Summer School on Cohen-Macaulay Modules

Notes of the Talks (taken by Jan Geuenich)

BIREP

12–16 August 2019

Contents

1 R-orders and Krull–Schmidtness

Monday 12$^{\text{th}}$ 13:00 – Biao Ma (Bielefeld, Germany)

Notation.

- $(R, \mathfrak{m}, k)$ is always a commutative noetherian local ring
- $\operatorname{mod}(R)$ the category of finitely generated R-modules
- $\operatorname{proj}(R)$ the category of finitely generated projective R-modules
- Λ is a module-finite R-algebra
- $\operatorname{mod}(\Lambda)$ the category of finitely generated left Λ-modules
- $\operatorname{proj}(\Lambda)$ the category of finitely generated left projective Λ-modules

1.1 Krull-Schmidt categories

Definition 1.1. *An additive category $\mathcal{A}$ is called a Krull-Schmidt category if each $A \in \mathcal{A}$ can be written as a finite direct sum of objects having local endomorphism ring.*

Remark 1.2. Let $\mathcal{A}$ be a Krull-Schmidt category.

(1) $\operatorname{End}_{\mathcal{A}}(A)$ local $\Leftrightarrow$ A indecomposable

Recall that S is (not necessarily commutative) local if $S/J(S)$ is a division ring.

(2) The Krull-Schmidt Theorem holds in $\mathcal{A}$.

(3) Any morphism $f\colon A \to B$ in $\mathcal{A}$ has a right minimal version (and similarly also a left minimal version), i.e. $f = (f' \ 0) : A = A' \oplus A'' \to B$ with right minimal f', meaning that $f'\theta = f'$ only if θ is invertible.

Definition 1.3. *A local ring $(R, \mathfrak{m}, k)$ is called Henselian if for every module-finite R-algebra Λ each idempotent in $\Lambda/J(\Lambda)$ lifts to an idempotent in Λ, i.e. for all idempotents $\overline{x}^2 = \overline{x} \in \Lambda/J(\Lambda)$ there exists an idempotent $e^2 = e \in \Lambda$ such that $\overline{x} = \overline{e}$.*

Theorem 1.4. *Let $(R, \mathfrak{m}, k)$ be Henselian. Then* $\operatorname{mod}(R)$ *is Krull-Schmidt.*

Proof. It is enough to show that $\Gamma = \operatorname{End}_R(M)$ is local for indecomposable modules M in $\operatorname{mod}(R)$. Note that Γ is module-finite. Nakayama's lemma implies $\mathfrak{m} \subseteq \operatorname{Ann}(\Gamma/J(\Gamma))$. Thus $\Gamma/J(\Gamma)$ is a finite-dimensional K-algebra, so semisimple. R is Henselian, so idempotents lift. Now M is indecomposable, so Γ has only the two idempotents 0, 1. Thus by Wedderburn–Artin $\Gamma/J(\Gamma)$ is a division ring. $\square$

Corollary 1.5. *Let $(R, \mathfrak{m}, k)$ be complete local. Then:*

(1) $\operatorname{mod}(R)$ *is Krull-Schmidt.*

(2) $\operatorname{mod}(\Lambda)$ *is Krull-Schmidt for every module-finite R-algebra Λ.*

Proof. (1) complete $\Rightarrow$ Henselian

(2) Let $M \in \operatorname{mod}(\Lambda)$ indecomposable. Now $\Gamma = \operatorname{End}_\Lambda(M) \subseteq \operatorname{End}_R(M)$ is module-finite. Repeat the proof of Theorem 1.4. $\square$

1.2 R-orders

From now on $(R, \mathfrak{m}, k)$ is a commutative noetherian complete regular local ring with Krull dimension $\dim(R) = d$ (e.g. $R = k[[x_1, \dots, x_d]]$). In this case

$$\operatorname{gl.dim}(R) = \operatorname{inj.dim}({}_R R) = \operatorname{proj.dim}({}_R k) = \dim(R) = d\,.$$

Definition 1.6.

(i) *A module-finite R-algebra Λ is called an <u>R-order</u> if ${}_R\Lambda \in \operatorname{proj}(R)$.*

(ii) *Let Λ be an R-order. A finitely generated Λ-module M is called <u>(maximal) Cohen-Macaulay (CM)</u> if ${}_R M \in \operatorname{proj}(R)$.*

Example 1.7.

(1) Any finite-dimensional algebra over a field k is a k-order.

(2) Any commutative complete CM local ring containing a field is an R-order.

Denote by $\operatorname{CM}(\Lambda)$ the category of CM Λ-modules.

Proposition 1.8. *Let Λ be an R-order. Then:*

(1) $\operatorname{mod}(\Lambda)$ *and* $\operatorname{CM}(\Lambda)$ *are Krull-Schmidt.*

(2) $\operatorname{CM}(\Lambda)$ *is a <u>resolving</u> subcategory of* $\operatorname{mod}(\Lambda)$, *i.e. it contains* $\operatorname{proj}(\Lambda)$ *and is closed under extensions and kernels of epimorphisms.*

(3) $\operatorname{Hom}_R(-, R)\colon \operatorname{CM}(\Lambda) \xrightarrow{\sim} \operatorname{CM}(\Lambda^{\mathrm{op}})$ *is a duality.*

${}_\Lambda\omega = \operatorname{Hom}_R(\Lambda_\Lambda, R)$ *and* $\omega_\Lambda = \operatorname{Hom}_R({}_\Lambda\Lambda, R)$ *are called the <u>canonical modules</u>.*

(4) $\operatorname{CM}(\Lambda)$ *is an exact category with enough projectives* $\operatorname{add}({}_\Lambda\Lambda)$ *and enough injectives* $\operatorname{add}({}_\Lambda\omega)$.

Proof. (1) $\operatorname{CM}(\Lambda)$ is closed under summands.

(2) $\operatorname{proj}(\Lambda) \subseteq \operatorname{CM}(\Lambda)$ and for $0 \to L \to M \to N \to 0$ we clearly have $L, N \in \operatorname{CM}(\Lambda) \Rightarrow M \in \operatorname{CM}(\Lambda)$ and $M, N \in \operatorname{CM}(\Lambda) \Rightarrow L \in \operatorname{CM}(\Lambda)$.

