk.



12.1 Definitions

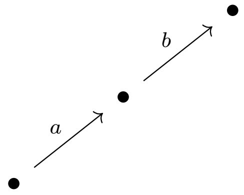
Definition 12.1. *Let $A = kQ/\langle\rho\rangle$ be a special biserial algebra of finite dimension over k. Recall that A is a* string algebra *if ρ is composed only of paths.*

Let A be a string algebra.

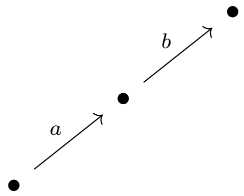
Definition 12.2.

- $C = a_n \cdots a_1$ *is a* non-trivial permitted thread *iff Cb or bC lies in $\langle\rho\rangle$ for all $b \in Q_1$.*
- $\Pi = a_n \cdots a_1$ *is a* non-trivial forbidden thread *iff $a_{i+1}a_i \in \rho$ for all $i \in [1, n-1]$ and $a_1 b$ and $b a_n$ lie in ρ for all $b \in Q_1$.*

For every $v \in Q_0$ such that



and $ba \neq 0$ we formally consider a trivial permitted thread h_v. For every $v \in Q_0$ such that



and $ba = 0$ we formally consider a trivial forbidden thread p_v.

Notation 12.3. $\mathcal{H}_A = \{\text{permitted threads}\}$

Let $\sigma, \varepsilon : Q_1 \to \{\pm 1\}$ be such that:

(1) If $b_1 \neq b_2 \in Q_1$, $s(b_1) = s(b_2)$, then $\sigma(b_1) = -\sigma(b_2)$.

(2) If $b_1 \neq b_2 \in Q_1$, $t(b_1) = t(b_2)$, then $\varepsilon(b_1) = -\varepsilon(b_2)$.

(3) If $b, c \in Q_1$, $cb \in \rho$, $s(c) = t(b)$, then $\sigma(c) = -\sigma(b)$.

We extend ε, σ to $\mathcal{H}_A$. For $H = a_n \cdots a_1$ non-trivial in $\mathcal{H}_A$ define

(1) $\sigma(H) := \sigma(a_1)$, $\varepsilon(H) := \varepsilon(a_n)$,

(2) for trivial threads h_v by connectivity of Q (i.e. $v \xrightarrow{c} \rightsquigarrow \sigma(h_v) = -\varepsilon(h_v) = -\sigma(c)$ and $\xrightarrow{b} v \rightsquigarrow \sigma(h_v) = -\varepsilon(h_v) = -\varepsilon(b)$),

(3) for trivial threads p_v similarly (i.e. $v \xrightarrow{c} \rightsquigarrow \sigma(p_v) = -\varepsilon(p_v) = -\sigma(c)$ and $\xrightarrow{b} v \rightsquigarrow \sigma(p_v) = -\varepsilon(p_v) = -\varepsilon(b)$).

12.2 The Algorithm

(1)

a) First consider $H_0 \in \mathcal{H}_A$.

b) Suppose that H_i is defined. Consider the forbidden thread Π_i which ends in $t(H_i)$ such that $\varepsilon(H_i) = -\varepsilon(\Pi_i)$.

c) $H_{i+1} :=$ permitted thread starting in $s(\Pi_i)$ with $\sigma(\Pi_i) = -\sigma(H_{i+1})$.

This process stops when $H_n = H_0$. Define (n, m) and $n = \sum_{i=1}^{n} \ell(\Pi_{i-2})$.

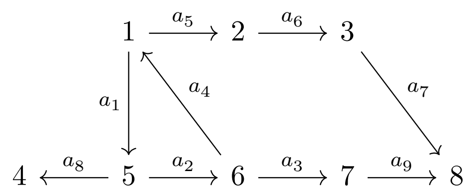
(2) Repeat (1) while all permitted threads haven't been considered.

(3) Add $(0, |C|)$ for every directed cycle C such that each consecutive pair of arrows is a relation.

(4) Define $\phi_A : \mathbb{N}^2 \to \mathbb{N}$ by

$$(n, m) \mapsto \text{number of times } (n, m) \text{ appeared in the previous process}\,.$$

Example 12.4.



with relations a_1a_4, a_4a_2, a_6a_5, a_8a_1 and

$$\begin{aligned}
&\sigma(a_1) = \sigma(a_2) = \sigma(a_3) = \sigma(a_7) = \sigma(a_9) = 1\,,\\
&\sigma(a_4) = \sigma(a_5) = \sigma(a_6) = \sigma(a_8) = -1\,,\\
&\varepsilon(a_4) = \varepsilon(a_7) = 1\,,\\
&\varepsilon(a_1) = \varepsilon(a_2) = \varepsilon(a_3) = \varepsilon(a_5) = \varepsilon(a_6) = \varepsilon(a_8) = \varepsilon(a_9) = -1\,.
\end{aligned}$$

12.3 Interpretation of Permitted Threads of $\overline{\widehat{A}}$

$A = kQ/\langle\rho\rangle$ gentle algebra $\rightsquigarrow$ $\widehat{A} = k\widehat{Q}/\langle\widehat{\rho}\rangle$ repetitive algebra ($\nu : a[z] \mapsto a[z+1]$)

Definition 12.5. *In $(\widehat{Q}, \widehat{\rho})$ a* full path *is a path p not involving any relation in $\widehat{\rho}$ such that $t(p) = \nu^{-1}(s(p))$.*

Define

- $\overline{\widehat{\rho}} = \widehat{\rho} \cup \{$*full paths*$\}$,
- $\overline{\widehat{A}} = k\widehat{Q}/\langle\overline{\widehat{\rho}}\rangle$ *the* expansion *of A.*

Remark 12.6. $\overline{\widehat{A}}$ is a string algebra, isomorphic to $\widehat{A}$/socle of inj.-proj.

Theorem 12.7 (Ringel, Butler)**.** *The vertices of the stable AR-quiver $\Gamma_{\widehat{A},s}$ of $\widehat{A}$ which are the end of AR-sequences with indecomposable middle term are in one-to-one correspondence with $\mathcal{H}_{\overline{\widehat{A}}}$.*

Remark 12.8. We get an easy description of $\tau_{\widehat{A}}$ through this correspondence.

Proposition 12.9. *If (Q, ρ) is not a tree (and gentle) with $A = kQ/\langle\rho\rangle$, we have that*

- *infinite τ-orbits $\leftrightarrow$ $\mathbb{Z}\mathbb{A}_\infty$-components in $\Gamma_{\widehat{A},s}$*
- *finite τ-orbits $\leftrightarrow$ $\mathbb{Z}\mathbb{A}_\infty/\langle\tau^n\rangle$-components in $\Gamma_{\widehat{A},s}$ coming from string modules*

12.4 Action of the Cosyzygy Functor

Let $A = kQ/\langle\rho\rangle$ be gentle, not a tree. Define

$$\Omega^{-1}(M) \; = \; \mathrm{Coker}(M \to E(M)) \text{ the cokernel of the injective hull as object in } \underline{\mathrm{mod}}(\widehat{A})\,.$$

Remark 12.10. $\Omega \circ \tau = \tau \circ \Omega \;\; \Rightarrow \;\; \Omega^{-1}$ permutes the components of $\Gamma_{\widehat{A},s}$

Definition 12.11. *The* characteristic components *of* $\Gamma_{\widehat{A},s}$ *are those of the form* $\mathbb{Z}\mathbb{A}_\infty$ *or* $\mathbb{Z}\mathbb{A}_\infty/\langle\tau^n\rangle$ *with* $n \geq 1$ *coming from string modules.*

Proposition 12.12. *All components* $\mathbb{Z}\mathbb{A}_\infty$ *and* $\mathbb{Z}\mathbb{A}_\infty/\langle\tau^n\rangle$ *with* $n \geq 2$ *come from string modules.*

Definition 12.13. *We say that two characteristic components* C_1 *and* C_2 *are* equivalent *iff they belong to the same* Ω^{-1}*-orbit.*

An equivalence class is called a series of components*.*

Remark 12.14. Since Ω^{-1} is an equivalence, it preserves the type of components.

$\Rightarrow$ Only one type of component in each series of components.

Proposition 12.15 (Avella-Alaminos–Geiß)**.** $\Gamma_{\widehat{A},s}$ *has only finitely many* $\mathbb{Z}\mathbb{A}_\infty$*-components.*

Let C be of type $\mathbb{Z}\mathbb{A}_\infty$ in $\Gamma_{\widehat{A},s}$.

