WHAT SHOULD A NON-COMMUTATIVE RESOLUTION OF SINGULARITIES BE, AND WHY SHOULD IT INVOLVE MCM MODULES?

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ABSTRACT. These lectures will describe the role of maximal Cohen-Macaulay modules over Cohen-Macaulay rings in the theory of non-commutative resolutions of singularities.

A lot of the background will (probably?) be covered in the lectures by participants in the summer school; this includes things like orders, AR theory, MCM approximations, finite CM type, and the McKay Correspondence (a key motivating example).

Note to the reader: These are the notes I prepared for my 3 talks at the Bad Driburg Summer School in the summer of 2019. The topic of the summer school was Maximal Cohen-Macaulay modules, and I knew that I wanted to talk about the connections between MCM modules and NCCRs, but I wasn't sure about exactly what I wanted to say, or what would be said before me by participants in the School. Therefore I prepared a lot more material than would fit in three lectures. The talks I actually gave were roughly:

(1) Working toward a first definition (Part 1) and preliminary improvements (the first couple of pages of Part 2) via the properties "Gorenstein", "symmetric", and "nonsingular"
(2) Orders, and the relationships between various combinations of these properties; ending with Van den Bergh's definition of NCCR (Part 2)
(3) Two examples and the non-existence of "strong" NCCRs (Part 3), plus connections with algebraic geometry (most of Part 5)

I've gone through the notes and added some marginal comments, as well as fixing a few of the typos, but I haven't rewritten them to correspond exactly to what I said in my lectures.

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Many thanks to the organizers for doing me the honor of inviting me to this summer school.

My goal in these lectures is to explain and motivate the notion of "non-commutative resolution of singularities". I will emphasize the role of maximal Cohen-Macaulay modules over Cohen-Macaulay rings, i.e. Cohen-Macaulay representation theory.

Part 1. Working toward a first definition

One might ask for a representation-theoretic notion of resolution of singularities. In modern algebra, we often treat two rings (algebras, etc.) as "the same" if they have the same module category, that is, they're Morita-equivalent. Even more modernly, we sometimes don't distinguish between rings that have the same derived category of modules.

From this point of view, what should a resolution of singularities look like? We should frame our answer in terms of $\operatorname{mod} R$ and $\operatorname{coh} X$, or even in terms of $D^b(\operatorname{mod} R)$ and $D^b(\operatorname{coh} X)$.

For any ring Λ, we have $\operatorname{mod} \Lambda \simeq \operatorname{mod} M_n(\Lambda)$ by Morita equivalence. We take this to mean that $\operatorname{mod}(-)$ "can't tell Morita-equivalent rings apart", so in particular it's blind to commutativity.

Similarly, $D^b(\operatorname{mod}(-))$ is blind to commutativity as well. So any "categorical" desingularization will automatically be a "non-commutative" desingularization.

There is a general definition of categorical desingularization:

Definition (Bondal-Orlov)**.** A categorical desingularization of a triangulated category D is an abelian category C of finite global dimension, and a triangulated subcategory $K \subseteq D^b(C)$, closed under direct summands, such that

$$D \simeq D^b(C)/K\,.$$

This is too general to do much with, though it does have the advantage of always existing (given Hironaka):

Theorem (Kuznetsov-Lunts)**.** *The derived category of any singularity over a field of characteristic 0 has a categorical resolution in this sense*

That was an OK answer, but I would prefer that it was more concrete, so we should insist that the abelian category C is $\operatorname{mod}\Lambda$ for some ring Λ.

Recall that a usual ("geometric") resolution of singularities is a map $\pi\colon \widetilde{X} \longrightarrow X$ such that

(1) $\widetilde{X}$ is non-singular ("smooth" – I won't worry about the distinction between regular and smooth points)
(2) π is proper, in particular surjective
(3) π is birational

Since the dictionary between algebra and geometry reverses all the arrows, it's reasonable to think that $\widetilde{X} \longrightarrow X$ should turn into a ring homomorphism in the other direction.

In my lecture I started out with a homomorphism $R \longrightarrow S$ where S is a commutative ring, and pointed out that there are easy examples of rings R that have no module-finite birational extensions which are regular rings, so we are forced to allow $S = \Lambda$ to be noncommutative.

How do the three properties in the definition of a resolution of singularities translate for a ring homomorphism $R \longrightarrow \Lambda$, where R is a commutative ring and Λ is not necessarily commutative (but R maps into $Z(\Lambda)$)?

(iii) $\pi\colon \widetilde{X} \longrightarrow X$ is birational. This essentially means that π induces an isomorphism on quotient fields. Morally, this means

$$\mathcal{O}_{\widetilde{X}} \otimes_{\mathcal{O}_X} K = K$$

where K is the quotient field of $\mathcal{O}_X$.

Since we only care about module categories, we should weaken this equality by replacing it with "Morita equivalent to" (or even "derived equivalent to"). But the only rings Morita equivalent to a field K are the matrix rings $M_n(K)$. So we ask

$$\boxed{\Lambda \otimes_{\mathcal{O}_X} K \cong M_n(K)\,.}$$

Call Λ birational (over R) if this is true.

If we were to replace "Morita" with "derived", we would arrive at

$$\boxed{\Lambda \otimes_{\mathcal{O}_X} K \text{ is semisimple}\,.}$$

As far as I know, this possibility has not gotten any attention. It would have interesting connections with the classical theory of orders, which I will mention later.

(ii) π is proper. If we expect to have the boxed isomorphism above, then we should ask that

$$\boxed{\Lambda \text{ is a finitely generated } R\text{-module.}}$$

Since finite maps of algebraic varieties are proper, this is at least as strong as (ii).

(i) $\widetilde{X}$ is smooth. Clearly we should (at least) insist that

$$\boxed{\Lambda \text{ has finite global dimension.}}$$

(There are other notions of smooth that could be used here. For example, one might ask that Λ have finite projective dimension over the enveloping algebra $\Lambda^{\mathsf{op}} \otimes_R \Lambda$.)

So we arrive at a first definition.