(3) Use the duality $\operatorname{Hom}_R(-, R)\colon \operatorname{proj}(R) \xrightarrow{\sim} \operatorname{proj}(R^{\mathrm{op}})$.

(4) $\operatorname{CM}(\Lambda)$ is closed under extensions, so $\operatorname{CM}(\Lambda)$ is an exact category. Then use the duality in (3). $\square$

Proposition 1.9. *Let Λ be an R-order. Then:*

(1) $\operatorname{inj.dim}({}_\Lambda\Lambda) \geq \operatorname{inj.dim}({}_R R)$.

(2) If $\operatorname{gl.dim}(\Lambda) < \infty$*, then* $\operatorname{gl.dim}(\Lambda) = \operatorname{inj.dim}({}_\Lambda\Lambda) \geq \operatorname{inj.dim}({}_R R) = \dim(R) = d$.

Definition 1.10. *Let Λ be an R-order.*

(1) Λ is called *non-singular* *if* $\operatorname{gl.dim}(\Lambda) = \dim(R) = d$.

(2) Λ is called an *isolated singularity* *if* $\operatorname{gl.dim}(\Lambda_\mathfrak{p}) = \dim(R_\mathfrak{p})$ *for all* $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$ *where* $\Lambda_\mathfrak{p} = \Lambda \otimes_R R_\mathfrak{p}$.

Remark 1.11. Λ non-singular $\Rightarrow$ Λ isolated singularity

Example 1.12. Let $R = k[[x_1, \ldots, x_d]]$ and G be a finite subgroup of $\mathrm{GL}_d(k)$ such that $|G| \neq 0$ in k. Then G acts linearly on R by permuting the variables and the skew group algebra $R\#G$ is a non-singular R-order.

Proposition 1.13. *The following are equivalent for an R-order Λ:*

(1) Λ is non-singular.

(2) $\mathrm{CM}(\Lambda) = \operatorname{proj}(\Lambda)$.

Proof. (1) $\Rightarrow$ (2) Let $x_1, \ldots, x_d$ be a regular system of parameters of R and $M \in \mathrm{CM}(\Lambda)$. Then $\operatorname{gl.dim}(\Lambda) = d \geq \operatorname{proj.dim}_\Lambda (M/(x_1, \ldots, x_d)M) = d = \operatorname{proj.dim}({}_\Lambda M)$, which implies that $M \in \operatorname{proj}(\Lambda)$.

(2) $\Rightarrow$ (1) For each $M \in \operatorname{mod}(\Lambda)$ there is a projective resolution

$$0 \to \Omega^d M \to P_{d-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

of ${}_\Lambda M$ (also ${}_R M$). So $\operatorname{gl.dim}(R) = d \Rightarrow \Omega^d M \in \operatorname{proj}(R) \cap \operatorname{mod}(\Lambda) \Rightarrow \Omega^d M \in \operatorname{proj}(\Lambda)$. $\square$

AR-formulas for Λ-orders

R-**dual.** $D_d := \operatorname{Hom}_R(-, R)\colon\ \mathrm{CM}(\Lambda) \xrightarrow{\sim} \mathrm{CM}(\Lambda^{\mathrm{op}})$ induces a duality

$$\underline{\mathrm{CM}}(\Lambda) = \mathrm{CM}(\Lambda)/\operatorname{add}(\Lambda) \xrightarrow{\quad D_d \quad} \overline{\mathrm{CM}}(\Lambda^{\mathrm{op}}) = \mathrm{CM}(\Lambda^{\mathrm{op}})/\operatorname{add}(\omega_\Lambda)\,.$$

For $X, Y \in \underline{\mathrm{CM}}(\Lambda)$ then

$$\underline{\operatorname{Hom}}_\Lambda(X, Y) \;=\; \{X \xrightarrow{f} Y : f \text{ doesn't factor through } \operatorname{proj}(\Lambda)\}\,.$$

Matlis dual. $D := \operatorname{Hom}_R(-, E) \cong \operatorname{Ext}^d_R(-, R)$ with $E := E(k)$ the injective envelope of ${}_R k$ gives a duality

$$\mathrm{f.l.}(R) \xrightarrow{\quad D \quad} \mathrm{f.l.}(R^{\mathrm{op}})\,.$$

Λ-**dual.** There exists a duality $(-)^* := \operatorname{Hom}_\Lambda(-, \Lambda)\colon\ \operatorname{proj}(\Lambda) \xrightarrow{\sim} \operatorname{proj}(\Lambda^{\mathrm{op}})$.

Auslander-Bridger transpose. There exists a duality

$$\underline{\mathrm{mod}}(\Lambda) \xrightarrow{\quad \mathrm{Tr} \quad} \overline{\mathrm{mod}}(\Lambda^{\mathrm{op}})$$

given by $M \mapsto \mathrm{Tr}(M) := \mathrm{coker}(f^*)$ where $P_1 \xrightarrow{f} P_0 \to M \to 0$ is a projective presentation (and $0 \to M^* \to P_0^* \xrightarrow{f^*} P_1^*$).

Theorem 1.14. *Let Λ be an isolated singularity. Then:*

(1) $$\begin{aligned} \mathrm{CM}(\Lambda) &= \{M \in \mathrm{mod}(\Lambda) : \mathrm{Ext}^i_\Lambda(M, {}_\Lambda\omega) = 0 \textit{ for all } i > 0\} \\ &= \{\mathrm{Tr}(X) : X \in \mathrm{mod}(\Lambda^{\mathrm{op}}) \textit{ such that } \mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(X, \Lambda_\Lambda) = 0 \textit{ for all } i = 1, \dots, d\}. \end{aligned}$$

(2) *There is a duality $\Omega^d \,\mathrm{Tr} : \underline{\mathrm{CM}}(\Lambda) \to \overline{\mathrm{CM}}(\Lambda^{\mathrm{op}})$.*

(3) *There is an equivalence $\tau := D_d \Omega^d \,\mathrm{Tr} : \underline{\mathrm{CM}} \to \overline{\mathrm{CM}}(\Lambda)$.*

(4) *<u>(AR-formula)</u> There exists an isomorphism*

$$\underline{\mathrm{Hom}}_\Lambda(\tau^-(N), M) \cong D\,\mathrm{Ext}^1_\Lambda(M, N) \cong \overline{\mathrm{Hom}}_\Lambda(N, \tau(M))$$

natural for any $M, N \in \mathrm{CM}(\Lambda)$.