$\rightsquigarrow i_{[C]} = (n, m)$ such that $|n - m| = \#[C]$ and $\Omega^{n-m}_{\widehat{A}}(M) = \tau^{n}_{\widehat{A}}(M)$ for all $M \in [C]$

Let C be of type $\mathbb{Z}\mathbb{A}_\infty/\langle\tau^n\rangle$ with $n \geq 1$.

$\rightsquigarrow i_{[C]} = (n, n)$ such that $(\Omega^{n-n}_{\widehat{A}}(M) =)M = \tau^{n}_{\widehat{A}}(M)$ for all $M \in [C]$

Define $N_A : \mathbb{N}^2 \to N$ by

$$(n, m) \; \mapsto \; \#\{[C] \,|\, i_{[C]} = (n, m)\}\,.$$

Fact 12.16. $N_A = \phi_A$

12.5 End of Proof

Let $A = kQ/\langle\rho\rangle$ and $B = kQ'/\langle\rho'\rangle$ be gentle algebras.

If Q is a tree, then $D^b(A) \cong D^b(\mathbb{A}_{\#Q_0})$. $\rightsquigarrow \phi_A = \phi_{\mathbb{A}_{\#Q_0}}$

Now assume that neither A nor B is a tree and $D^b(A) \simeq_\Delta D^b(B)$.

Theorem 12.17 (Asashiba)**.** $D^b(\widehat{A}) \simeq_\Delta D^b(\widehat{B})$.

Theorem 12.18 (Rickard)**.** *For self-injective finite-dimensional algebras:*

derived equivalence $\Rightarrow$ *stable equivalence*

$\widehat{A}$- mod $\cong_\Delta$ $\widehat{B}$- mod

$\rightsquigarrow [\mathbb{Z}\mathbb{A}_\infty]$ in $\Gamma_{\widehat{A},s} \leftrightarrow [\mathbb{Z}\mathbb{A}_\infty]$ in $\Gamma_{\widehat{B},s}$

$\rightsquigarrow [\mathbb{Z}\mathbb{A}_\infty/\langle\tau^n\rangle]$ in $\Gamma_{\widehat{A},s} \overset{n\geq 2}{\leftrightarrow} [\mathbb{Z}\mathbb{A}_\infty/\langle\tau^n\rangle]$ in $\Gamma_{\widehat{B},s}$

$\rightsquigarrow \sum_{(n,m)} \phi_A(n,m)m = \#Q_0$ is a derived invariant $\rightsquigarrow$ recover $\phi_A(1,1) = \phi_B(1,1)$

13 Derived Discrete Algebras

Thursday 17th 8:30 – Toshitaka Aoki (Nagoya, Japan)

References.

- D. Vossieck. *The algebras with discrete derived category.*

Structure.

(1) Introduction

(2) Main result in [Vossieck] and sketch of proof

(3) Derived equivalences

13.1 Introduction and Notation

Aim.

Introduce the algebras with discrete derived category and classify them up to Morita equivalences / up to derived equivalences.

Notation.

- $k = \overline{k}$ an algebraically closed field
- A a finite-dimensional k-algebra
- mod-A the category of finitely generated A-modules
- $D^b(A)$ the bounded derived category of mod-A
- $D^b(A)_{\mathrm{perf}}$ the subcategory of $D^b(A)$ formed by perfect complexes
- $K_0(A)$ the Grothendieck group of mod-A

Definition 13.1. *For $X \in D^b(A)$ define*

$$\underline{\mathrm{Dim}}X \; := \; (\dim H^i(X))_{i\in\mathbb{Z}} \; \in \; K_0(A)^{(\mathbb{Z})}$$

the sequence of dimension vectors *of $H^i(X)$.*

Definition 13.2 (Vossieck)**.** *We say $D^b(A)$ is* discrete *if for all positive $x \in K_0(A)^{(\mathbb{Z})}$*

$$\#\{X \in D^b(A) \,|\, X \text{ indecomposable with } \underline{\mathrm{Dim}}X = x\}/iso. < \infty\,.$$

Example 13.3. The path algebra A of a quiver of Dynkin type $\mathbb{A}_m$, $\mathbb{D}_n$ $(n \geq 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ has a discrete derived category.

Proof.

- A is representation-finite.
- Any indecomposable complex is a shift of an indecomposable A-module up to isomorphism (see [Happel]).

$\square$

13.2 Main Result and "Proof"

Theorem 13.4 (Vossieck)**.** *Let A be a connected basic finite-dimensional k-algebra. Then the following are equivalent:*

(i) The repetitive algebra $\widehat{A}$ is representation discrete, *i.e. for every positive $m \in K_0(\widehat{A})$*

$$\#\{M \in \operatorname{mod}(\widehat{A}) \mid M \text{ indecomposable with } \underline{\dim} M = m\}/\text{iso.} < \infty \,.$$

(ii) $D^b(A)$ is discrete.

(iii) $D^b(A)_{\text{perf}}$ is discrete.

(iv) A is either derived hereditary *of Dynkin type or there is a presentation $A \xrightarrow{\simeq} kQ/I$ where*

- *(Q, I) is a gentle quiver,*
- *Q contains exactly one cycle,*
- *Q does not satisfy the* clock-condition

$$\#\{\text{clockwise relations } C \in I\} = \#\{\text{counter-clockwise relations } C \in I\} \,.$$

Remark 13.5.

- Derived hereditary algebras of type $\mathbb{A}_n$ are precisely the gentle tree algebras [Assem–Happel].
- Derived hereditary algebras of type $\widetilde{\mathbb{A}}_m$ (not discrete) are precisely the gentle one–cycle algebras satisfying the clock-condition [Assem-Skowroński].

Proof.
"(i) ⇒ (ii)". Use the Happel functor $H : D^b(A) \to \underline{\operatorname{mod}}(\widehat{A})$.
"(ii) ⇒ (iii)". Trivial.
"(iv) ⇒ (i)". Assume A is derived hereditary of Dynkin type. Then $\widehat{A}$ is locally representation finite, i.e. for each vertex v of the quiver of $\widehat{A}$

$$\#\{M \in \operatorname{mod}(\widehat{A}) \mid M \text{ indecomposable with } Me_v \neq 0\}/\text{iso.} < \infty \,.$$

Thus $\widehat{A}$ is representation discrete.

Assume now $A \xrightarrow{\simeq} kQ/I$ is a gentle algebra. Then $\widehat{A}$ is special biserial.

The indecomposables in $\operatorname{mod}(\widehat{A})$ are

- non-uniserial projective-injectives,
- string modules
- band modules

Note: If there are no bands for $\widehat{A}$, then $\widehat{A}$ is representation-discrete.

Let $\overline{A} = \widehat{A}/\operatorname{soc}(\text{non-uniserial proj.-inj.})$.

Recall: Each band corresponds to a cyclic word b such that b is not a proper power of a cyclic word and $b^m \neq 0$ for any $m \in \mathbb{N}$.

Lemma 13.6 (Ringel '97)**.** *Let $\widehat{Q}$ be the quiver with*

- *vertices $v[z]$ for $v \in Q_0$ and $z \in \mathbb{Z}$,*
- *arrows $d[z] : v[z] \to w[z]$ for $d : v \to w$ and $\widehat{p} : w[z] \to v[z]$ for maximal paths p.*

Then

$$\{\textit{cyclic words } w \textit{ in } Q \textit{ with } \text{cyclic defect } \delta_c(w) = 0\} \xleftrightarrow{1:1} \{\textit{cyclic words } \widehat{w}\}$$

where

$$\delta_c(w) := \#\{\textit{clockwise relations } w \in I\} - \#\{\textit{counter-clockwise relations } w \in I\}.$$

If (Q, I) satisfies the additional condition, then the left set is empty.

$\rightsquigarrow \widehat{A}$ does not have any band modules.

$\rightsquigarrow \widehat{A}$ is representation discrete.

"(iii) $\Rightarrow$ (iv)".

Lemma 13.7 (V.4.1)**.** *If $D^b(A)_{\text{perf}}$ is discrete, then A is representation finite.*

To prove this part, we need "covering theory" (see Gabriel and Roiter) and the "cleaving method" (see "Algebra V III. Rep. of fin. dim. algebras") for k-categories or bound quivers.

Assume $D^b(A)_{\text{perf}}$ is discrete. We regard A as a k-category with

- objects: $\{e_1, \ldots, e_n\}$ a complete set of pairwise orthogonal idempotents in A,
- $\operatorname{Hom}(e_i, e_j) = e_j A e_i$ for all $1 \leq i, j \leq n$.