Weakest possible definition: Let R be a commutative Noetherian ring. A very weak non-commutative resolution of R (or of $\operatorname{Spec} R$) is a module-finite algebra Λ which is birational and has finite global dimension.

There are several motivating examples. Here are two. These particular two are chosen because they both predate the notion of a non-commutative resolution of singularities.

Example (The McKay Correspondence[1]). Let $S = k[\![x_1, \ldots, x_n]\!]$ for some $n \geqslant 2$ (or the localized polynomial ring), and let $G \subset \mathrm{GL}_n(k)$ be a finite group with order invertible in k. Then G acts by linear changes of variables on S; set $R = S^G$, the ring of invariants.

Technical assumption: we assume that G contains no pseudo-reflections (elements fixing a hyperplane pointwise) but the identity. We say G is small.

Then R is a (complete) normal domain of dimension n; it is Cohen-Macaulay always and (since G is small) R is Gorenstein iff $G \subset \mathrm{SL}_n(k)$. The ring S is a MCM R-module of rank $|G|$.

The twisted group algebra $S\#G$ is like the usual group ring $S[G]$, but with "twisted" multiplication.

Definition. The skew group ring (twisted group ring, crossed product ring, ...) $S\#G$ is

- the free S-module on $\{\sigma\}_{\sigma \in G}$
- multiplication defined by $(s\sigma)(t\tau) = s\sigma(t)\sigma\tau$ (twisted!)

Observe that a (left) $S\#G$ module is nothing but an S-module with a compatible action of G. An easy computation shows that

$$\mathrm{Hom}_{S\#G}(-,-) = \mathrm{Hom}_S(-,-)^G$$

on $S\#G$-modules. Since we made sure $|G| \in k^\times$, taking G-invariants is exact, so

$$\mathrm{Ext}^i_{S\#G}(-,-) = \mathrm{Ext}^i_S(-,-)^G$$

for all i.

In particular

(1) An $S\#G$-module is projective iff it is projective (=free) as an S-module.
(2) $\mathrm{gldim}\, S\#G = n = \dim S$. (Above gives $\leqslant$, Koszul complex on variables gives $\geqslant$.)

Theorem (Auslander 1962). *The natural map $\gamma \colon S\#G \longrightarrow \mathrm{End}_R(S)$, sending $s\sigma$ to the R-linear endomorphism $s\sigma$, is an isomorphism (because G is small).*

[1] How much I say here depends a lot on what is covered in the summer school prior to my talk.

Sarah stated this theorem, but didn't mention the part about $S\#G$ having finite global dimension.

Corollary. *R has an algebra $E = \mathrm{End}_R(S)$ which is of finite global dimension and is birational.*

In fact E is much better than that: as an R-module, E is isomorphic to a direct sum of copies of S, so is a MCM R-module. Also, its global dimension is not just finite, it is the smallest possible value $\dim R$.

There's a lot more to say here.

For example, in dimension 2 we have $\mathrm{MCM}(R) = \mathrm{add}_R(S)$ by a result of J. Herzog, as mentioned in Sarah's talk.

In fact, the non-commutative desingularizations provided by the McKay Correspondence are just about the nicest ones, and the analogy with algebraic geometry works the best of any examples we know.

The endomorphism ring E "knows about" the singularities of $\mathrm{Spec}\, R$, in the following sense. It is known that $\mathrm{Spec}\, R$ has a unique minimal resolution of singularities $\pi \colon Y \longrightarrow X$ with exceptional fiber a bunch of $\mathbb{P}^1$'s [Du Val].

Artin-Verdier: There is a one-one correspondence $\mathrm{ind}\, \underline{\mathrm{MCM}}(R) \leftrightarrow$ exceptional curves (this means omit the regular module R) sending M to the unique $\mathbb{P}^1$ intersecting $c_1(M)$, and this induces an isomorphism between the stable AR quiver (doubled) and the dual graph of the resolution.

The result of Artin-Verdier has a "derived version". To state it, we need to know that the minimal resolution of $\mathrm{Spec}\, R$ can be realized as Nakamura's G-Hilbert scheme $H = \mathrm{Hilb}^G(\mathbb{C}^2) = \{I \subset S \mid S/I \cong \mathbb{C}G\}$.

Kapranov-Vasserot: There is an equivalence

$$D^b(\mathrm{coh}\, H) \simeq D^b_G(\mathrm{coh}\, \mathbb{C}^2)\,,$$

where the thing on the right is the bounded derived category of G-equivariant coherent sheaves on $\mathbb{C}^2$. But that's nothing but the bounded derived category of fin.gen. $S\#G$-modules, $D^b(\operatorname{mod} S\#G) \simeq D^b(\operatorname{mod} E)$.

So this example actually achieves our goal of a "representation-theoretic" resolution of singularities! (In some cases.)

Q: Can one extend Kapranov-Vasserot to $n \geqslant 3$?

- For $n \geqslant 3$, $\operatorname{Hilb}^G(\mathbb{C}^n)$ is no longer the minimal resolution.
- For $n \geqslant 4$, it is not even smooth.

However,

Bridgeland-King-Reid: For $G \subset \mathrm{SL}_3(\mathbb{C})$, there are equivalences

$$D^b(\operatorname{coh} \operatorname{Hilb}^G(\mathbb{C}^3)) \simeq D^b_G(\operatorname{coh} \mathbb{C}^3) \simeq D^b(\operatorname{mod} S\#G)$$

So even though S no longer contains all the MCMs, $E = \operatorname{End}_R(S)$ still "is" the resolution.

Conjecture (BKR, Nakamura, Douglas). $G \subset \mathrm{SL}_n(\mathbb{C})$. *If $Y \longrightarrow \mathbb{C}^n/G = \operatorname{Spec} R$ is a crepant resolution of singularities, $\pi^*\omega_X = \omega_Y$, then*

$$D^b(\operatorname{coh} Y) \simeq D^b_G(\mathbb{C}^n)\,.$$

In particular $D^b(\operatorname{coh} Y) \simeq D^b(\operatorname{mod} \operatorname{End}_R(S))$ is independent of Y!