2 Maximal and hereditary orders

Monday 12^{th} 14:15 – Yuta Kimura (Bielefeld, Germany)

Notation.

- $R = k[[x]]$ with maximal ideal $\mathfrak{m} = (x)$ (or complete DVR such as $\widehat{\mathbb{Z}}_p$).
- $K := \operatorname{Quot}(R)$ fractional field of R (so $K = k((x))$).

commutative ring S	order Λ $(\dim(R) = 1)$	f.d. algebra A
$\mathrm{CM}(S) = \mathrm{proj}(S)$ regular $(\operatorname{gl.dim}(S) < \infty)$	$\mathrm{CM}(\Lambda) = \mathrm{proj}(\Lambda)$ non-singular $(\operatorname{gl.dim}(\Lambda) = 1)$	$\mathrm{mod}(A) = \mathrm{proj}(A)$ semisimple $(\operatorname{gl.dim}(A) = 0)$
$\underline{\mathrm{CM}}(S)$ triangulated Gorenstein $(\operatorname{inj.dim}(S) < \infty)$	$\underline{\mathrm{CM}}(\Lambda)$ triangulated Gorenstein $(\operatorname{inj.dim}(\Lambda) = 1)$	$\underline{\mathrm{mod}}(A)$ triangulated selfinjective $(\operatorname{inj.dim}(A) = 0)$
	$\Lambda \subseteq \Lambda'$ overorder $\Rightarrow \mathrm{CM}(\Lambda') \hookrightarrow \mathrm{CM}(\Lambda)$.	$A \twoheadrightarrow A/I$ $\Rightarrow \mathrm{mod}(A/I) \hookrightarrow \mathrm{mod}(A)$.

Example 2.1. $\Lambda = \begin{pmatrix} R & \mathfrak{m}^\ell \\ R & R \end{pmatrix}$ is an R-order.

- For any $\ell \in \mathbb{Z}$ let $\mathfrak{m}^\ell = (x^\ell) = Rx^\ell \subseteq K$.
- There is an isomorphism ${}_R\mathfrak{m}^\ell \xrightarrow[\cdot x^{-\ell}]{\sim} {}_RR$.
- $\mathrm{CM}(\Lambda)$ has an AR-quiver, which will now be computed.
- $(-)^* = \operatorname{Hom}_R(-, R)$.
- With $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ we have $\Lambda e_1 = \left(\begin{smallmatrix} R \\ R \end{smallmatrix}\right)$, $\Lambda e_2 = \left(\begin{smallmatrix} \mathfrak{m}^\ell \\ R \end{smallmatrix}\right)$.

(1) $\left(\begin{smallmatrix} R \\ R \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \mathfrak{m} \\ R \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \mathfrak{m}^2 \\ R \end{smallmatrix}\right)$, $\ldots$, $\left(\begin{smallmatrix} \mathfrak{m}^\ell \\ R \end{smallmatrix}\right) \in \mathrm{CM}(\Lambda)$.

(2) Applying $\tau = (-)^* \circ \Omega_{\Lambda^{\mathrm{op}}} \circ \operatorname{Tr}_\Lambda$ gives $\tau\left(\begin{smallmatrix} \mathfrak{m}^i \\ R \end{smallmatrix}\right) \cong \left(\begin{smallmatrix} \mathfrak{m}^i \\ R \end{smallmatrix}\right)$ for all $1 \leq i \leq \ell - 1$.

- $\operatorname{rad}\left(\begin{smallmatrix} R & \mathfrak{m}^\ell \\ R & R \end{smallmatrix}\right) = \left(\begin{smallmatrix} \mathfrak{m} & \mathfrak{m}^\ell \\ R & \mathfrak{m} \end{smallmatrix}\right)$, $\operatorname{rad}\left(\begin{smallmatrix} \mathfrak{m}^i \\ R \end{smallmatrix}\right) = \left(\begin{smallmatrix} \mathfrak{m}^{i+1} \\ R \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \mathfrak{m}^i \\ R \end{smallmatrix}\right) / \operatorname{rad}\left(\begin{smallmatrix} \mathfrak{m}^i \\ R \end{smallmatrix}\right) \cong S_1 \oplus S_2$.

$$\begin{tikzcd} \left(\begin{smallmatrix} R \\ R \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} \mathfrak{m}^\ell \\ R \end{smallmatrix}\right) \arrow[rr, "M"] \arrow[dr, "(x^{\ell-i}\ 1)"'] & & \left(\begin{smallmatrix} R \\ R \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} \mathfrak{m}^\ell \\ R \end{smallmatrix}\right) \arrow[r, "(x^i\ 1)"] & \left(\begin{smallmatrix} \mathfrak{m}^i \\ R \end{smallmatrix}\right) \arrow[r] & 0 \\ & \left(\begin{smallmatrix} \mathfrak{m}^{\ell-i} \\ R \end{smallmatrix}\right) \arrow[ur, "\left(\begin{smallmatrix} 1 \\ -x^i \end{smallmatrix}\right)"'] & & & \end{tikzcd}$$

where $M = \begin{pmatrix} x^{\ell-i} & 1 \\ -x^\ell & -x^i \end{pmatrix}$.