(1) [Vossieck, Lemma 4.2]: If A is simply connected, then A is derived hereditary of Dynkin type. The converse also holds.

(2) If A is not simply connected, we can show that A is a gentle algebra.

Now, there is a presentation $A \xrightarrow{\simeq} kQ/I$ where (Q, I) is a gentle quiver. If Q is a gentle tree, then $\widehat{A}$ is derived hereditary of type $\mathbb{A}_m$ by Remark 13.5, a contradiction.

So Q contains at least one cycle.

Lemma 13.8 (Ringel)**.** *If Q contains at least two cycles, then there exists a cyclic word with cyclic defect* 0.

Consequently, Q contains exactly one cycle.

If (Q, I) satisfies the clock condition, A is derived equivalent to an algebra of type $\widetilde{\mathbb{A}}_n$, a contradiction.

Therefore (Q, I) does not satisfy the clock condition. □

Theorem 13.9 (Bobiński–Geiß–Skowroński)**.** *Let A be a connected finite-dimensional algebra which is not of Dynkin type. Then the following are equivalent:*

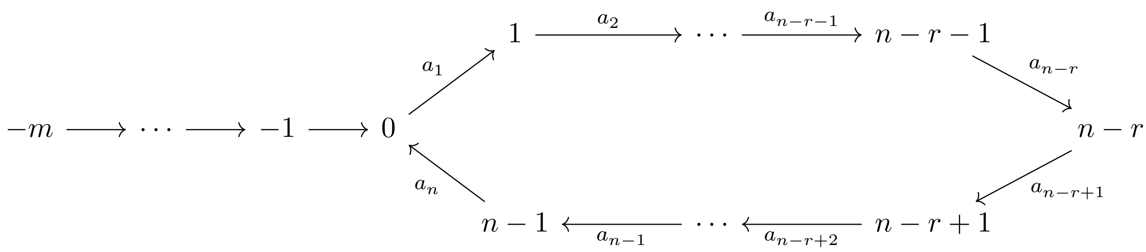
(i) $D^b(A)$ *is discrete.*

(ii) $D^b(A) \xrightarrow{\sim} D^b(\Lambda(r, n, m))$ *for some* (r, n, m)*.*

(iii) *A is tilting-cotilting equivalent to* $\Lambda(r, n, m)$

Moreover, $D^b(\Lambda(r, n, m)) \xrightarrow{\sim} D^b(\Lambda(r', n', m'))$ *if and only if* $(r, n, m) = (r', n', m')$*.*

The algebra $\Lambda(r, n, m)$ is given by the quiver



with relations $a_1 a_n$, $a_n a_{n-1}$, $\dots$, $a_{n-r+2} a_{n-r+1}$.

14 Singularity Categories of Gentle Algebras

Thursday 17th 10:00 – David Pauksztello (Verona, Italy)

References.

(1) Geiß, Reiten. *Gentle algebras are Gorenstein.*

(2) Kalck. *Singularity categories of gentle algebras.*

Notation.

- Λ a finite-dimensional k-algebra

14.1 Gorenstein Algebras, Motivation

Definition 14.1. Λ *is* Gorenstein *if* $\operatorname{inj.dim}{}_\Lambda\Lambda < \infty$ *and* $\operatorname{inj.dim}\Lambda_\Lambda < \infty$.

Example 14.2.

- Λ with $\operatorname{gl.dim}\Lambda < \infty$
- Λ self-injective

Properties of Gorenstein Algebras.

- [Happel] $K^b(\operatorname{proj}\Lambda) = K^b(\operatorname{inj}\Lambda) \Leftrightarrow \Lambda$ Gorenstein.
- $K^b(\operatorname{proj}\Lambda)$ satisfies Serre duality, i.e. has AR-triangles.
- The full subcategory of *Gorenstein projective modules* is defined by

$$\mathrm{GP}(\Lambda) \;=\; \{M \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(M,\Lambda) = 0 \,\forall\, i > 0\},$$

an exact Frobenius category whose projective-injectives are the projective Λ-modules.

Theorem 14.3 (Buchweitz)**.** *Let Λ be Gorenstein. The embedding* $\mathrm{GP}(\Lambda) \hookrightarrow D^b(\Lambda)$ *induces a triangle equivalence*

$$\mathrm{GP}(\Lambda)/\operatorname{proj}\Lambda \xrightarrow{\sim} D_{\mathrm{sg}}(\Lambda) \;:=\; D^b(\Lambda)/K^b(\operatorname{proj}\Lambda)$$

Remark 14.4.

- GPs are often called *maximal Cohen-Macaulay modules.*

  Simple hypersurface singularities $\Leftrightarrow$ finitely indecomposable GPs.

- When Λ is self-injective, all modules are GP, so the singularity category is $\underline{\operatorname{mod}}\Lambda$.

14.2 Gentle Algebras Are Gorenstein

Let $\Lambda = kQ/I$ be a gentle algebra.

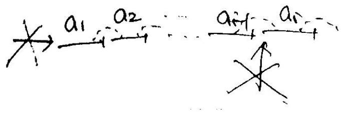
- An arrow $b \in Q_1$ is *gentle* if there is no $a \in Q_1$ with $ba \in I$.
- A direct walk $w = a_n \cdots a_1$ is *critical* if $a_{i+1}a_i \in I$ for $1 \leq i < n$.
  It is called a *critical cycle* if $s(a_1) = t(a_n)$ and $a_1 a_n \in I$.

Note.

- There exists at most one arrow a_0 such that $a_n \cdots a_1 a_0$ is critical.
- There exists at most one arrow a_{n+1} such that $a_{n+1}a_n \cdots a_1$ is critical.

Lemma 14.5. *There is a bound $n(\Lambda) \leq |Q_1|$ for the maximal lengths of critical paths starting with a gentle arrow.*

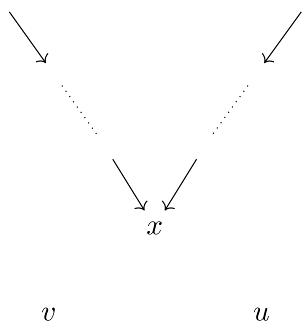
Proof. Assume $a_{n+1}a_n \cdots a_1$ is critical with a_1 gentle and $a_1, \ldots, a_n$ pairwise different. Draw a picture …



□

Injectives and Projectives.

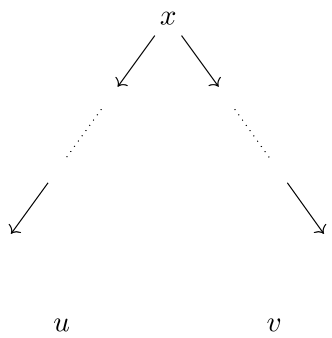
The injective I_x is



$$= M(v^{-1}u)$$

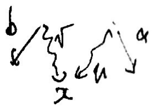
where u, v distinct maximal directed paths ending (resp. starting) at $x \in Q_0$.

Similary, the projective P_x looks as follows:



$= M(uv^{-1})$

For $I_x = M(v^{-1}u)$ consider the unique (if they exist!) arrows a and b such that $v^{-1}ua^{-1}$ and/or $bv^{-1}u$ are defined as strings. Then a, b are gentle arrows.



Definition 14.6. *For each $a \in Q_1$ define*

$$r(a) := \textit{ the unique maximal direct string such that } r(a)a \textit{ is defined as a string}.$$

Define $R(a) := M(r(a))$.

Proposition 14.7. *Let $I_x = M(v^{-1}u)$. For $j \geq 1$ each indecomposable non-projective summand of $\Omega^j M(v^{-1}u)$ is of the form $R(a_j)$ for a critical path $a_j \cdots a_1$ with a_1 gentle.*

Proof. Take the projective cover of I_x. $\rightsquigarrow P_t \oplus P_s \to I_x \rightsquigarrow$ Draw a picture ... □

Theorem 14.8 (Geiß–Reiten)**.**

$$\operatorname{inj.dim}(\Lambda) = \begin{cases} n(\Lambda) = \operatorname{proj.dim}_\Lambda D(\Lambda^{\mathrm{op}}) & \textit{if } n(\Lambda) > 0 \\ \operatorname{proj.dim}_\Lambda D(\Lambda^{\mathrm{op}}) \leq 1 & \textit{if } n(\Lambda) = 0. \end{cases}$$

In particular, Λ is Gorenstein.

Proof. $\operatorname{proj.dim}_\Lambda D(\Lambda^{\mathrm{op}}) \leq n(\Lambda) + \delta_{n(\Lambda),0}$.