Example (Finite CM type[2]). Let $(R, \mathfrak{m})$ be a CM local ring of finite CM type and dimension $d \geqslant 2$, and let $R = M_0, M_1, \ldots, M_n$ be a complete list of representatives for the isomorphism classes of indecomposable MCM R-modules. Set $G = M_0 \oplus \cdots \oplus M_n$; we call G a representation generator for ($\mathrm{MCM}(R)$ or) R. Let $\Lambda = \operatorname{End}_R(G)$. Sometimes Λ is called an Auslander algebra for R.

I claim that Λ is a birational R-algebra of finite global dimension. Since $\dim R \geqslant 2$, we know that R is a normal domain (it's an isolated singularity by Auslander's theorem

This wasn't mentioned before my talk, so I said a few words about it.

[2]Again, some of this may be covered in the summer-school talk before mine.

) and G is torsion-free. This gives birationality.

The proof that $\operatorname{gldim}\Lambda < \infty$ is essentially due to Auslander, who proved it for Artin algebras [1971]. Here it is. Let X be a Λ-module, with projective resolution

$$P_{d-1} \xrightarrow{\varphi_{d-1}} P_{d-2} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\varphi_1} P_0 \longrightarrow X \longrightarrow 0\,. \tag{1}$$

By the principle of "projectivization", the projective $\operatorname{End}_R(G)$-modules P_i are of the form $\operatorname{Hom}_R(G, G_i)$ for $G_i \in \operatorname{add}(G) = \operatorname{MCM}(R)$. So there is a sequence of maps of R-modules

$$G_{d-1} \xrightarrow{f_{d-1}} G_{d-2} \longrightarrow \cdots \longrightarrow G_1 \xrightarrow{f_1} G_0 \tag{2}$$

so that applying $\operatorname{Hom}_R(G, -)$ returns the exact sequence $P_{d-1} \longrightarrow \cdots \longrightarrow P_0$. Note, importantly, that G has a direct summand isomorphic to R; this implies that (2) is <u>also exact</u>.

Let $G_d = \ker\left(G_{d-1} \xrightarrow{f_{d-1}} G_{d-2}\right)$. Then G_d is also MCM by the Depth Lemma, so is in $\operatorname{add}(G)$. Since $\operatorname{Hom}_R(G, -)$ is left-exact, applying it gives

$$0 \longrightarrow \operatorname{Hom}_R(G, G_d) \longrightarrow P_{d-1} \xrightarrow{\varphi_{d-1}} P_{d-2} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\varphi_1} P_0\,.$$

The cokernel of φ_1 is isomorphic to X and $\operatorname{Hom}_R(G, G_d)$ is projective, so this shows that $\operatorname{pd}_\Lambda X \leqslant d$.

One can be a little more careful and say something more precise [Iyama 2007, Quarles 2005]: The simple Λ-modules $S_0, \ldots, S_n$ are quotients of $\operatorname{Hom}_R(G, M_i)$ for $i = 0, \ldots, n$. One can show that in fact $\operatorname{pd}_\Lambda S_i = 2$ for all $i \neq 0$, while the simple S_0 corresponding to the R-summand has projective dimension d. It follows that $\operatorname{gldim}\Lambda = d$.

Corollary. *R has an algebra $\Lambda = \operatorname{End}_R(G)$ which is of finite global dimension and is birational.*

In this case Λ is not quite as nice as in the previous example. It has the smallest possible global dimension, but its simples have different projective dimensions if $d \geqslant 3$. Also, who knows whether it's an MCM R-module?

As these examples indicate, endomorphism rings are a useful source of non-commutative algebras. We'll come back to this.

Side note: The best case of this example is when $d = 2$, for several reasons. First, Λ is MCM in this case (we'll come back to this later). Second, all the simples have the same projective dimension. But third and most important, Auslander proved that the two-dimensional complete CM local $\mathbb{C}$-algebras are precisely the rings of invariants $\mathbb{C}[\![u, v]\!]^G$ from the previous example! So when $d = 2$ these two examples coincide.

Part 2. Improving the definition: nonsingular symmetric orders

Even though the examples above are encouraging, there are a number of problems with this definition. The main issue is that finite global dimension is a very weak, and badly-behaved, property for non-commutative rings. Among other problems,

- there is no analog of the Auslander-Buchsbaum formula, which in particular says that if M has finite projective dimension then $\mathrm{pd}_R(M)$ is bounded by the (Krull) dimension.
- finite global dimension does not imply any Gorenstein or Cohen-Macaulay property as in the commutative case.
- it does not localize well.

We will solve all of these by adding in three hypotheses on Λ and the R-structure of Λ.

First, we strengthen the property of finite global dimension so that it does localize well.

Definition. An R-algebra Λ is non-singular if $\mathrm{gldim}\,\Lambda_\mathfrak{p} = \dim R_\mathfrak{p}$ (Krull dimension) for every prime ideal $\mathfrak{p} \in \mathrm{Spec}\, R$.

This property was defined earlier in Biao's talk for orders. The definition he gave was $\mathrm{gldim}\,\Lambda =$. These are equivalent for orders, but not in general. See below.

This is inconvenient to check in practice, so we will be on the lookout for situations when finite global dimension automatically implies non-singularity.

In the commutative world, regular $\Longrightarrow$ Gorenstein $\Longrightarrow$ Cohen-Macaulay. That's no longer true for non-commutative rings. So we will add in Gorenstein and CM hypotheses.

From now on, we assume our base ring R is a Cohen-Macaulay local domain. This is relatively mild, and still includes many examples of interest, such as hypersurfaces, complete intersections, rings of invariants, determinantal rings. . . . It's a reasonable class of rings to restrict to. The "domain" part is to avoid technicalities: some results require R to be "equidimensional" and I don't want to talk about that.

We also assume that R has a canonical module ω_R. This is very mild.

The most important thing to know about canonical modules is that R is Gorenstein if and only if R itself is a canonical module. The canonical module is a generalization of the injective hull of the sum of the simples; it gives a similar duality theory for MCM modules.

