BIREP SUMMER SCHOOL ON GENTLE ALGEBRAS

BAD DRIBURG, AUGUST 2017

1. Overview: Gentle algebras

We can find gentle algebras in many places in mathematics. The study of gentle algebras was initiated by Assem and Skowroński [6] (who introduced them in order to study iterated tilted algebras of type A) and became an active area of research in representation theory in the recent years. Gentle algebras form an interesting class of examples because of the simple combinatorial definition and because of many connections to other classes of algebras such as string algebras, biserial algebras, special biserial algebras, tame algebras, Gorenstein algebras, Koszul algebras, Brauer graph algebras, and surface algebras.

One goal of the summer school is to understand the classification of the indecomposable modules over a gentle algebra up to isomorphism. As every gentle algebra is a string algebra, we can describe the indecomposable modules combinatorially by string modules and band modules. Especially, every gentle algebra is tame. One can extend the classification of indecomposable modules to the larger class of biserial algebras (introduced by Tachikawa [57]). During the school we wish to study the relationship between gentle algebras, special biserial algebras, and biserial algebras. In particular, a theorem of Ringel [47] and Schröer [51] asserts that an algebra is gentle if and only if its repetitive algebra is special biserial. A theorem of Skowroński and Waschbüsch [55] states that a representation-finite biserial algebra is always special biserial.

The second goal of the summer school is to learn about derived categories of gentle algebras. Schröer-Zimmermann's theorem [52] asserts that the class of gentle algebras is closed under derived equivalence. A classification of the indecomposable objects in the derived category of a gentle tree algebra is due to Assem and Happel [5]. On the other hand, gentle one-cycle algebras are related to Voßieck's derived discrete algebras [59]. On a related note, Avella-Alaminos and Geiß [7] constructed a function which turns out to be a derived invariant. The singularity category of a gentle algebra was described by Kalck [34].

The third goal of the summer school is to understand how gentle algebras arise from surfaces. To a triangulation of a Riemannian surface together with marked points Fomin–Shapiro–Thurston [25] (generalizing work of Fock–Goncharov [23, 24] and Gekhtman–Shapiro–Vainshtein [29]) assigned a quiver such that a flip in the triangulation corresponds to a so-called mutation of the quiver. In addition, Labardini-Fragoso [40] associated with this quiver a potential, from which we obtain a corresponding Jacobian algebra. If the marked points are only on the boundary of the surface, then these algebras are gentle due to a result of Assem–Brüstle–Charbonneau-Jodoin–Plamondon [4]. Ladkani [42] showed that certain invariants determine their derived equivalence classes. Later David-Roesler–Schiffler [18] introduced surface cut algebras. These algebras can be regarded as the degree-zero part of the Jacobian algebra with respect to a grading that is defined in terms of a cut [2]. The authors proved that an admissible surface cut algebra is always gentle and of global dimension at most 2.

2. Gentle algebras and their relations to biserial, special biserial and string algebras

Talk 1 (M. Kaniecki)**.** Introduction to gentle algebras, string algebras, special biserial algebras and biserial algebras

Abstract: Give definitions and explain why every gentle algebra is a string algebra, every string algebra is a special biserial algebra and every special biserial algebra is a biserial algebra. A possible source is the overview article by Schröer [50]. Illustrate the definitions by appropriate examples, see for example Külshammer [38].

Talk 2 (P. Lampe)**.** The representation theory of the Lorentz group (after Gelfand and Ponomarev)

Abstract: We study indecomposable representations of the Lorentz group following Gelfand and Ponomarev [30]. At first we introduce the Lorentz group as the group of symmetries in special relativity. It is a non-compact real Lie group so that it is not possible to classify all indecomposable modules using classical theory. Instead, we study Harish-Chandra modules which are defined by a finiteness condition with respect to the maximal compact subgroup of space rotations. Then we discuss Gelfand–Ponomarev's classification of Harish-Chandra modules who reduce the problem to the classification of nilpotent modules over the algebras $k[x]$ and $k[x,y]/(xy)$. Gelfand–Ponomarev's techniques lead to the notion of string algebras.