- Apply $\mathrm{Hom}_\Lambda(-,\Lambda)$, then $\mathrm{Hom}_\Lambda(\Lambda e_i,\Lambda)\cong e_i\Lambda$ and with $N=\begin{pmatrix} x^{\ell-i} & -x^{-\ell} \\ 1 & -x^i \end{pmatrix}$

$$\begin{tikzcd}
 & & & & & 0 \\
0 \arrow[r] & {}_\Lambda\left(\begin{pmatrix} \mathfrak{m}^i \\ R \end{pmatrix},\Lambda\right) \arrow[r] & \begin{pmatrix} R & \mathfrak{m}^\ell \end{pmatrix}\oplus\begin{pmatrix} R & R \end{pmatrix} \arrow[dr, "N"] \arrow[dd] & & \mathrm{Tr}\begin{pmatrix} \mathfrak{m}^\ell \\ R \end{pmatrix} \arrow[ur] & \\
 & & & \begin{pmatrix} R & \mathfrak{m}^\ell \end{pmatrix}\oplus\begin{pmatrix} R & R \end{pmatrix} \arrow[ur] & & \\
 & & \Omega\,\mathrm{Tr}\begin{pmatrix} \mathfrak{m}^i \\ R \end{pmatrix} \arrow[ur] \arrow[dd] & & & \\
 & 0 \arrow[ur] & & & & \\
 & & 0 & & &
\end{tikzcd}$$

- $\mathrm{add}(\Lambda)=\mathrm{add}(\omega)$.
- $\Omega\,\mathrm{Tr}\begin{pmatrix} \mathfrak{m}^i \\ R \end{pmatrix}=\mathrm{im}\,N=\begin{pmatrix} R & \mathfrak{m}^i \end{pmatrix}$.
- $\Lambda\subseteq\begin{pmatrix} R & \mathfrak{m}^{\ell-1} \\ R & R \end{pmatrix}$ for $\ell\geq 1$.
- $\tau\begin{pmatrix} \mathfrak{m}^i \\ R \end{pmatrix}\cong\begin{pmatrix} R & \mathfrak{m}^i \end{pmatrix}^*\cong\begin{pmatrix} R \\ \mathfrak{m}^{-i} \end{pmatrix}\cong\begin{pmatrix} \mathfrak{m}^i \\ R \end{pmatrix}$.

(3) $0\to\begin{pmatrix} \mathfrak{m}^i \\ R \end{pmatrix}\xrightarrow{\begin{pmatrix} -x \\ 1 \end{pmatrix}}\begin{pmatrix} \mathfrak{m}^{i+1} \\ R \end{pmatrix}\oplus\begin{pmatrix} \mathfrak{m}^{i-1} \\ R \end{pmatrix}\xrightarrow{(1\ x)}\begin{pmatrix} \mathfrak{m}^i \\ R \end{pmatrix}\to 0$ is an AR-sequence.

- So the AR-quiver of $\mathrm{CM}(\Lambda)$ is

$$\begin{pmatrix} R \\ R \end{pmatrix}\leftrightarrows\begin{pmatrix} \mathfrak{m} \\ R \end{pmatrix}\leftrightarrows\quad\cdots\quad\leftrightarrows\begin{pmatrix} \mathfrak{m}^{\ell-1} \\ R \end{pmatrix}\leftrightarrows\begin{pmatrix} \mathfrak{m}^\ell \\ R \end{pmatrix}$$

Definition 2.2. *Let Λ, Λ' be R-orders and A a finite-dimensional K-algebra.*

(1) Λ <u>R-order</u> in A $:\Leftrightarrow K\otimes_R\Lambda\cong A$

(Remark: $\Lambda\hookrightarrow K\otimes_R\Lambda\cong A$)

(2) Λ' <u>overorder</u> of Λ in A $:\Leftrightarrow\Lambda\subseteq\Lambda'\subseteq A$

(3) Λ <u>maximal order</u> in A $:\Leftrightarrow$ there is no proper overorder of Λ in A

Example 2.3.

(a) G finite group $\rightsquigarrow RG$ is an R-order in KG.

(b) For $\ell \geq 2$:

$$\begin{tikzcd}
 & A = M_2(K) & \\
 & \left(\begin{smallmatrix} R & R \\ R & R \end{smallmatrix}\right) \arrow[u, hook] & \\
\left(\begin{smallmatrix} R & \mathfrak{m} \\ R & R \end{smallmatrix}\right) \arrow[ur, hook] & & \left(\begin{smallmatrix} R & \mathfrak{m}^{\ell-2} \\ \mathfrak{m} & R \end{smallmatrix}\right) \arrow[ul, hook'] \\
 & \left(\begin{smallmatrix} R & \mathfrak{m}^{\ell-1} \\ R & R \end{smallmatrix}\right) \arrow[ul, hook'] \arrow[ur, dash, "\cong"'] &
\end{tikzcd}$$

Proposition 2.4. *Let Λ' be an overorder of Λ. Then:*

(a) The functor f in the following diagram is fully faithful:

$$\begin{tikzcd}
\operatorname{mod}(\Lambda') \arrow[r, "\text{res}"] & \operatorname{mod}(\Lambda) \\
\operatorname{CM}(\Lambda') \arrow[u, hook] \arrow[r, "f"] & \operatorname{CM}(\Lambda) \arrow[u, hook]
\end{tikzcd}$$

(b) f dense $\Rightarrow \Lambda = \Lambda' \Rightarrow \Lambda'\Lambda \subseteq \Lambda$

Hereditary orders

Theorem 2.5. *Let Λ be an R-order in A. The following are equivalent:*

(1) Λ' overorder of Λ in A with $\operatorname{rad}(\Lambda) \subseteq \operatorname{rad}(\Lambda') \Rightarrow \Lambda = \Lambda'$.

(2) $\operatorname{CM}(\Lambda) = \operatorname{proj}(\Lambda)$.

(3) ${}_\Lambda\Lambda$ is an hereditary algebra.

(4) $\operatorname{rad}(\Lambda) \in \operatorname{proj}(\Lambda)$.

Corollary 2.6. *Maximal orders are hereditary.*

Theorem 2.7. *Let A be a finite-dimensional K-algebra. The following are equivalent:*

(a) A contains a maximal order.

(b) A contains a hereditary order.

(c) A is semisimple and the integral closure of R in $Z(A)$ is finitely generated over R.

Example 2.8. $\Lambda = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$ is maximal in $A = \begin{pmatrix} K & 0 \\ K & K \end{pmatrix}$.

3 Bäckström orders

Monday 12^{th} 15:45 – Sebastian Eckert (Bielefeld, Germany)

Setting.