Set $\omega_\Lambda = \mathrm{Hom}_R(\Lambda, \omega_R)$. It's a bimodule.

If Λ were a commutative ring, then it would be Gorenstein if and only if $\omega_\Lambda \cong \Lambda$. In the non-commutative world this name is given to a (very) slightly weaker property.

Definition. An R-algebra Λ is Gorenstein if ω_Λ is a projective left Λ-module.

A related property is that Λ be a symmetric algebra.

Definition. We say Λ is a symmetric R-algebra if $\mathrm{Hom}_R(\Lambda, R) \cong \Lambda$ as bimodules.

If R is Gorenstein, then clearly symmetric $\implies$ Gorenstein (but not conversely, because of the "as bimodules" part). In general, neither implies the other.

Finally we add in a Cohen-Macaulayness assumption. When S is a commutative module-finite R-algebra (and R is CM), it is the same to say that S is CM as a ring or as an R-module. So in the non-commutative setting we choose the latter.

Definition. An R-algebra Λ is an R-order if it is maximal Cohen-Macaulay as R-module, that is $\operatorname{depth}_R \Lambda = \dim R$. Equivalently, $\operatorname{Ext}^i_R(\Lambda, \omega_R) = 0$ for all $i > 0$.

Now, this is a central issue so I want to spend some time on it. Cohen-Macaulay modules are the topic of this workshop, but why is this an appropriate property to impose on Λ? What is the point of an order? Here are a few supporting observations.

(1) [Iyama-Wemyss 2010, they credit Auslander 1984] The following are equivalent for an order Λ over a CM local ring R with canonical module ω_R:
   (a) Λ is non-singular;
   (b) $\operatorname{gldim} \Lambda = \dim R$;
   (c) $\operatorname{gldim} \Lambda < \infty$ and Λ is a Gorenstein algebra (ω_Λ is projective);
   (d) Every Λ-module which is MCM over R is projective ($\operatorname{MCM}(\Lambda) = \operatorname{proj} \Lambda$).

   So for orders, we don't have to assume that the global dimension has the correct value locally: it's enough for it to have the correct value just once. And it's even enough for it just to be finite, as long as the order is Gorenstein.

   Biao proved (a) $\iff$ (d) in his talk.

(2) [Stangle 2016, generalizing Iyama-Reiten 2008] Orders of finite global dimension over CM local rings satisfy an inequality like the Auslander-Buchsbaum equality:

$$\dim R \leqslant \operatorname{pd}_\Lambda X + \operatorname{depth}_R X \leqslant \dim R + n\,,$$

   where n is the projective dimension of the right Λ-module ω_Λ. In particular, if Λ is a Gorenstein algebra, then we get the AB equality on the nose.

(3) [Van den Bergh 2004] If Λ is a non-singular order, then every simple Λ-module has the same projective dimension. (We say Λ is homologically homogeneous.)

So it really seems that non-singular orders are a good place to live: we get the Gorenstein property for free, the Auslander-Buchsbaum equality, a complete understanding of the MCM Λ-modules, and we rule out some pathologies.

Stronger definition: A (medium strength) non-commutative desingularization of a commutative Noetherian ring R is a non-singular, birational R-order Λ.

What about the examples? Let's check in.

Example (The McKay Correspondence)**.** The R-algebra $E = \mathrm{End}_R(S) \cong S\#G$ is always an order. It has the right global dimension, so it is non-singular. It's even Gorenstein, since $\mathrm{Hom}_R(E, \omega_R)$ is a MCM R-module, so projective by the result of Iyama-Wemyss.

So this one ticks all the boxes.

Example. When R has finite CM type, the Auslander algebra $\Lambda = \mathrm{End}_R(G)$ may or may not be an order.

For particular examples, first consider the (A_1) singularity in dimension 2: Let $R = k[\![x, y, z]\!]/(xy - z^2)$. Then R has only one non-free indecomposable MCM module up to isomorphism, $I = (x, z)$, and

$$\mathrm{End}_R(R \oplus I) \cong \begin{bmatrix} R & I \\ I* \cong I & R \end{bmatrix}$$

is a sum of MCM modules, so is an order.

Now consider the (A_1) singularity in dimension 3: let $R = k[\![x, y, u, v]\!]/(xy - uv)$. Then R has two non-free indecomposable MCM modules up to isomorphism: $\mathfrak{p} = (x, u)$ and $\mathfrak{q} = (x, v)$. As R-modules, we have

$$\mathrm{End}_R(R \oplus \mathfrak{p} \oplus \mathfrak{q}) \cong \begin{bmatrix} R & \mathfrak{p} & \mathfrak{q} \\ \mathfrak{q} & R & \mathrm{Hom}(\mathfrak{p}, \mathfrak{q}) \\ \mathfrak{p} & \mathrm{Hom}(\mathfrak{q}, \mathfrak{p}) & R \end{bmatrix}$$

(Or maybe the transpose?)

The module $\mathrm{Hom}(\mathfrak{p}, \mathfrak{q})$ is isomorphic to the fractional ideal $(x, v, \frac{v}{x})$, which has depth 2. So this is not an order.

Depths of Hom modules

It's worth investigating the depth of Hom and End modules a little more carefully.

I omitted this section, since I didn't end up talking about cluster tilting at all.

Unfortunately, there is no useful relationship in general between M having good depth and $\mathrm{End}_R(M)$ having good depth. We always have $\operatorname{depth} \mathrm{End}_R(M) \geqslant 2$ as long as $\operatorname{depth} M \geqslant 2$.

This is because applying $\mathrm{Hom}_R(-, M)$ to a free presentation $R^n \longrightarrow R^m \longrightarrow M \longrightarrow 0$ gives an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, M) \longrightarrow M^m \longrightarrow M^n$$

and the Depth Lemma says that the depth of the kernel is at least $\min\{2, \operatorname{depth} M\}$.

Other than that anything can happen: M can be MCM and Λ have low depth, or vice versa.

In practice, M will often have a free summand. When this happens, M is a direct summand of $\mathrm{End}_R(M)$, so we have $\operatorname{depth} \mathrm{End}_R(M) \leqslant \operatorname{depth} M$. So that's at least something.