Suppose $n(\Lambda) > 0$. Let $a_n \cdots a_1$ be a critical path with a_1 gentle. If there is $b \in Q_1$ such that $s(b) = s(a_1)$ then $I_{t(b)}$ looks like

$$\overset{v}{\rightsquigarrow} t(b) \overset{b}{\longleftarrow} \overset{a_1}{\longrightarrow}$$

by Proposition 14.7 and $\operatorname{proj.dim} I_{t(b)} \geq 1$.

If there is no such b, then $I_{s(a_1)}$ looks like

$$\overset{v}{\rightsquigarrow} s(a_1) \overset{a_1}{\longrightarrow}$$

and $\operatorname{proj.dim} I_{s(a_1)} \geq n$.

Note: $n(\Lambda) = n(\Lambda^{\mathrm{op}}) \Rightarrow \Lambda$ is Gorenstein □

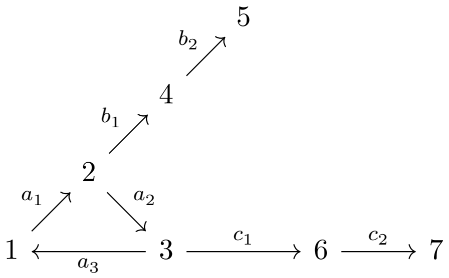
Theorem 14.9 (Kalck)**.**

(1) $\operatorname{ind}\mathrm{GP}(\Lambda) = \operatorname{ind}\operatorname{proj}(\Lambda) \cup \{R(a_1), \dots, R(a_n) \mid c = a_n \cdots a_1 \in \mathcal{C}(\Lambda)\}$

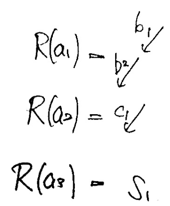
(2) $D_{\mathrm{sg}}(\Lambda) \cong \prod_{c\in\mathcal{C}(\Lambda)} D^b(k\mathbb{A}_1)/\Sigma^{\ell(c)}$ *"product of orbit categories" [Keller]*

where $\ell(c)$ is the length of the cycle c.

Example 14.10. Let Λ be the algebra given by the quiver



with relations a_1a_3, a_2a_1, a_3a_2, c_2c_1. Then:



There are short exact sequences

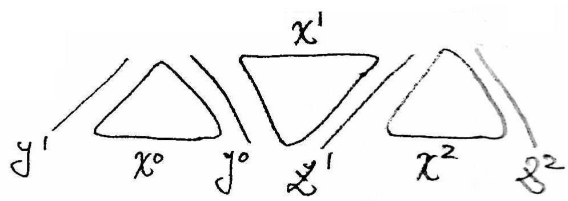
$$0 \to R(a_i) \to P_i \to R(a_{i-1}) \to 0\,.$$

For example,

$$0 \to (\xleftarrow{c_1}) \to (\xleftarrow{c_1}\xleftarrow{a_2}\xrightarrow{b_1}\xrightarrow{b_2}) \to (\xrightarrow{b_1}\xrightarrow{b_2}) \to 0\,.$$

In particular, $\Omega R(a_{i-1}) = R(a_i)$ and $\Sigma R(a_i) = R(a_{i-1})$ in $\underline{\mathrm{GP}}(\Lambda)$.

$D^b(\Lambda)$ looks like:



where

- $\Delta : \mathcal{X}^0, \mathcal{X}^1, \mathcal{X}^2$ are $\mathbb{Z}\mathbb{A}_\infty$ components of $K^b(\operatorname{proj}\Lambda)$,
- $\backslash/ : \mathcal{Z}^0, \mathcal{Z}^1, \mathcal{Z}^2$ are $\mathbb{A}_\infty^\infty$ components of $D^b(\Lambda) \setminus K^b(\operatorname{proj}\Lambda)$

  (one of an irreducible morphism in a $\mathcal{Z}$ component lies on the boundary of an $\mathcal{X}$ component, i.e. each $\mathcal{Z}$ component is identified in $D_{\mathrm{sg}}(\Lambda)$).

Sketch.

Use the following facts to show $R(a_i)$ are all the GPs:

- A GP Λ-module is either projective or of infinite projective dimension.
- M is GP $\Leftrightarrow \Omega M \cong \Omega^d N$ for some $N \in \operatorname{mod}\Lambda$, where $d = \operatorname{inj.dim}_\Lambda \Lambda$

  ($\Rightarrow$ every GP module is a submodule of a projective)

The short exact sequences $0 \to R(a_i) \to P_i \to R(a_{i-1}) \to 0$ for $a_i \in c \in \mathcal{C}(\Lambda)$ show $R(a_i)$ are GP.

No submodule of a projective can have a subword of the form $\to\leftarrow$.
So the worst case is $\leftarrow\to$. $\rightsquigarrow$ Get a projective.

The remaining GPs are uniserial. The only way to embed into a projective is if they have the form $R(a)$ for some $a \in Q_1$.
By Proposition 14.7 if $a \notin c \in \mathcal{C}(\Lambda)$ then $\operatorname{proj.dim} R(a) < \infty$.

Second Statement.

We have $\Sigma R(a_i) = R(a_{i-1})$, so $\Sigma^{\ell(c)} R(a_i) = R(a_i)$.

Fact 14.11. *Any semisimple abelian category with autoequivalence Σ admits a unique triangulated structure with shift Σ.*

$$\underline{\operatorname{Hom}}(R(a), R(a')) = \delta_{a,a'} k \,.$$

Remark 14.12. [Chen–Shen–Zhou] have more general versions of these statements for quadratic monomial algebras.

15 Quivers with Potential from Surface Triangulations

Thursday 17th 14:00 – Toshiya Yurikusa (Nagoya, Japan)

Aim.

To introduce a new class of gentle algebras.

- Quivers with potential (QP) and QP-mutations
- QPs from surface triangulations (unpunctured case)

15.1 Quivers with Potential

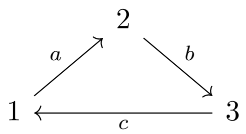
Notation.

- k a field
- Q a finite quiver without loops

Definition 15.1. *A* potential S *on* Q *is a linear combination of cyclic paths up to cyclical equivalence (i.e.* $a_d \cdots a_1 \sim a_1 a_d \cdots a_2$*).*

The pair (Q, S) *is called a* quiver with potential (QP)*.*

Example 15.2.



Potential (3-cycle case): $S = cba \sim bac \sim acb, 0, cbacba, \ldots$

Definition 15.3. *The* cyclic derivative ∂_a *at* $a \in Q_1$ *is defined by*

$$\partial_a(a_d \cdots a_1) \;=\; \sum_{i=1}^{d} \partial_{a,a_i} a_{i-1} \cdots a_1 a_d \cdots a_{i+1}$$

where $a_d \cdots a_1$ *is a cyclic path.*

The ideal

$$J(S) \;:=\; \langle \partial_a(S) \mid a \in Q_1 \rangle$$

of the completed path algebra of Q *is called the* Jacobian ideal.

Following [DWZ '08] we define the Jacobian algebra

$$\mathcal{P}(Q, S) \;:=\; \textit{the completed path algebra}/J(S)\,.$$

15.2 QP-Mutations

Let (Q, S) be a QP and $v \in Q_0$.

Theorem 15.4 (and Definition). *If Q has no 2-cycles incident to v, we obtain a new QP*

$$(Q', S') \;=\; \widetilde{\mu}_v(Q, S) \qquad \text{“QP-premutation at } v\text{”}$$

constructed as follows:

(1) For each $i \xleftarrow{b} v \xleftarrow{a} j$ add an arrow $i \xleftarrow{[ba]} j$.

(2) Reverse all arrows incident to v ($\xleftarrow{a} v \rightsquigarrow \xrightarrow{a^*} v$).

Let

$$S' \;:=\; [S] + \sum_{i \underset{b}{\leftarrow} v \underset{a}{\leftarrow} j \text{ in } Q} a^* b^* [ba]$$

where $[S]$ is obtained from S by replacing all $i \xleftarrow{b} v \xleftarrow{a} j$ with $[ba]$.

By [DWZ, Theorem 4.6] (“splitting theorem”) there exists a QP

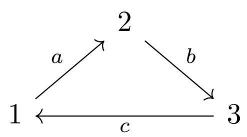
$$(Q^*, S^*) \;=\; \mu_v(Q, S)$$

such that S^* has no 2-cycles and $\mathcal{P}(Q^*, S^*) \cong \mathcal{P}(Q', S')$.

“Remove 2-cycles in S' and the corresponding arrows.”