- R complete discrete valuation ring
- k the residue field of R, i.e. $k = R/\pi R = R/\mathfrak{m}$
- K field of quotients of R
- A finite-dimensional separable K-algebra
- Λ R-order in A
- $\mathrm{CM}(\Lambda)$ category of finitely generated left Λ-lattices

Definition 3.1. *An R-order Λ is a Bäckström order provided there exists a hereditary order Γ such that* $\operatorname{rad}\Gamma = \operatorname{rad}\Lambda \subseteq \Lambda \subseteq \Gamma$.

Proposition 3.2. Λ *is Bäckström if* Λ *is a subhereditary order* (Λ, Γ) *and for any indecomposable projective* Λ*-lattice* P

$$\operatorname{rad}(P) \;\cong\; {}_{\Lambda}X$$

for some indecomposable projective Γ*-lattice* X*.*

Lemma 3.3. *The class of Bäckström orders is closed under Morita equivalence.*

Remark 3.4. We can thus restrict to basic Bäckström orders Λ, i.e. $\Lambda/\operatorname{rad}(\Lambda)$ is a product of skew fields.

Aim. Understand when in this situation $\mathrm{CM}(\Lambda)$ is of finite type.

We need some algebraic structure associated to Λ:

Tensor algebras and valued graphs

Given Λ and Γ we put

$$\mathcal{A} \;=\; \Lambda/\operatorname{rad}(\Lambda) \;=\; \prod_{i=1}^{s} D_i \quad \text{and} \quad \mathcal{B} \;=\; \Gamma/\operatorname{rad}(\Gamma) \;=\; \prod_{j=s+1}^{t} M_{n_j}(D_j)\,.$$

Then:

- $\mathcal{A}$ and $\mathcal{B}$ are finitely generated k-algebras with an algebra homomorphism $\mathcal{A} \hookrightarrow \mathcal{B}$ induced by $\Lambda \subseteq \Gamma$.
- D_i are finite-dimensional skew fields over k.

- Let S_j with $s+1 \leq j \leq t$ be a full set of simple $\mathcal{B}$-modules with $\mathrm{End}_\Gamma(S_j) = D_j$. Then ${}_iS_j = D_i \otimes_k S_j$ with $1 \leq i \leq s$ and $s+1 \leq j \leq t$ are (D_i, D_j)-bimodules.
- $d_{ij} = \dim_{D_i}({}_iS_j)$ for $1 \leq i \leq s$ and $s+1 \leq j \leq t$ and $d_{ij} = 0$ else.
- $d'_{ij} = \dim_{D_j}({}_iS_j)$ for $1 \leq i \leq s$ and $s+1 \leq j \leq t$ and $d_{ij} = 0$ else.

$\rightsquigarrow$ Valued graph with vertices k with $1 \leq k \leq t$ and whenever ${}_iS_j \neq 0$ an edge

$$i \xrightarrow{(d_{ij}, d'_{ij})} j \quad .$$

Example 3.5.

i)

$$\Lambda = \begin{pmatrix} R & \mathfrak{m} & R \\ \mathfrak{m} & R & R \\ \mathfrak{m} & \mathfrak{m} & R \end{pmatrix} \qquad \Gamma = \begin{pmatrix} R & R & R \\ R & R & R \\ \mathfrak{m} & \mathfrak{m} & R \end{pmatrix}$$

$$\mathrm{rad}(\Lambda) = \mathrm{rad}(\Gamma) = \begin{pmatrix} \mathfrak{m} & \mathfrak{m} & R \\ \mathfrak{m} & \mathfrak{m} & R \\ \mathfrak{m} & \mathfrak{m} & \mathfrak{m} \end{pmatrix}$$

$$\mathcal{A} = \prod_{i=1}^{3} D_i \qquad \mathcal{B} = M_2(D_4) \times D_5 \qquad D_i = k\,.$$

$$S_4 = \begin{pmatrix} D_4 \\ D_4 \end{pmatrix} \qquad S_5 = D_5$$

The valued graph is

$$1 \xrightarrow{(1,1)} 4 \xleftarrow{(1,1)} 2 \qquad 3 \xrightarrow{(1,1)} 5\,.$$

Tensor algebra

Consider the tensor algebra

$$\mathcal{D} = \begin{pmatrix} \mathcal{B} & {}_{\mathcal{B}}\mathcal{B}_{\mathcal{A}} \\ 0 & \mathcal{A} \end{pmatrix}$$

with ${}_{\mathcal{B}}\mathcal{B}_{\mathcal{A}}$ viewed as $(\mathcal{B}, \mathcal{A})$-bimodule.

$\rightsquigarrow$ $\mathrm{rad}^2(\mathcal{D}) = 0$, $\mathcal{D}$ is the tensor algebra of a species $\mathbb{S} = \mathbb{S}(\Lambda, \Gamma)$ and $\mathrm{mod}(\mathcal{D}) \cong \mathrm{rep}(\mathbb{S})$.

We write $\mathcal{D}$-modules as triples (U, V, φ) where

- U is a $\mathcal{A}$-module,

- V is a $\mathcal{B}$-module,
- $\varphi\colon \mathcal{B} \otimes_k U \to V$ is a $\mathcal{B}$-module homomorphism.

Theorem 3.6 (Ringel–Roggenkamp). *The functor $F\colon \mathrm{CM}(\Lambda) \to \mathrm{mod}(\mathcal{D})$ induced by*

$$M \mapsto (M/\operatorname{rad}(\Lambda)M,\, \Gamma M/\operatorname{rad}(\Gamma)M,\, \varphi)$$

where φ is induced by the natural inclusion $M \hookrightarrow \Gamma M \subseteq A \otimes_\Lambda M$ is a representation equivalence between $\mathrm{CM}(\Lambda)$ and the category $\mathcal{C}$ of all finitely generated $\mathcal{D}$-modules without simple direct summands.