There is one useful sufficient condition for $\mathrm{Hom}_R(M, N)$ to be MCM, the proof of which is essentially the same as the remark above.

Proposition. *Let M, N be MCM modules over a CM local ring R. If $\mathrm{Ext}^i_R(M, N) = 0$ for $i = 1, \ldots, \dim R - 2$, then $\mathrm{Hom}_R(M, N)$ is MCM.*

If R is an isolated singularity then the converse is true: $\mathrm{Hom}_R(M, N)$ is MCM if and only if $\mathrm{Ext}^{1,\ldots,d-2}_R(M, N) = 0$.

In particular this partly explains the behavior of the example above: If R is a hypersurface ring, then $\operatorname{Ext}^2(M, M) \cong \underline{\operatorname{End}}_R(M)$, the stable endomorphism ring of M. If M is non-free then that is non-zero. So once the dimension of R is at least 4, $\operatorname{End}_R(M)$ has no chance to be MCM. In the particular example of the (A_1) singularity in dimension 3, we have $\mathfrak{q} \cong \Omega\mathfrak{p}$, so $\operatorname{Ext}^1(\mathfrak{p}, \mathfrak{q}) \cong \operatorname{Ext}^2(\mathfrak{q}, \mathfrak{q}) \neq 0$ and so $\operatorname{Hom}(\mathfrak{p}, \mathfrak{q})$ is not MCM.

The strongest combination we could ask for, in terms of the words defined so far, is that Λ is a symmetric non-singular order. This turns out to have intimate connections with the classical theory of orders. (Unfortunately orders $\neq$ orders.) As an aside, here are some comments about that.

I neglected to mention in these notes or in my lectures that endomorphism rings are automatically symmetric. This is a result of Auslander (and Goldman?).

Recall from classical commutative algebra that if R is a domain with quotient field K, then:

- A classical order over R is a module-finite algebra Λ contained in a semisimple K-algebra D, with $\Lambda \otimes_R K = D$.
- Such a thing is a maximal order if it is maximal under inclusion.

Yuta defined orders for $R = k[\![x]\!]$ and $K = k(\!(x)\!)$. In that case non-singular means hereditary.

Maximal orders are the best-behaved ones, and are well-studied over, for instance, Dedekind domains.

Proposition. *Let R be a normal domain with quotient field K. Let Λ be a symmetric, non-singular R-order contained in $M_n(K)$ and such that $\Lambda \otimes_R K = M_n(K)$. Then Λ is a maximal order in $M_n(K)$.*

Proof. Since Λ is reflexive over R, you can check maximality at the height-one primes. So assume that R is a DVR. Then Λ, being non-singular, is a hereditary ring, so is Morita equivalent to a product of rings $T_m(\Delta)$, where Δ is the unique maximal order in $M_n(K)$ and $T_m(K)$ is the ring of matrices $(a_{ij}) \in M_n(\Delta)$ with $a_{ij} \in \operatorname{rad}\Delta$ for all $i > j$.

But such a product of rings is symmetric only if $m = 1$ and there is only one factor. So $\Lambda = \Delta$. □

Here is the point.

Theorem (Auslander-Goldman)**.** *Let R be a normal domain, K its quotient field. If Λ is a maximal order in $M_n(K)$, then $\Lambda \cong \operatorname{End}_R(M)$ for some reflexive R-module M.*

Corollary. *The following are equivalent for a module-finite algebra Λ over a Gorenstein local normal domain R:*

(1) Λ is a symmetric, birational, non-singular R-order.
(2) $\Lambda \cong \operatorname{End}_R(M)$ for some reflexive R-module, Λ is MCM as an R-module, and Λ is homologically homogeneous.
(3) $\Lambda \cong \operatorname{End}_R(M)$ as above, and $\operatorname{gldim}\Lambda < \infty$.

So finally we arrive at Van den Bergh's definition.

Definition (Van den Bergh)**.** Let R be a Gorenstein normal domain. A non-commutative crepant resolution of R is an algebra $\Lambda = \operatorname{End}_R(M)$, where M is a reflexive R-module, such that Λ has finite global dimension and is MCM as R-module.

Equivalently, Λ is a symmetric birational non-singular R-order.

Part 3. Variant definitions

You may be unhappy or surprised that R suddenly became Gorenstein. The definition of Van den Bergh (in either of the two equivalent [for Gorenstein rings!!] formulations above) makes sense more generally.

The point is that the implication

$$\boxed{\text{symmetric} + \text{finite gldim} \implies \text{nonsingular}\,.}$$

for orders fails when R is only Cohen-Macaulay and not Gorenstein. Here is an example.

Example. Let $R = \mathbb{C}[\![x, y, z, u, v]\!]/I$, where I is the ideal generated by the 2×2 minors of the matrix $\left[\begin{smallmatrix} x & y & u \\ y & z & v \end{smallmatrix}\right]$. Then R is a 3-dimensional normal domain and has finite CM type [Yoshino, 16.12]. In particular, its indecomposable MCM modules are

$$R, \qquad \omega_R \cong (x, y), \qquad \Omega^1\omega, \qquad \Omega^2\omega, \qquad (\Omega^1\omega)^\vee\,.$$

(The $^\vee$ is the canonical dual.)

We know that the endomorphism ring $\Lambda_1 = \operatorname{End}_R(R \oplus \omega_R \oplus \Omega^1\omega \oplus \Omega^2\omega \oplus (\Omega^1\omega)^\vee)$ has global dimension 3 by the previous example. However it has bad depth: several of the summands have depth 2.

We can partially fix this by deleting some summands:

The endomorphism ring $\Lambda_2 = \operatorname{End}_R(R \oplus \omega)$ is MCM as an R-module. (The only thing to check is that $\operatorname{Hom}_R(\omega, R) \cong \Omega^1\omega$.) It's symmetric by the result of Auslander-Goldman. But Quarles & Smith show that $\operatorname{gldim} \Lambda_2 = 4$, not 3, so Λ is not nonsingular.