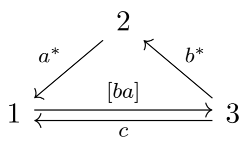
Then $\mu_k(Q, S)$ is a QP-mutation *of (Q, S) at v.*

Example 15.5. Let (Q, S) be the QP with

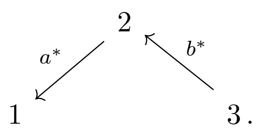
$Q =$ 

and $S = cba$.

Then $\widetilde{\mu}_2(Q, S)$ is the QP (Q', S') with

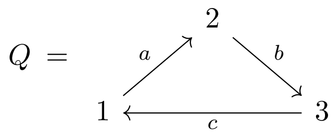
$Q' =$ 

and $S' = c[ba] + a^* b^* [ba]$. Then $\mu_2(Q, S)$ is the QP (Q^*, S^*) with

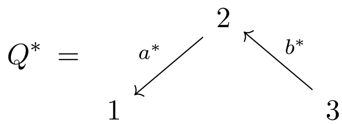
$Q^* =$ 

The 2-acyclicity of Q is essential to apply the QP-mutation for every vertex of Q. But 2-acyclicity is not invariant under QP-mutation.

Example 15.6. For $(Q, S)$with



and $S = 0$ the QP $\mu_2(Q, S) = \widetilde{\mu}_2(Q, S)$ is (Q^*, S^*) with



and $S^* = a^*b^*[ba]$.

Theorem 15.7 (DWZ, Corollary 7.4)**.** *Let k be an uncountable field. Any 2-acyclic quiver has a potential S such that the quiver obtained from (Q, S) after any sequence of QP-mutations is 2-acyclic. Such a potential S is called* non-degenerate.

15.3 Surface Triangulations (Unpunctured Case)

Let Σ be a connected oriented Riemann surface with boundary $\partial\Sigma$ and M a finite set of marked points on $\partial\Sigma$ containing at least one point from each connected component of $\partial\Sigma$.

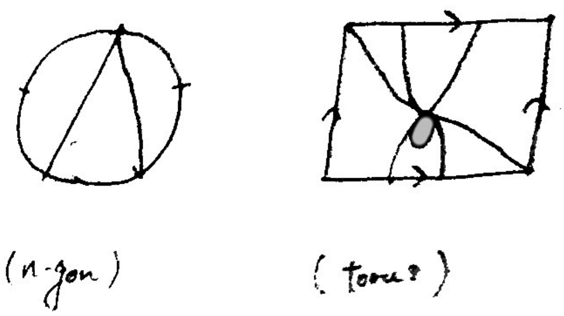
Then (Σ, M) is called a *marked surface* (without punctures).

Definition 15.8. *An* arc *on (Σ, M) is a curve up to isotopy on Σ satisfying:*

- *Its endpoints lie in M.*
- *It has no self-intersection (except in the endpoints).*
- *It is neither contractible nor a boundary segment.*

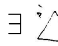
A triangulation *of a marked surface is given by a maximal collection of arcs which do not intersect each other.*

Example 15.9.



Definition 15.10. *Let (Σ, M) be a marked surface and τ a triangulation of (Σ, M).*

Define a QP $(Q(\tau), S(\tau))$ as follows:

- $Q(\tau)_0 = \{arcs\ of\ \tau\}$
- $Q(\tau)_1 = \{i \to j \mid \exists$  $in\ \tau\}$
- $S(\tau) = \sum_{internal\ triangles\ of\ \tau}$ 

Remark 15.11. If $Q(\tau)$ is 2-acyclic, then

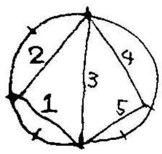
$$J(S(\tau)) = \Big\langle$$  ,  ,  $$\mid \Delta \text{ internal triangle of } \tau \Big\rangle .$$

By [LF '09, Theorem 3.6]

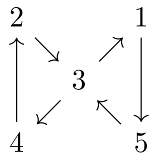
$$\mathcal{P}(Q(\tau), S(\tau)) = kQ(\tau)/J(S(\tau)) .$$

is finite-dimensional. This is a gentle algebra [ABCP, '09, Theorem 2.7] (next talk).

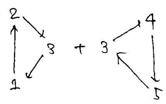
Example 15.12. For

$\tau =$ 

we have

$Q(\tau) =$ 

and

$S(\tau) =$  .

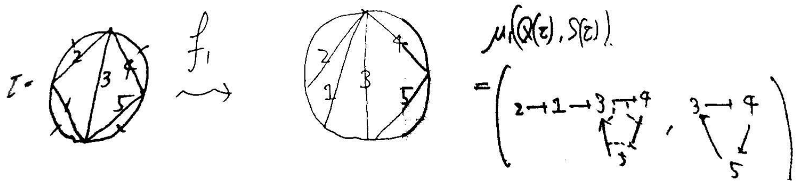
Theorem 15.13 (LF, Theorem 3.0). *QP-mutations of $(Q(\tau), S(\tau))$ are compatible with flips of τ where a* flip *of τ at an arc v is*

$$f_v(\tau) = (\tau \setminus \{v\}) \cup \{v'\}$$

such that $f_v(\tau)$ is a triangulation with $v \neq v'$.

Since $Q(\tau)$ has no 2-cycles for any triangulation τ, the potential $S(\tau)$ is non-degenerate.

Example 15.14.



Theorem 15.15 (GLFS '16, Theorem 1.4). *If (Σ, M) is not a torus with $|M| = 1$, then $S(\tau)$ is the only non-degenerate potential up to right equivalence.*

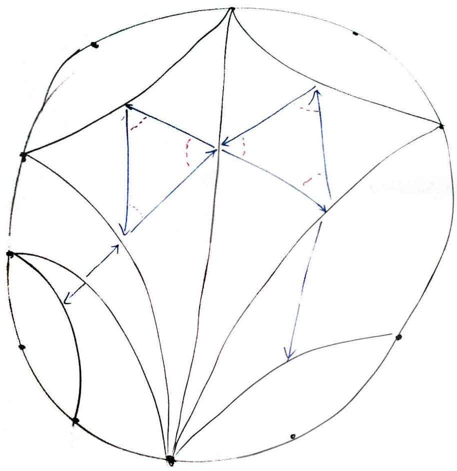
16 Gentle Algebras Arising from Surface Triangulations

Thursday 17^{th} 15:15 – Alexander Garver (Montreal, Canada)

References.

- [Assem–Brüstle–Charbonneau-Jodoin–Plamondon]

Let (S,M) be an unpunctured surface and Γ a triangulation of (S,M).



$\rightsquigarrow (Q(\Gamma), W(\Gamma))$

$\rightsquigarrow A(\Gamma) = kQ(\Gamma)/I(\Gamma)$ where $I(\Gamma) = J(W(\Gamma))$

Questions.

- Properties of $A(\Gamma)$
- Which $A(\Gamma)$ are cluster-tilted?
- Which gentle algebras are cluster-tilted?

16.1 Properties of $A(\Gamma)$

Theorem 16.1. *The following hold:*

(i) $A(\Gamma)$ *is gentle.*

(ii) $A(\Gamma)$ *is Gorenstein of dimension one.*

(iii) If $ab \in I(\Gamma)$ *where* $x \xrightarrow{a} z \xrightarrow{b} y$, *then there is an arrow* $y \to x$ *in* $Q(\Gamma)$.

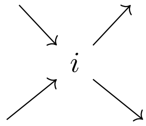
(iv) There is a Galois covering $k\widetilde{Q}/\widetilde{I}$ *of* $A(\Gamma)$ *such that:*

  (T1) Every chordless cycle in $\widetilde{Q}$ *is a* 3*-cycle with full relations.*

  (T2) These are the only relations.

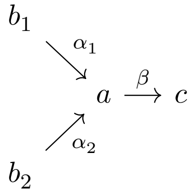
Proof. (i)

- $A(\Gamma)$ is finite-dimensional [LF].
- $I(\Gamma)$ is generated by 2-paths.
- Any vertex i of $Q(\Gamma)$ has

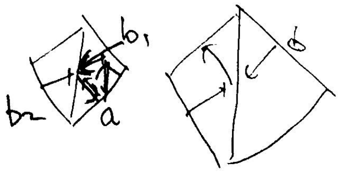


since the corresponding arc appears in exactly 2 triangles.