Example 3.7.

ii)

$$\Lambda = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a-d, b, c \in \mathfrak{m} \right\} \qquad \Gamma = \begin{pmatrix} R & R \\ R & R \end{pmatrix}$$

$$\operatorname{rad}(\Lambda) = \operatorname{rad}(\Gamma) = \begin{pmatrix} \mathfrak{m} & \mathfrak{m} \\ \mathfrak{m} & \mathfrak{m} \end{pmatrix}$$

$$\mathcal{A} = D_1 \qquad \mathcal{B} = M_2(D_2) \qquad D_1 = D_2 = k$$

$$S_2 = \begin{pmatrix} k \\ k \end{pmatrix}$$

$${}_1S_2 = D_1 \otimes_k \begin{pmatrix} k \\ k \end{pmatrix} = k \oplus k$$

$$\rightsquigarrow \text{ valued graph } 1 \xrightarrow{(2,2)} 2$$

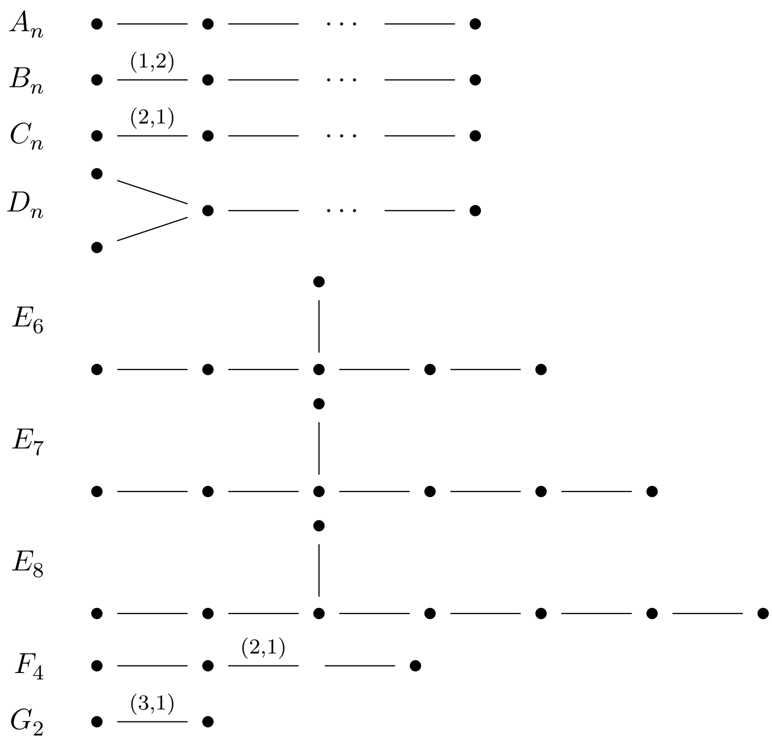
iii)

$$\Lambda = \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) : a-a', b-b', c, c' \in \mathfrak{m},\ d, d' \in R \right\}$$

$$\Gamma = \begin{pmatrix} R & R \\ \mathfrak{m} & R \end{pmatrix} \times \begin{pmatrix} R & R \\ \mathfrak{m} & R \end{pmatrix}$$

$$\rightsquigarrow \text{ valued graph } \quad 3 \leftarrow 1 \rightarrow 4 \qquad 5 \leftarrow 2 \rightarrow 6 \qquad A_3 \amalg A_3$$

Theorem 3.8 (Dlab–Ringel). *A tensor algebra $\mathcal{D}$ is of finite type iff the associated valued graph is a finite union of Dynkin diagrams, i.e. one of the following:*



Theorem 3.9 (Ringel—Roggenkamp). $\mathrm{CM}(\Lambda)$ *is of finite type iff:*

$$\left\{ \begin{array}{l} \textit{isoclasses of} \\ \textit{ind. } \Lambda\textit{-lattices} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \textit{non-simple positive roots} \\ \textit{of an associated root system} \end{array} \right\}$$

Remark 3.10.

- Finiteness doesn't depend on R.
- The indecomposable Λ-lattices are determined uniquely by ΓM and $M/\operatorname{rad}(M)$.

Proposition 3.11. *For arbitrary R-orders Λ a necessary condition to be of finite type is that the associated valued graph is a disjoint union of Dynkin diagrams.*
(Λ is contained in a Bäckström order.)

Aim. Understand indecomposable Λ-lattices and the AR-quiver.

The AR-species of $\mathcal{D}$ has as its vertices the isoclasses of indecomposable $\mathcal{D}$-modules and irreducible maps correspond to valued edges.

We need more data: Denote by P_j the indecomposable projective Γ-modules, by σ the permutation of $\{s+1,\dots,t\}$ with $\operatorname{rad}(P_j) = P_{\sigma(j)}$, by S_j the simple projective $\mathcal{D}$-modules with $s+1 \leq j \leq t$. Define $\phi(S_j) = Q_j$ iff the $\mathcal{D}$-socle of the indecomposable non-simple injective $\mathcal{D}$-module Q_j is $\mathcal{D}$-isomorphic to S_j.

Example 3.12. Continuation of iii) above: $\sigma = (3\ 5)(4\ 6)$

$$\phi(S_3) = Q_3 \qquad \phi(S_4) = Q_4 \qquad \phi(S_5) = Q_5 \qquad \phi(S_6) = Q_6$$

Theorem 3.13 (Roggenkamp)**.** *Let Λ be a Bäckström order with tensor algebra $\mathcal{D}$. Then the AR-species of* $\mathrm{CM}(\Lambda)$ *is obtained from that of $\mathcal{D}$ by*

- *deleting all simple injective $\mathcal{D}$-modules S_i with $1 \leq i \leq s$ (and arrows ending there),*
- *identifying Q_j with $S_{\sigma(j)}$ where $\phi(\mathcal{S}_j) = Q_j$.*

Example 3.14. Continuing iii), delete S_1, S_4 and identify along dotted lines:

$$\begin{tikzcd}
S_3 \arrow[dr] \arrow[rrrrrr, bend left, dotted, no head] & & \overline{Q}_3 \arrow[dr] \arrow[rr, bend left, dotted, no head] & & S_5 \arrow[dr] & & \overline{Q}_5 \arrow[dr] & \\
& P_1 \arrow[ur] \arrow[dr] & & S_1 & & P_2 \arrow[ur] \arrow[dr] & & S_4 \\
S_4 \arrow[ur] \arrow[rrrrrr, bend right, dotted, no head] & & \overline{Q}_4 \arrow[ur] \arrow[rr, bend right, dotted, no head] & & S_6 \arrow[ur] & & \overline{Q}_6 \arrow[ur] &
\end{tikzcd}$$

4 Tiled orders

Monday 12th 17:00 – Jan Geuenich (Bielefeld, Germany)

1

Tiled orders

R complete DVR with
- quotient field K
- residue field k

§1 Preliminaries

Def. • An R-lattice in a f.d. K-vsp. V is a f.g. R-submodule of V.
It's full if it generates V as a K-vsp.
• An R-order in a f.d. K-alg. A is a full R-lattice in A that is a subring of A.