Talk 3 (A. Gottwald)**.** The classification of indecomposable modules over special biserial algebras and string algebras

Abstract: We describe the classification of indecomposable modules over special biserial algebras by strings, bands and non-uniserial projective-injective modules by Butler–Ringel [13] and Wald–Waschbüsch [60]. Also the classification of indecomposable modules over string algebras, initiated by Gelfand–Ponomarev [30] and Ringel [46], finalized by Donovan–Freislich [20], Wald–Waschbüsch [60] and Butler–Ringel [13] via the functorial filtration. A possible way to proceed is to follow Erdmann [21, II.1-II.3]; details are explained well in Pyro's thesis [45].

Talk 4 (Ö. Eiriksson)**.** Irreducible morphisms for string and band modules

Abstract: The talk concerns morphisms between string and band modules as studied by Butler–Ringel [13], Crawley-Boevey [16] and Krause [36]. A possible way to proceed is to follow Erdmann who considers irreducible morphisms for band modules [21, II.4] via the functors from the previous talk and irreducible morphisms for string modules [21, II.5] using hooks and cohooks. Moreover, we describe Auslander–Reiten sequences over string algebras [21, II.6].

Talk 5 (M. Flores Galicia)**.** The structure of biserial algebras

Abstract: The aim of the talk is to discuss Vila-Freyer–Crawley-Boevey's description of basic biserial algebras and to show that a representation-finite biserial algebra is special biserial. For the first part we follow [58], see also [39]. For the second part, the speaker could try to apply the results of the first part or proceed as in Skowroński–Waschbüsch [55]: Recall that special biserial algebras are biserial, and state the corollary in [55, p.4]. Briefly introduce distributive algebras and state the necessary properties. Explain that representation-finite algebras are distributive (cf. Jans [33]). Then go on to state and prove Lemma 2. Time permitting define $\alpha(A)$ and $\beta(A)$, as well as primitive V-sequences. Then one can state Theorem 1 and its Corollary.

Talk 6 (omitted)**.** Biserial algebras are tame

Abstract: Crawley-Boevey's theorem [15] asserts that biserial algebras are always tame. We wish to understand the proof which features arguments from algebraic geometry and deformation theory.

Talk 7 (J. McMahon)**.** The repetitive algebra of a gentle algebra is special biserial

Abstract: We define the repetitive algebra $\hat{A}$ of a finite-dimensional algebra A as introduced in [32, § 2] and briefly state how it is related to the trivial extension $T(A)$ (cf. [56, Remark 3.10] and [31, II 2.4]). If $A = KQ/\langle\rho\rangle$ is given by a quiver with zero and commutativity relations, we describe the repetitive quiver $\hat{Q}$ and relations $\hat{\rho}$ and show that $\hat{A} \cong K\hat{Q}/\langle\hat{\rho}\rangle$. Possible sources are [51] and (for gentle A) [44, 47]. Following [47, § 4] and [51, § 4] we finally prove that A is gentle if and only if $\hat{A}$ (equivalently $T(A)$ according to [44]) is special biserial.

Talk 8 (omitted)**.** The representation dimension of a special biserial algebra is at most 3

Abstract: The talk concerns Erdmann–Holm–Iyama–Schröer's result [22] that the representation dimension of a special biserial algebra is bounded above by 3. Begin with an introduction to Auslander's representation dimension, see for example Oppermann [43] or Ringel [49]. Our result about special biserial algebras is Corollary 1.3 of the main Theorem 1.1 in [22]. The crucial step in the proof of Theorem 1.1 concerns the global dimension of the endomorphism algebra of a module M. In the situation of Corollary 1.3, such a module M can be chosen to have a special form, see the remark after the proof of the corollary. It might be possible to shorten the proof by computing the global dimension of $\mathrm{End}(M)$ in this situation explicitly. Otherwise give an overview about the proof in [22]; another reference is Ringel [48].