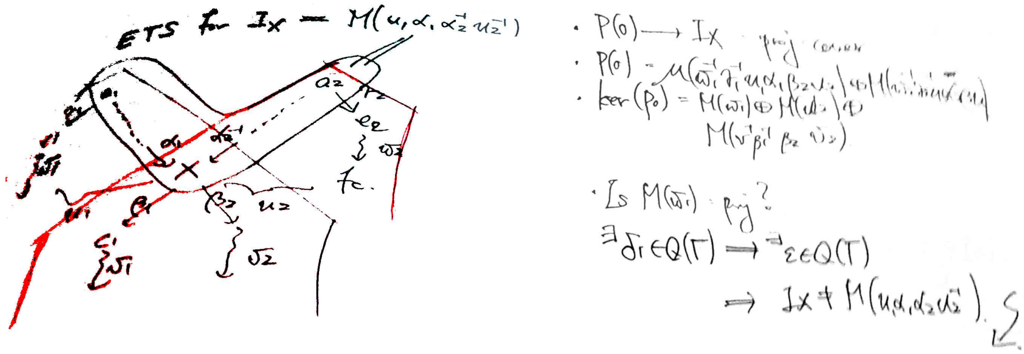
- Suppose



then draw some picture …



(ii) Drawing a picture …



□

16.2 Which $A(\Gamma)$ are cluster-tilted?

Recall that if Q is acyclic, one defines its *cluster category*

$$\mathcal{C}_Q \;=\; D^b(kQ)/\tau^{-1}[1]\,.$$

Then $\operatorname{ind}\mathcal{C}_Q = \operatorname{ind} kQ \,\dot\cup\, P_i[1]_{i\in Q_0}$.

If $T = T_1 \oplus \cdots \oplus T_n$ is a *cluster-tilting object* (i.e. $\mathrm{Ext}^1_{\mathcal{C}_Q}(T,T) = 0$ and $n = \#Q_0$), then $\mathrm{End}_{\mathcal{C}_Q}(T)$ is a *cluster-tilted algebra.*

Theorem 16.2. *The following are equivalent:*

(1) $A(\Gamma)$ is cluster-tilted.

(2) $A(\Gamma)$ is cluster-tilted of type $\mathbb{A}$ or $\widetilde{\mathbb{A}}$.

(3) S is a disc or an annulus.

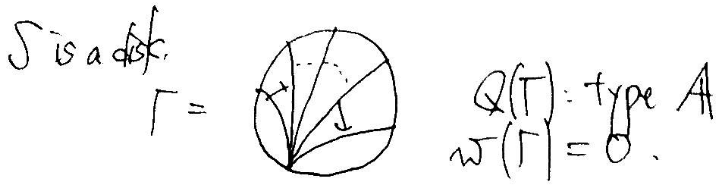
Moreover, all cluster-tilted algebras of these types are realizable as $A(\Gamma)$.

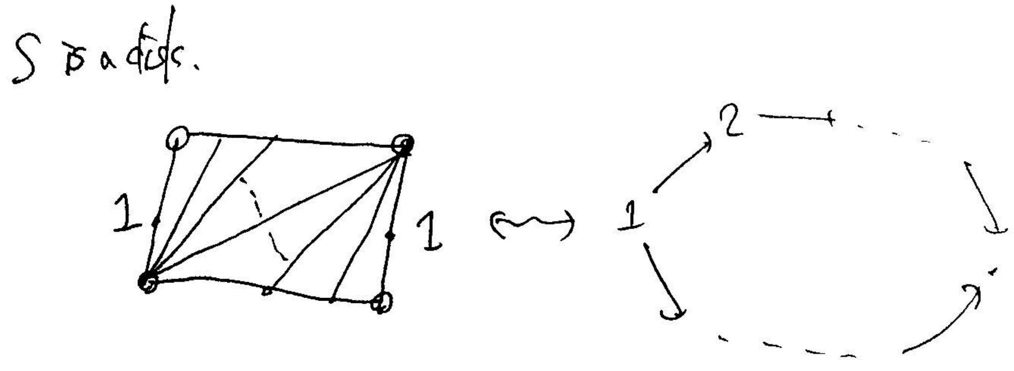
Proof. "(2) ⇒ (1)": Trivial.

"(1) ⇒ (2)": Let $(Q(\Gamma), W(\Gamma))$ be the QP corresponding to $A(\Gamma)$.
$(Q(\Gamma), W(\Gamma)) \rightsquigarrow (Q', 0)$ (under a sequence of QP mutations)
$\Rightarrow A(\Gamma') = kQ'$ hereditary
$\Rightarrow$ Since $A(\Gamma')$ is gentle, it is of type $\mathbb{A}$ or $\widetilde{\mathbb{A}}$.

"(3) ⇒ (2)": Any two triangulations of (S, M) are flip-equivalent [Hatcher, 1991].

Since flips correspond to mutations, it is easy to show that "(3) ⇒ (2)" for a particular triangulation:





□

16.3 Which gentle algebras are cluster-tilted?

Theorem 16.3 (Assem–Brüstle–Schiffler 2008)**.** *An algebra Λ is cluster-tilted iff there exists a tilted algebra C (i.e. $C = \mathrm{End}_{kQ}(T)$ for a tilting object in* $\mathrm{mod}\, kQ$*) such that*

$$\Lambda \cong \widetilde{C} := C \ltimes \mathrm{Ext}^2_C(DC, C)\,.$$

As abelian group

$$\widetilde{C} \;=\; C \oplus \operatorname{Ext}^2_C(DC, C)$$

with addition $(c, e) + (c', e') = (c + c', e + e')$ where $e + e'$ is the Baer sum in $\operatorname{Ext}^2_C(DC, C)$ and multiplication $(c, e)(c', e') = (cc', ce' + ec')$ with $e_1 = ce'$ and

$$\begin{array}{llllllllllll} e: & 0 & \longrightarrow & P & \longrightarrow & M & \longrightarrow & N & \longrightarrow & I & \longrightarrow & 0 \\ e': & 0 & \longrightarrow & P' & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & I' & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{llllllllllll} e': & 0 & \longrightarrow & P' & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & I' & \longrightarrow & 0 \\ & & & \big\downarrow{\scriptstyle c} & & \big\downarrow & & \| & & \| & & \\ e_1: & 0 & \longrightarrow & ce'C & \longrightarrow & M_1 & \longrightarrow & N' & \longrightarrow & I' & \longrightarrow & 0 \end{array}$$

where the left-hand square is a pushout.

Theorem 16.4. *Let $C = kQ_C/I_C$ be a tilted algebra and $\widetilde{C}$ the trivial extension. The following are equivalent:*

(1) C is gentle.

(2) C is tilted of type $\mathbb{A}$ or $\widetilde{\mathbb{A}}$.

(3) $\widetilde{C}$ is gentle.

(4) $\widetilde{C}$ is cluster-tilted of type $\mathbb{A}$ or $\widetilde{\mathbb{A}}$.

Proof.

"(1) $\Rightarrow$ (2)": [Schröer 1999]

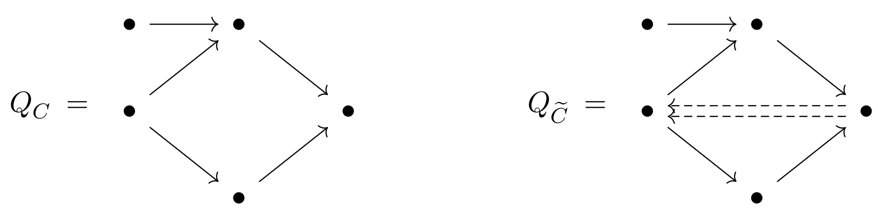
"(3) $\Rightarrow$ (1)": [Assem–Coelho–Trepode]

"(2) $\Leftrightarrow$ (4)": [Assem–Brüstle–Schiffler]

"(2) $\Rightarrow$ (3)": Not quite easy.

Important part here is saying what is $I_{\widetilde{C}}$ where $\widetilde{C} = kQ_{\widetilde{C}}/I_{\widetilde{C}}$. □

Example 16.5.



17 Surface (Cut) Algebras

Thursday 17th 17:00 – Raquel Coelho Simoes (Lisbon, Portugal)

References.

- [David-Roesler–Schiffler]

17.1 Cuts of Triangulated Surfaces

Fix (S, M, T) where ...

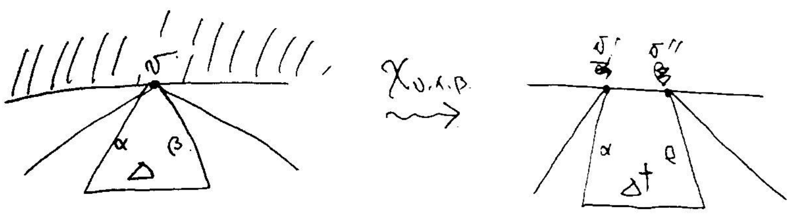
- S is a connected oriented unpunctured Riemann surface with boundary ∂S,
- M is a set of marked points in ∂S intersecting each connected component of ∂S,
- T is a triangulation of (S, M).