And here is another example which is not an order, but which is also fixable.

Example. Let $R = \mathbb{C}[\![x^2, xy, y^2, xz, yz, z^2]\!]$. Then R is also a 3-dimensional normal domain of finite CM type [Yoshino, 16.10]: its indecomposable MCM modules are

$$R, \qquad \omega \cong (x^2, xy, xz), \qquad \Omega^1\omega\,.$$

The endomorphism ring $\Lambda = \operatorname{End}_R(R \oplus \omega \oplus \Omega^1\omega)$ has global dimension 3, but only has depth 2, so is not an order.

Notice that $\Lambda' = \operatorname{End}_R(R \oplus \omega)$ is MCM: it must be, by the McKay Correspondence, since $\mathbb{C}[\![x, y, z]\!] \cong R \oplus \omega$ as R-modules.

By the way, these are the only two known examples of CM local rings of finite CM type in dimension at least 3, which are not Gorenstein/hypersurfaces.

Auslander and Reiten proved that they have finite CM type, and also that they are the only ones of their kind: no other "scrolls" or "Veronese rings" in dimension $\geqslant 3$ have finite CM type.

In a lecture at the Fields Institute in 2015 (and in a grant proposal around that time), I suggested that perhaps a stronger hypothesis would work better instead of assuming that Λ is MCM as an R-module.

Recall that an R-module M is called totally reflexive (or of G-dimension zero) if $M \cong M^{**}$ and

$$\operatorname{Ext}_R^{>0}(M, R) = \operatorname{Ext}_R^{>0}(M^*, R) = 0\,.$$

Over a Gorenstein ring, this is equivalent to being MCM. Over general CM rings, this is a stronger property than being MCM. So I proposed:

Definition. A strong NCR of a CM normal domain R is an R-algebra $\Lambda = \operatorname{End}_R(M)$, where M is a reflexive R-module, with $\operatorname{gldim}\Lambda < \infty$ and Λ totally reflexive as an R-module.

The expectation was that this stronger hypothesis would enable us to recover the good behavior of nonsingular orders in the Gorenstein case. Unfortunately,

Theorem (Stangle 2015)**.** *If a CM local ring R has a strong NCR, then R is Gorenstein.*

So rather than “strong”, it should be called “too strong”.

Proof. It is enough to show that $\mathrm{Ext}_R^{\gg 0}(k, R) = 0$, where $k = R/\mathfrak{m}$ is the residue field. Since Λ is a finitely generated R-module, $\Lambda/\mathfrak{m}\Lambda$ is a finite-dimensional vector space, so it suffices to show that $\mathrm{Ext}_R^{\gg 0}(\Lambda/\mathfrak{m}\Lambda, R) = 0$.

But one can show, using the fact that $\mathrm{Ext}_R^{>0}(\Lambda, R) = 0$ (collapsing spectral sequence, or just directly), that

$$\mathrm{Ext}_R^i(\Lambda/\mathfrak{m}\Lambda, R) \cong \mathrm{Ext}_\Lambda^i(\Lambda/\mathfrak{m}\Lambda, \mathrm{Hom}_R(\Lambda, R)) .$$

Since Λ has finite global dimension, in particular the Λ-module $\mathrm{Hom}_R(\Lambda, R)$ has finite injective dimension, and so those Exts vanish for large enough indices. We win. □

More recent work [Iyama-Wemyss if R is CM, Dao-Faber-Ingalls in general] has adopted a weaker version of this definition without the assumptions that R is Gorenstein or normal:

Definition. Let R be a Noetherian commutative ring. A non-commutative crepant resolution (NCCR) of R is an algebra $\Lambda = \mathrm{End}_R(M)$, where M is a torsion-free module with full support ($\mathrm{Supp}\, M = \mathrm{Spec}\, R$) and such that Λ is a non-singular R-order.

(Actually, D-F-I show that for a reduced ring, an NCCR in this sense is the same thing as an NCCR of the normalization, so you don't get anything new in that case. Gorenstein-ness is a different story.)

We'll explain the name in a moment.

Dao-Iyama-Takahashi-Vial and Dao-Faber-Ingalls also define a weaker version.

Definition. Let R be a Noetherian commutative ring. A non-commutative resolution (NCR) of R is an algebra $\Lambda = \mathrm{End}_R(M)$, where M is an R-module with full support ($\operatorname{Supp} M = \operatorname{Spec} R$) and such that Λ has finite global dimension.

Part 4. Connections with cluster tilting

I gave the participants the chance to vote before the third talk: would they rather hear about connections with cluster tilting, or connections with algebraic geometry? The vote was essentially even, so I broke the tie. I skipped this part entirely.

I should probably say a few words about the connections with cluster tilting, since I know that's a very popular topic in the representation theory world.

This is from [Iyama-Wemyss, Inv. math 2013].

Let R be a d-dimensional CM local (for simplicity) ring with canonical module ω.

Definition. A MCM module M is called CT if

$$\begin{aligned}\operatorname{add} M &= \{X \in \mathrm{MCM}(R) \mid \operatorname{Hom}_R(M,X) \in \mathrm{MCM}(R)\} \\ &= \{X \in \mathrm{MCM}(R) \mid \operatorname{Hom}_R(X,M) \in \mathrm{MCM}(R)\}\end{aligned}$$

Note that this is not exactly the same as a cluster tilting module. Indeed, M is called n-cluster tilting if

$$\begin{aligned}\operatorname{add} M &= \left\{X \in \mathrm{MCM}(R) \mid \operatorname{Ext}_R^{1,\dots,n-1}(M,X) = 0\right\} \\ &= \left\{X \in \mathrm{MCM}(R) \mid \operatorname{Ext}_R^{1,\dots,n-1}(X,M) = 0\right\}\end{aligned}$$

Under certain circumstances (for example, if R is an isolated singularity) they are related. Indeed, recall that

Proposition. *Let M, N be MCM modules over a CM local ring. If $\operatorname{Ext}_R^i(M,N) = 0$ for $i = 1,\dots,\dim R - 2$, then $\operatorname{Hom}_R(M,N)$ is MCM.*

If R is an isolated singularity then the converse is true: $\operatorname{Hom}_R(M,N)$ is MCM if and only if $\operatorname{Ext}_R^{1,\dots,d-2}(M,N) = 0$.