Talk 9 (W. Gnedin)**.** Symmetric special biserial algebras are Brauer graph algebras

Abstract: Brauer *tree* algebras naturally occur in modular representation theory as blocks with cyclic defect groups (see [1, § 17]). Firstly, we give the definition of Brauer graph algebras and observe that they are symmetric and special biserial. An accessible reference could be [54]. We present Schroll's [53, § 2.3] construction of the Brauer graph G_A of a symmetric special biserial algebra A and then prove that A is isomorphic to the Brauer graph algebra defined by G_A. In particular, the trivial extension of a gentle algebra is a Brauer graph algebra. Its underlying Brauer graph can be explicitly described as done in [54, § 3.1].

3. Derived equivalences and derived invariants

Talk 10 (K. Jacobsen)**.** Introduction to triangulated categories

Abstract: Give an introduction to triangulated categories, for example following Happel [31]. As examples of triangulated categories, we consider derived categories and stable categories.

Talk 11 (G. Bocca)**.** Happel's functor

Abstract: The talk concerns Happel's functor. Let Λ be a finite-dimensional algebra. Happel's functor is a full and faithful triangle functor $H\colon D^b(\operatorname{mod}\Lambda) \to \underline{\operatorname{mod}}\,\widehat{\Lambda}$, where $\widehat{\Lambda}$ is the repetitive algebra of Λ. There are several different constructions of the functor due to Happel [31], Keller–Voßieck [35] and Barot–Mendoza [8].

Talk 12 (S. Opper)**.** Classification of indecomposable objects in the derived category of a gentle algebra

Abstract: Describe the indecomposable objects in the derived category of Arnesen–Laking–Pauksztello's *running example* [3]. It is possible to classify the indecomposable objects in the derived category of every gentle algebra, as conducted by Bekkert–Merklen [9], but for simplicity this talk restricts to this particular example.

Talk 13 (F. Yuliawan)**.** The class of gentle algebras is closed under derived equivalence

Abstract: The goal of this talk is to show that every finite-dimensional algebra that is derived equivalent to a gentle algebra A is itself gentle. The Happel functor, Rickard's theorem (compare e.g. [37, Chapter 3]), and the special biseriality of the repetitive algebra $\hat{A}$ are used to reduce this problem to the verification of the following statement: The stable endomorphism algebra of a module without self-extensions over a special biserial algebra is gentle. We sketch the steps of the (technical) proof of this fact given by Schröer–Zimmermann [52], perhaps illustrating each of them with an example.

Talk 14 (N. Berkouk)**.** Combinatorial derived invariants of gentle algebras

Abstract: The talk discusses the Avella-Alaminos–Geiß invariant [7]. We want to understand the combinatorial algorithm defining the invariant, illustrated by suitable examples, and why it is a derived invariant. The speaker is encouraged to implement the algorithm.

Talk 15 (T. Aoki)**.** Derived discrete algebras

Abstract: Introduce Voßieck's [59] derived discrete algebras. Show that they are always derived equivalent to gentle one-cycle algebras, see Voßieck [59]. An explicit list of these algebras up to derived equivalence was obtained by Bobiński–Geiß–Skowroński [12].

Talk 16 (omitted)**.** Classification of the indecomposable objects in the derived category of a gentle tree algebra

Abstract: Explain Assem–Happel's derived equivalence classification of gentle tree algebras [5]. Give a short survey about the generalizations by Assem–Skowroński [6] to gentle one-cycle algebras and by Bobiński [11] to gentle two-cycle algebras.

Talk 17 (D. Pauksztello)**.** Singularity categories of gentle algebras

Abstract: Introduce and motivate Gorenstein algebras. Prove the theorem of Geiß–Reiten [28], according to which every gentle algebra is Gorenstein. Note that Chen–Shen–Zhou [17] give a similar proof of a more general statement. Introduce the singularity category and state Buchweitz's result concerning the relation to the stable category of Gorenstein projective R-modules. Present Kalck's description [34] of the singularity category of a gentle algebra.