Motivation

Study R-orders in $A = KG$ for finite groups G where $R = \mathbb{Z}_p$.
(⇝ Plesken's "Group Rings of Finite Groups Over p-adic Integers")

Def. An R-order Λ in a f.d. separable K-alg. A is said to be tiled if $e_i \Lambda e_i$ are maximal R-orders in $e_i A e_i$ for a complete set $e_1, \dots, e_n$ of primitive orthogonal idempotents of Λ.

Reduce (w.l.o.g.) to

$A = M_n(D)$ for a f.d. K-division algebra D

Facts • D contains a unique maximal R-order Δ (= integral closure of R in D).
• Δ is a noncommutative DVR, i.e. a local left and right PID with $\mathfrak{p} := \operatorname{rad}(\Delta) \neq 0$.
• $\mathfrak{p} = t\Delta = \Delta t$ for some $t \in \Delta$.

§2 Classification of Tiled Orders

Notation For $M = (m_{ij}) \in M_r(\mathbb{Z})$, $\underline{n} = (n_1, \dots, n_r) \in \mathbb{Z}_{>0}^r$ with $n_1 + \dots + n_r = n$:

$$\Lambda_{M,\underline{n}} := \left(M_{n_i, n_j}(\mathfrak{p}^{m_{ij}}) \right) \subseteq A$$

$$\Lambda_M := \Lambda_{M,(1,\dots,1)}$$

Theorem [Zassenhaus, Plesken, Rump]

(a) $\Lambda_{M,\underline{n}}$ tiled R-order in A $\iff$ $\begin{cases} ① \; m_{ij} + m_{jk} \geq m_{ik} \\ ② \; m_{ij} + m_{ji} \geq m_{ii} = 0 \end{cases}$ $\forall i,j,k$

(b) A tiled R-order $\Lambda_{M,\underline{n}}$ is basic iff ② holds with strict inequality and $\underline{n} = (1, \dots, 1)$.

(c) There is a bijection between the set of isoclasses (= conjugacy classes) of tiled R-orders in A and the set of equivalence classes of triples $(r, \underline{n}, \underline{m})$ with $\underline{n} = (n_1, \dots, n_r) \in \mathbb{Z}_{>0}^r$, $\underline{m} = (m_{ijk}) \in \mathbb{Z}^{r \times r \times r}$ satisfying $n_1 + \dots + n_r = n$ and

$$\begin{array}{llll} \boxed{1} & m_{ijk} = m_{ij\ell} + m_{jk\ell} - m_{ik\ell} & \geq 0 & \forall i,j,k \\ \boxed{2} & m_{iji} & > 0 & \forall i \neq j \\ \boxed{3} & m_{iii} & = 0 & \forall i \end{array}$$

where $(r, \underline{n}, \underline{m}) \sim (r', \underline{n}', \underline{m}')$ iff $r = r'$ and $n'_i = n_{\pi(i)}$, $m'_{ijk} = m_{\pi(i)\pi(j)\pi(k)}$ for some permutation π of $\{1, \dots, r\}$. ②

The correspondence is given by:

$$\Lambda_{M, \underline{n}} \longmapsto (r, \underline{n}, \underline{m}) \quad \text{with } m_{ijk} = m_{ij} + m_{jk} - m_{ik}$$

$$\Lambda_{(m_{ijk})_{i,j}, \underline{n}} \longleftarrow\!\!\shortmid \; (r, \underline{n}, \underline{m})$$

for any fixed k

<u>Facts</u> Let $\Lambda = \Lambda_M$ be a basic tiled R-order in A with $M \in M_n(\mathbb{Z}_{\geq 0})$.

[Jategaonkar] • $\operatorname{gldim} \Lambda < \infty \implies \left(m_{ij} \leq n-1 \ \forall i,j \text{ and } M = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \implies \operatorname{gldim} \Lambda \leq n-1 \right)$

[Kirichenko] • Λ 1-Iwanaga-Gorenstein $\iff m_{ij\pi(i)} = 0 \ \forall i,j$ for some permutation π

<u>Example</u>

• $\Lambda_{\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}} \cong \Lambda_{\begin{pmatrix} 0 & a+b \\ 0 & 0 \end{pmatrix}}$ $\forall a, b \in \mathbb{Z}$ with $a + b > 0$ is 1-Iwanaga-Gorenstein.

It has finite global dimension iff it is hereditary iff $a + b = 1$.

§3 <u>CM modules</u>

Λ basic R-order in A

$\operatorname{mod}(\Lambda)$ category of f.g. Λ-modules

$CM(\Lambda) = \{ X \in \operatorname{mod}(\Lambda) : X \ R\text{-lattice} \}$

$S = (D \cdots D)$ simple A-module

$M = (m_{ij})$ satisfying ① and satisfying ② strictly

$\mathcal{L}_\Lambda = \{ X \in CM(\Lambda) : 0 \neq X \subseteq S \}$

$P_\Lambda = \mathcal{L}_\Lambda \cap \operatorname{proj}(\Lambda)$

$\mathcal{L}_M = \{ (m_1, \dots, m_n) \in \mathbb{Z}^n : m_i + m_{ij} \geq m_j \}$ ordered by $\underline{m} \leq \underline{m}' \iff m_i \geq m'_i \ \forall i$

$P_M = \{1, \dots, n\} \times \mathbb{Z}$ ordered by $(i,a) \leq (j,b) \iff m_{ji} \leq a - b$

<u>Remark</u> $\mathbb{Z}$ acts on P_Λ and $\mathcal{L}_\Lambda$ by multiplication with t, on $\mathcal{L}_M$ by addition of $(1, \dots, 1)$ and on P_M by addition of 1 in the second component.