This shows that $d-1$ cluster tilting modules are CT over CM local rings, and they are the same thing for isolated singularities. In general they are different.

Anyway.

Theorem (IW)**.** *For a MCM R-module, the following are equivalent.*

(1) M is a CT module.
(2) $R \in \operatorname{add}_R(M)$ and $\operatorname{End}_R(M)$ is a non-singular R-order.
(3) $\omega_R \in \operatorname{add}_R(M)$ and $\operatorname{End}_R(M)$ is a non-singular R-order.

The CT modules are therefore precisely the MCM generators which give NCCRs.

Proof of (1) implies (2). By assumption we already know $\operatorname{End}_R(M)$ is MCM, and $R \in \operatorname{add}_R(M)$ since M is MCM. Let Y be an $\operatorname{End}_R(M)$-module with a projective resolution

$$P_{d-1} \longrightarrow \ldots \longrightarrow P_0 \longrightarrow Y \longrightarrow 0$$

By Yoneda, there is an exact sequence $M_{d-1} \longrightarrow \ldots \longrightarrow M_0$ with each $M_i \in \operatorname{add}_R(M)$ and such that applying $\operatorname{Hom}_R(M,-)$ gives the resolution above. Let $K_d = \ker(M_{d-1} \longrightarrow M_{d-2})$. Then K is MCM by the Depth Lemma, and so is $\operatorname{Hom}_R(M,K)$, which sits in an exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(M,K) \longrightarrow P_{d-1} \longrightarrow \ldots \longrightarrow P_0 \longrightarrow Y \longrightarrow 0\,.$$

Since M is CT, this means $K \in \operatorname{add}_R(M)$, so $\operatorname{pd}_{\operatorname{End}_R(M)}(Y) \leqslant d$. This shows $\operatorname{gldim}\operatorname{End}_R(M) \leqslant d < \infty$, and we saw at some point that non-singularity can be checked locally. □

A source of CT modules: The McKay Correspondence gives CT modules (since we know it gives non-singular R-orders).

Specifically, if $R = k[[x_1,\ldots,x_n]]^G$ (or polynomial ring) where G is a finite subgroup of $\operatorname{GL}_n(k)$ and k is a field of characteristic zero, then $k[[x_1,\ldots,x_n]]$ is a CT R-module.

(Notice that we don't require G to be small.)

Of course, this example is not particularly useful to get new examples of NCCRs. Here's a better one.

Example (Burban-Iyama-Keller-Reiten, 2007). Let $R = k[\![x,y]\!]/(f)$ be a one-dimensional reduced hypersurface ring, where k is an algebraically closed field of characteristic zero. Then there is a 1-cluster tilting module (in the stable category $\underline{\mathrm{MCM}}(R)$) if and only if f factors as a product of linear factors $f = f_1 \cdots f_n$, and in this case there are precisely $n!$ basic 1-cluster tilting modules.

One specific one is

$$M = \bigoplus_{i=1}^{n} k[\![x,y]\!]/(f_1 f_2 \cdots f_i)\,.$$

The others are obtained by a “mutation” process.

Part 5. Connections with algebraic geometry

The definition of Van den Bergh had a word in it that came out of nowhere: crepant. Next we want to explain that.

Recall that we wanted to impose a "Gorenstein-ness" assumption on an R-algebra Λ, that $\mathrm{Hom}_R(\Lambda, \omega_R)$ be projective as Λ-module. If R is Gorenstein, this is almost as strong as asking that Λ be symmetric: $\mathrm{Hom}_R(\Lambda, R) \cong \Lambda$. (And we know that this latter property holds if Λ is an endomorphism ring.)

What does this mean in terms of the motivating geometry?

Let X be an algebraic variety, and let ω_X be the canonical sheaf (aka dualizing sheaf, if X is CM) for X.

Suppose we have a resolution of singularities $\pi\colon \widetilde{X} \longrightarrow X$. There is also a canonical sheaf upstairs, $\omega_{\widetilde{X}}$. In fact, by basic properties of the canonical sheaf, we have

$$\omega_{\widetilde{X}} \cong \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_{\widetilde{X}}, \omega_X) .$$

Now assume that X is Calabi-Yau, that is, $\omega_X \cong \mathcal{O}_X$. This is a strong version of Gorenstein-ness, which would just say that ω_X is locally free. Then the above says

$$\omega_{\widetilde{X}} \cong \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_{\widetilde{X}}, \mathcal{O}_X) .$$

and if we assume that $\mathcal{O}_{\widetilde{X}}$ is a symmetric $\mathcal{O}_X$-algebra, then this means that

$$\omega_{\widetilde{X}} \cong \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_{\widetilde{X}}, \mathcal{O}_X) \quad \cong \quad \mathcal{O}_{\widetilde{X}} \quad \cong \quad \pi^*\omega_X ,$$

so that $\widetilde{X}$ is also Calabi-Yau, and the resolution π "preserves the canonical sheaf".

Such resolutions are called crepant. Specifically, π is a crepant resolution if

$$\pi^*\omega_X = \omega_{\widetilde{X}} .$$

Informally, this means that the two natural $\mathcal{O}_X$-module structures on $\omega_{\widetilde{X}}$ coincide: the "induced", via $\otimes$, and the "co-induced", via $\mathscr{H}om$.

This is why we defined the non-commutative crepant resolution.

How far does this analogy with algebraic geometry go? Well, here's one thing we would like:

Fact. If a complex ($\mathbb{C}$) algebraic variety X is Gorenstein and has a crepant resolution of singularities, then X has rational singularities.

(This means that $R\pi_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$; equivalently X is normal and $R^i\pi_*\mathcal{O}_{\widetilde{X}} = 0$ for $i > 0$.)

(This follows directly from Grauert-Riemenschneider vanishing, which is why I needed the complex numbers.)