4. Algebras from surface triangulations

Talk 18 (T. Yurikusa)**.** Quivers with potential from surface triangulations

Abstract: We restrict to the surfaces (S, M) with marked points M on the boundary and ideal triangulations of S. Following Labardini-Fragoso [40] one defines the quiver with potential (QP) associated to an ideal triangulation Γ of (S, M). In this case the Jacobian algebras are finite-dimensional and it is not necessary to complete the path algebra, see [41, Thm 3.2]. The mutations of QPs associated to flips in the triangulation should be explained (in examples). Then, the uniqueness result of the non-degenerate potential should be explained, see Labardini-Fragoso [41, Thm 4.1] or Geiß–Labardini-Fragoso–Schröer [27].

Talk 19 (A. Garver)**.** Gentleness and cluster tilted algebras of type A

Abstract: We follow Assem–Brüstle–Charbonneau-Jodoin–Plamondon [4]. This talk should explain Theorem 2.7, Theorem 3.3 and Theorem 4.1. The definition of $A(\Gamma)$ is already given in the previous talk. As a corollary of the gentleness one obtains that $A(\Gamma)$ has tame representation type.

Talk 20 (R. Coelho Simões)**.** Surface algebras

Abstract: Use cuts in a triangulation to define a grading on $A(\Gamma)$ and define a surface algebra arising from a cut as the subalgebra of degree zero. References are David-Roesler–Schiffler [18] and Amiot–Grimeland [2]. Prove that surface cut algebras are gentle and calculate their Avella-Alaminos–Geiß invariants.

Talk 21 (M. Pressland)**.** Derived equivalence classification of gentle algebras arising from surfaces

Abstract: The talk follows Ladkani [42]. It deals with the classification up to derived equivalence of the algebras $A(\Gamma)$ associated to an ideal triangulation of a surface with marked points on the boundary. First the notion of a good mutation needs to be introduced (Definition 1), AG-invariants need to be recalled (from an earlier talk), and Brenner–Butler tilts and cotilts need a short introduction. Explain Theorem A and Proposition 1. Give an example how the algorithm works.

5. Other talks

Talk 22 (omitted)**.** A gentle algebra is Koszul

Abstract: Koszul duality is a concept which originated in the study of quadratic algebras and occurs in various branches of pure mathematics. The talk concerns the theorem of Bessenrodt–Holm [10] according to which every gentle algebra is Koszul with respect to the grading given by the length of paths. Moreover, Bessenrodt–Holm describe the Koszul dual algebra. The speaker should give a short introduction to Koszul algebras and Koszul duality and then state Bessenrodt–Holm's results.

Talk 23 (A. Garcia Elsener)**.** The characterization of gentle 2-Calabi–Yau tilted algebras

Abstract: We prove that all gentle 2-Calabi-Yau tilted algebras (over an algebraically closed field of characteristic different from 3) are Jacobian, moreover their bound quiver can be obtained via block decomposition. To do this, we go over a property that characterizes the indecomposable syzygies over 2-CY tilted algebras, and we need some of the content on Talk 17: Kalck's description of the singularity category and Geiß–Reiten's theorem that all gentle algebras are Gorenstein. We follow [26].

6. On the Classification of Representations of Special Biserial Algebras

Christof Geiß will give a series of three talks on that topic.

Abstract: In this series of talks we outline how the matrix reduction methods, developed by the Kiev school of representation theory, allow in a very efficient way to classify the indecomposable representations of a special biserial algebra. This is essentially a special case of Bangming Deng's 2000 Ph.D. thesis [19]. In fact, Bangming Deng used those methods for the more difficult problem of classifying the indecomposable representations of clans. It seems that this method is more convenient than the previously used functorial filtrations.

If time permits, we will also discuss how the same methods allow to describe explicitly the indecomposable, right bounded, complexes in the derived category of a gentle algebra of possibly infinite global dimensions. This is based on work of Burban and Drozd [14].

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