Let Δ be an internal triangle in T, $v \in M$ one of the vertices of Δ, and α, β the arcs of Δ incident to v:

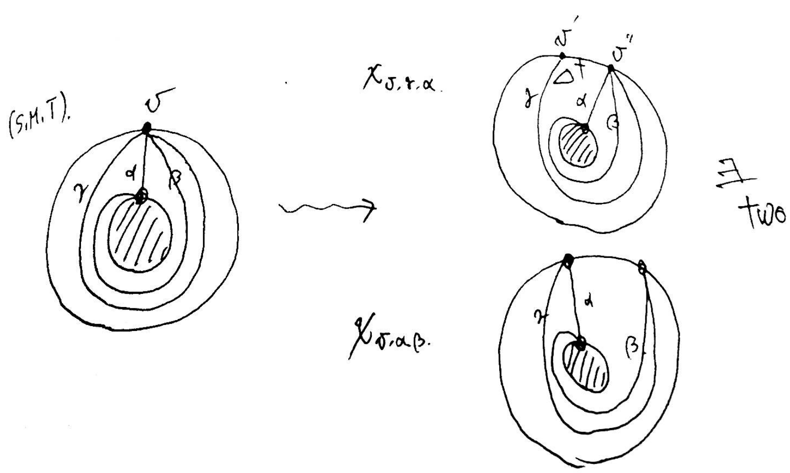
$(S, M, T) \overset{\text{cut at } v, \alpha, \beta}{\rightsquigarrow} (S, \chi_{v,\beta,\alpha}(M), \chi_{v,\beta,\alpha}(T))$ where

$$\chi_{v,\beta,\alpha}(M) \;=\; (M \setminus \{v\}) \cup \{v', v''\}$$
$$\chi_{v,\beta,\alpha}(T) \;=\; T \setminus \{\gamma \mid \gamma \text{ incident to } v\}) \cup \{\gamma^+ \mid \gamma \text{ incident to } v' \text{ or } v''\}$$

where γ^+ is the arc obtained from γ by replacing the end of $\overline{\gamma}$ by the concatenation of $\overline{\gamma}$ and δ' (resp. δ'') if $\overline{\gamma} = \overline{\alpha}$ or $\overline{\gamma\alpha\beta}$ (resp. $\overline{\gamma} = \overline{\beta}$ or $\overline{\alpha\beta\gamma}$).



Example 17.1.



Definition 17.2.

(1) *$\chi_{v,\beta,\alpha}(S, M, T)$ is called the* local cut *of (S, M, T) at vertex v relative to α and β.*

(2) *A* cut *of (S, M, T) is a partially triangulated surface (S, M^+, T^+) obtained by applying a sequence of local cuts $\chi_{v_1,\beta_1,\alpha_1}, \ldots, \chi_{v_t,\beta_t,\alpha_t}$ in such a way that we cut each internal triangle at most once.*

(3) *A cut is* admissible *if every internal triangle of T is cut exactly once.*

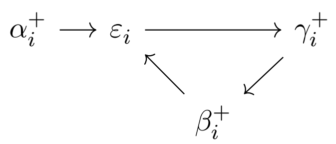
(4) Δ^+ quasi-triangles

17.2 Definition of Surface Algebras

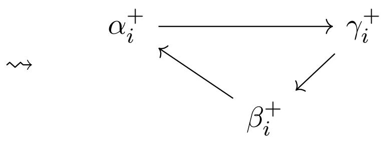
Let (S, M^+, T^+) be the cut of (S, M, T) given by $(\chi_{v_i,\beta_i,\alpha_i})_{i=1,\ldots,t}$.

First, complete T^+ to a triangulation $\overline{T}^+$ of (S, M^+).

Second, construct $Q_{\overline{T}^+}$ (see previous talk). Some picture here ...



Third, obtain Q_{T^+} from $Q_{\overline{T}^+}$ by deleting the vertices ε_i.



Locally: (again a picture ...)

Definition 17.3. *A* (cut) surface algebra *of type (S, M) is $A^+ = kQ^+/I^+$ (with I^+ as in the above figure) where (S, M^+, T^+) is a cut of a triangulated surface (S, M, T).*

$$\begin{array}{ccc} (S, M, T) & \xrightarrow{\text{cut}} & (S, M^+, T^+) \\ \downarrow & & \downarrow \\ A(T) & \xrightarrow{\text{cut ``edges''}} & A(T^+) \end{array}$$

Definition 17.4. *Let Q be a quiver and C an oriented cycle in Q.*

(1) *C is a* chordless cycle *if it is a full subquiver of Q and for each $v \in C$ there is a unique $a \in C$ and a unique $b \in C$ such that $s(a) = v$ and $t(b) = v$.*

(2) *A* cut *of Q is a subset of the set of arrows lying on chordless cycles such that no two arrows lie in the same cycle.*

(3) *A cut is* admissible *if it contains exactly one arrow of each chordless cycle in Q.*

(4) Let $A = kQ/I$. An algebra is said to be obtained from A by a cut *if it is isomorphic to $kQ/\langle I \cup \Gamma \rangle$ where Γ is a cut of Q.*

[Amiot–Grimeland] In other words, let d be a degree map assigning degree 0 or 1 to each arrow of Q such that:

- Chordless cycles have degree 1.
- Arrows not lying on a chordless cycle have degree 0.

$\rightsquigarrow$ d describes an admissible cut.

The cut algebra of A with respect to d is the degree zero subalgebra.

Observation 17.5. $\chi_{v,\beta,\alpha} \leftrightarrow$ *cutting the arrows between α and β in Q_T*

Theorem 17.6. *Every surface algebra is gentle.*

Proof. Let A be a surface algebra. Then $A = A(T^+)$ with (S, M^+, T^+) a cut of (S, M, T). Now:

- $A(T^+)$ is obtained from $A(T)$ by a cut.
- $A(T)$ is gentle.
- Any cut of a gentle algebra is gentle.

□

17.3 Motivation

- (see Wassilij's talk) gentle algebra $G \xrightarrow{\text{trivial extension}}$ BGA $T(G) = G \ltimes DG$
- [Schroll] Every gentle algebra is the admissible cut of a unique Brauer graph algebra (its trivial extension).
- The Brauer graph of $A(T^+)$ is T^+. But the BGA (i.e. $T(A(T^+))$) is not the Jacobian algebra $A(T)$.

Theorem 17.7 (DR–S)**.** *If (S, M^+, T^+)*

(1) $\operatorname{gl.dim}(A^+) \leq 2$

(2) $A(T) \cong A(T^+) \ltimes \operatorname{Ext}^2_{A(T^+)}(DA(T^+), A(T^+))$ *(compare [ABS])*

17.4 AG-Invariant

Example 17.8. A picture …

Notation. Let (S, M, T) be a triangulated surface, C the boundary components of S.

- $M_{C,T} = \{$marked points on C that are incident to at least one arc in T $\}$
- $n_{C,T} = \#M_{C,T}$
- $m_{C,T} = \#$boundary segments on C that have both endpoints on $M_{C,T}$

Theorem 17.9. *Let $A = A(T^+)$ be a surface algebra of type (S, M, T) given by a cut (S, M^+, T^+). The AG-invariant of A is given as follows:*

(a) $(0,3) \overset{1:1}{\leftrightarrow}$ *internal triangle in T^+, and $\nexists (0, m)$ with $m \neq 3$.*

(b) ordered pairs (n, m) in $\mathrm{AG}(A)$ with $n \neq 0$ $\overset{1:1}{\leftrightarrow}$ *boundary components of S.*

If C is a boundary component, the corresponding (n, m) is given by $n = n_{C,T} + \ell$ and $m = m_{C,T} + 2\ell$ where

$$\ell \;=\; \#\text{local cuts } \chi_{v,\beta,\alpha} \text{ in } (S, M^+, T^+) \text{ such that } v \text{ is a point on } C..$$

"Proof". permitted threads $\mathcal{H} \overset{1:1}{\leftrightarrow}$ non-empty complete fans of (S, M^+, T^+) (picture ...)

forbidden threads $\mathcal{F} \setminus$ cycles:

- length 2 $\overset{1:1}{\leftrightarrow}$ quasi-triangles
- length 1 $\overset{1:1}{\leftrightarrow}$ triangles with exactly one side on the boundary
- length 0 $\overset{1:1}{\leftrightarrow}$ triangles with exactly two sides on the boundary

(another picture ...) □

18 Derived Equivalence Classification of Surface Algebras

Friday 18^{th} 8:30 – Matthew Pressland (Stuttgart, Germany)

(d'après Ladkani)

Aim.