Theorem (Stafford-Van den Bergh)**.** *Let k be an algebraically closed field of characteristic 0 and Δ be a prime affine k-algebra that is finitely generated as a module over its center $Z(\Delta)$. If Δ is a non-singular $Z(\Delta)$-order then $Z(\Delta)$ has only rational singularities.*

In particular, suppose R is a Gorenstein normal affine k-algebra. If R has an NCCR, then $\mathrm{Spec}(R)$ has only rational singularities.

This makes us feel much better.

Also, the definition of NCR is apparently much weaker than that of NCCR, but it is already enough to obtain a version of this result:

Dao-Iyama-Takahashi-Vial: Let R be a Cohen-Macaulay standard graded $\mathbb{C}$-algebra. Assume that R has rational singularities away from the irrelevant ideal. If R has an NCR, then R has rational singularities.

Corollary. *Let R be a CM standard graded $\mathbb{C}$-algebra of finite CM representation type. Then R has only rational singularities.*

This follows from the fact that CM local rings of finite CM type always have NCRs (though rarely NCCRs).

The Corollary was already known by work of Eisenbud-Herzog, but using the classification of CM standard graded $\mathbb{C}$-algebras of finite CM representation type. This is a direct proof, which is new.

Historical digression

The results above encourage us to believe that an NCCR is "the same thing" as a crepant resolution of singularities. In fact this was the intuition behind Van den Bergh's original definition, as I will now explain briefly.

The Minimal Model Program is a strategy for carrying out the birational classification of smooth algebraic varieties. The details aren't important for us here. "Running the MMP" on a variety involves certain "moves", which are intended to simplify a variety until it you can't go any farther (you get to a "terminal" variety).

Bondal and Orlov suggest viewing the MMP as operations on the bounded derived category $D^b(\operatorname{coh} X)$.

Example. One of the moves is blowing up a smooth subvariety, which induces a fully faithful functor (even a semi-orthogonal decomposition!) on derived categories.

Example. Another of the moves is a "flop". This means replacing a smooth variety Y by Y', where

$$\begin{array}{ccc} Y & & Y' \\ & {}_{f}\searrow \quad \swarrow_{f'} & \\ & X & \end{array}$$

is a diagram of varieties, and f and f' are both crepant resolutions of singularities (and some other technical conditions).

Algebraically this is easy to describe, at least after passing to the completions of the local coordinate rings: a local Gorenstein terminal singularity is a hypersurface ring of multiplicity 2. So the defining equation can be written $u^2 + \cdots$; then Y' is just Y, and f' is the composition of $u \mapsto -u$ with f.

Bondal and Orlov make the following conjecture.

Conjecture. *If Y and Y' are related by a flop, then they are derived equivalent: $D^b(\operatorname{coh} Y) \simeq D^b(\operatorname{coh} Y')$.*

In the case $\dim Y = 3$, this was proved by Bridgeland in 2002, using Fourier-Mukai transforms.

At around the same time, Bridgeland-King-Reid described an approach to the McKay correspondence based on Fourier-Mukai transforms.

Van den Bergh observed that "an essential feature of the McKay correspondence is the appearance of the skew group algebra", and was able to show

Theorem (VdB '04)**.** *Let R be a Gorenstein normal $\mathbb{C}$-algebra, $X = \operatorname{Spec} R$, and $\pi\colon \widetilde{X} \longrightarrow X$ a crepant resolution of singularities. Assume π has at most one-dimensional fibres (e.g. $\dim R \leqslant 3$). Then R has an NCCR $\Lambda = \operatorname{End}_R(M)$.*

Furthermore, Λ is derived equivalent to $\widetilde{X}$.

The Bondal-Orlov conjecture in this case reduces to showing that the corresponding non-commutative rings Λ and, say, Λ', are derived equivalent, which Van den Bergh does, reproving Van den Bergh's result.

This raises a more general conjecture: Perhaps <u>all</u> crepant resolutions, commutative as well as non-commutative, are derived equivalent. Or (the "non-commutative Bondal-Orlov conjecture") at least all NCCRs of a ring are derived equivalent.

The ncBO conjecture is known for $\dim R \leqslant 3$ [Iyama-Wemyss, first for Gorenstein rings then for CM rings].

Actually, Iyama and Wemyss give a very tempting sufficient condition for the derived-equivalence of two NCCRs $\mathrm{End}_R(M)$ and $\mathrm{End}_R(N)$. Since $\mathrm{End}_R(M)$ has finite global dimension, $\mathrm{Hom}_R(M,N)$ has finite projective dimension. By Yoneda's Lemma, this means that N has a finite resolution by direct sums of direct summands of M, say $0 \longrightarrow M_{\dim R-2} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow N \longrightarrow 0$, which remains exact upon applying $\mathrm{Hom}_R(M,-)$. Say that (M,N) satisfies the depth condition if $\mathrm{Hom}_R(M_i,N)$ locally has depth at least $\dim R - i - 1$ for each i. Then (Iyama-Wemyss prove) $\mathrm{End}_R(M)$ and $\mathrm{End}_R(N)$ to be derived equivalent, it suffices that (M,N) and (N^*,M^*) both satisfy the depth condition.

The condition is clearly vacuous if $\dim R \leqslant 3$.

By the way, it is not known (even in dimension three) whether existence of a crepant resolution of singularities is equivalent to having an NCCR.

It's known to be true under some circumstances in dimension three, and VdB conjectures it to be true generally in dim three, but it's known to be false (both directions) in dimensions $\geqslant 4$.

Example. $S = \mathbb{C}[[x,y,z,t]]$ with C_2 acting by negating the variables. The invariant ring R has a NCCR by the McKay correspondence, but is known not to have a CR [Reid].

Example. Dao gives some examples of hypersurfaces with CRs but no NCCRs, for example, $\mathbb{C}[[x_1,\ldots,x_5]]/(x_1^5 + x_2^4 + x_3^4 + x_4^4 + x_5^4)$. More generally, he shows (using "Tor-rigidity") that a hypersurface ring with an isolated singularity has no NCCR if its dimension is even and at least 4.