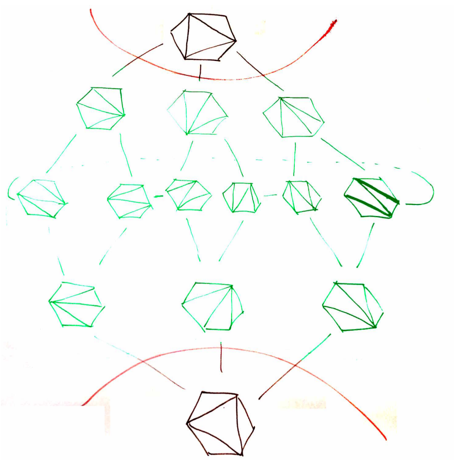
Classify surface algebras $A(\Gamma)$ up to derived equivalence.

Approach.

1) Separate non-equivalent algebras $\rightsquigarrow$ AG invariants

2) Exhibit derived equivalences $\rightsquigarrow$ good mutations

Example 18.1.

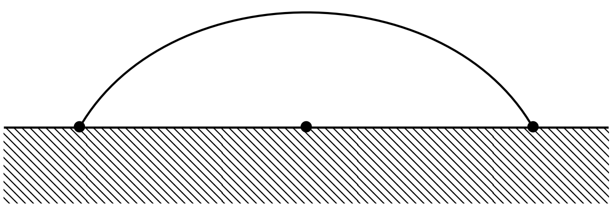


18.1 AG Invariants

Recall 18.2. *The AG invariant $\phi_{A(\Gamma)} : \mathbb{N}^2 \to \mathbb{N}$ is a function given by "path counting". In this case, computed by [David-Roesler–Schiffler].*

Let (S, M) be a surface with triangulation Γ.

Definition 18.3. *A* dome *in Γ is a triangle with two boundary arcs.*



Write d_C for the number of domes incident with the boundary component C and set

$$n_C = \#(M \cap C) .$$

Parameters of Γ:

- g the genus of S,
- b the number of boundary components,
- (n_C, d_C) for each boundary component C.

The parameters determine (S, M) up to homeomorphism.

Proposition 18.4 (David-Roesler–Schiffler, Ladkani)**.**

$$\phi_{A(\Gamma)} = \sum_{C \text{boundary component}} \mathbb{1}_{(n_C - d_C, n_C - 2d_C)} + t\mathbb{1}_{(0,3)}$$

where $t = 4(g-1) + 2b + \sum_C d_C$ *is the number of internal triangles of* Γ*.*

Since $n_C \neq d_C$ *for all* C*, the* AG *invariant* $\phi_{A(\Gamma)}$ *determines all the parameters.*

In particular, $A(\Gamma) \overset{\text{der.}}{\simeq} A(\Gamma')$ *means* Γ *and* Γ' *are triangulations of* the same *surface.*

18.2 Good Mutations

Recall 18.5. *Flipping an arc* v *of* Γ *induces a* mutation *of* $A(\Gamma)$ *to* $A(\mu_v(\Gamma))$.

Aim.

- Find *good* mutations such that $A(\Gamma) \overset{\text{der.}}{\simeq} A(\mu_v(\Gamma))$.
- Show that if Γ and Γ' have the same parameters $\left(\Leftrightarrow \phi_{A(\Gamma)} = \phi_{A(\Gamma')}\right)$, then they are linked by good mutations.

Definition 18.6. *Let* A *be an algebra. Then* $T^\bullet \in K^b(\operatorname{proj} A)$ *is a* tilting complex *if*

(i) $\operatorname{Hom}(T^\bullet, T^\bullet[i]) = 0$ *for all* $i \neq 0$*,*

(ii) $\operatorname{thick} T^\bullet = K^b(\operatorname{proj} A)$*.*

$$\overset{\text{[Rickard,Keller]}}{\Rightarrow} \quad A \overset{\text{der.}}{\simeq} \operatorname{End}(T^\bullet)^{\text{op}}$$

Example 18.7. Let T be a (classical) tilting module, i.e.

$$\operatorname{proj.dim} T \leq 1, \quad \operatorname{Ext}^1_A(T,T) = 0, \quad \exists\, 0 \to A \to T_0 \to T_1 \to 0 \text{ with } T_0, T_1 \in \operatorname{add} T.$$

Then [Brenner–Butler, Happel], $A \overset{\text{der.}}{\simeq} \operatorname{End}_A(T)^{\text{op}}$. Let $0 \to P_1 \to P_0 \to T \to 0$ be a projective resolution.

$\rightsquigarrow (\cdots 0 \to P_1 \to P_0 \to 0 \cdots) \in K^b(\operatorname{proj} A)$ is a tilting complex.

A vertex v of $A(\Gamma)$ determines complexes:

$$T_v^- = P_v \xrightarrow{(\cdot a)} \bigoplus_{a: j \to v} P_j \oplus \bigoplus_{i \neq v} P_i$$

$$T_v^+ = P_v \xrightarrow{(\cdot a)} \bigoplus_{a: v \to j} P_j \oplus \bigoplus_{i \neq v} P_i$$

Definition 18.8. *Say the mutation μ_v is* good *if T_k^ε is a tilting complex with*

$$\mathrm{End}_{A(\Gamma)}(T_k^\varepsilon) \overset{Morita}{\simeq} A(\mu_v(\Gamma))$$

for some $\varepsilon \in \{+, -\}$.

$$\Rightarrow \qquad A(\Gamma) \overset{\text{der.}}{\simeq} A(\mu_v(\Gamma))$$

Example 18.9. v a sink $\rightsquigarrow T_k^-$ tilting; v a source $\rightsquigarrow T_k^+$ tilting.

Proposition 18.10 (Ladkani)**.** *If $\mu_v(\Gamma)$ and Γ have the same parameters, then μ_v is good.*

Proof. The number of arrows in $A(\Gamma)$

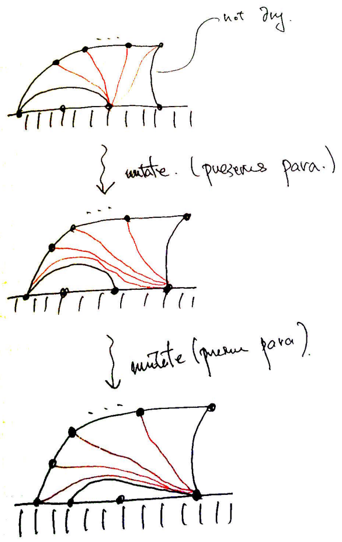
$$12(g-1) + 6b + \sum_C (n_C + d_C)$$

can be recovered from the parameters. [Ladkani] showed previously (with computer assistance) that mutations preserving the number of arrows are good. □

Theorem 18.11 (Ladkani)**.** *If Γ and Γ' have the same parameters, then $A(\Gamma) \overset{\text{der.}}{\simeq} A(\Gamma')$.*

Proof. Since they have the same parameters, Γ and Γ' are both triangulations of one surface (S, M).

Step 1: Adjust spacing of domes of Γ to match Γ'.
Idea:



Repeat this. $\rightsquigarrow$ There is an automorphism of (S, M) taking domes of Γ to those of Γ'.

Step 2: Apply this automorphism.

Step 3: Γ and Γ' have the same domes. We want a sequence of good mutations $\Gamma \rightsquigarrow \Gamma'$. Use a combinatorial recipe of [Mosher].
Idea: Pick an arc $a \in \Gamma' \setminus \Gamma$, orient it arbitrarily. Flip first arc of Γ that a intersects.

Observation: We can choose a carefully so that we never create or destroy domes:

(1) a cannot intersect an arc of a dome since Γ and Γ' have the same domes.

(2) To avoid creation of domes: (picture)

$\square$

Example 18.12. In Example 18.1 the green part corresponds to different orientations of $\mathbb{A}_3$:

$$\mathrm{gl.dim} = 1 \quad \text{and} \quad \phi_{A(\Gamma)} = \mathbb{1}_{(4,2)}$$

For the red part:

$$\mathrm{gl.dim} = \infty \quad \text{and} \quad \phi_{A(\Gamma)} = \mathbb{1}_{(3,0)} + \mathbb{1}_{(0,3)}$$