<u>Lemma</u> There is a commutative diagram of $\mathbb{Z}$-equivariant maps for $\Lambda = \Lambda_M$:

$$(i,0) \longmapsto e_i\Lambda =: P_i$$

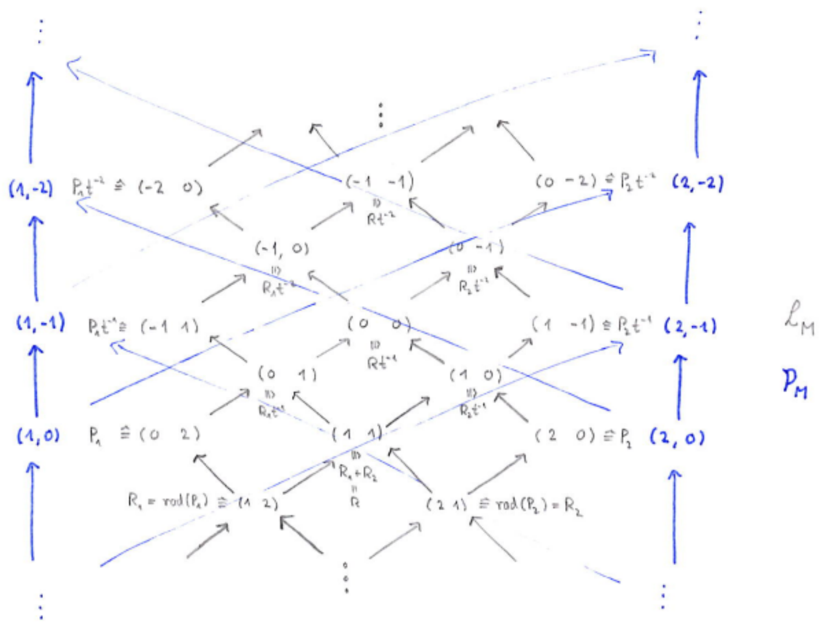
$$\begin{tikzcd}
(i,0) \arrow[d, mapsto] & P_M \arrow[r, "\simeq"] \arrow[d, hook] & P_\Lambda \arrow[d, hook] \\
e_iM & \mathcal{L}_M \arrow[r, "\simeq"] \arrow[d, two heads] & \mathcal{L}_\Lambda \arrow[d, two heads] \\
& \mathcal{L}_M/\mathbb{Z} \arrow[r, "\simeq"] & \mathcal{L}_\Lambda/\cong
\end{tikzcd}$$

$$\underline{m} \longmapsto S_{\underline{m}} := (p^{m_1} \cdots p^{m_n})$$

The upper two horizontal maps are isomorphisms of posets.

3

Example $\Lambda = \Lambda_M$ with $M = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$



Theorem [Plesken, Rump] For R-orders Λ in A t.f.a.e.:

(1) Λ is tiled in A.

(2) Λ is an intersection of max. R-orders in A and ℓ_Λ forms a distributive lattice.

(3) ——— " ——— and the Λ-ideals form —— " ——.

Fact A tiled R-order Λ in A is hereditary iff $\mathcal{P}_\Lambda$ is a chain.

§4 Finite-Type Classification

$\Lambda = \Lambda_M$ basic R-order in A

Def. $\operatorname{rep}^\circ(\mathcal{P}_M^{op})$ is the category with

- objects $V = (V_p)_{p \in \mathcal{P}_M \cup \{\infty\}}$ where V_p are subspaces of a f.d. k-vsp. V_∞
  - $V_p = 0 \quad \forall p = (i,a)$ with $a \ll 0$
  - $V_p \subseteq V_q \quad \forall p \gg q$ in $\mathcal{P}_M$
  - $V_p = V_\infty \quad \forall p = (i,a)$ with $a \gg 0$
- morphisms k-linear maps $V_\infty \xrightarrow{f} V'_\infty$ s.th. $f(V_p) \subseteq V'_p \;\; \forall p$

Remark $\operatorname{rep}^\circ(\mathcal{P}_M^{op})$ is Krull-Schmidt and has an autoequivalence σ given on objects V by

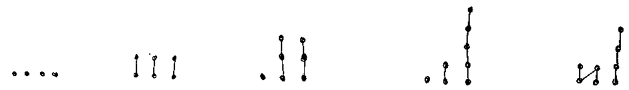
$$(\sigma V)_{(i,a)} = V_{(i,a-1)} .$$

<u>Theorem</u> [Zavadskij-Kirichenko] T.f.a.e.:

(1) $CM(\Lambda)$ is of finite type.

(2) $\mathrm{rep}^\circ(\mathcal{P}_M^{op})$ is of finite type up to the $\mathbb{Z}$-action of σ.

(3) $\mathcal{P}_M$ contains none of the following as a full subposet:



"KLEINER'S CRITICAL FIVE"

<u>Corollary</u> $n \leq 2 \Rightarrow CM(\Lambda)$ has finite type.

<u>Remark</u> The proof in [ZK] uses matrix problems and "differentiation of posets".

§5 Idea of proof for $R = \Delta = k[[t]]$ and $K = k((t))$

<u>Def.</u> $\mathrm{Latt}(M)$ is the category with

- <u>objects</u> $X = (X_1, \dots, X_n)$ tuples of full R-lattices in $KX_1 = \dots = KX_n$ satisfying $X_i t^{m_{ij}} \subseteq X_j \quad \forall i,j$,
- <u>morphisms</u> $F = (f_1, \dots, f_n)$ where the f_i are morphisms $X_i \to X_i'$.

<u>Lemma</u> There is an equivalence of categories

$$G: \mathrm{Latt}(M) \xrightarrow{\simeq} CM(\Lambda)$$
$$X \longmapsto \tilde{X} := \bigoplus_{i=1}^{n} X_i$$

where $C \in \Lambda$ acts on $x \in \tilde{X}$ as $(xC)_j = \sum_i x_i c_{ij}$.

<u>Def.</u> Denote by π the functor obtained by precomposing G with

$$F: \mathrm{rep}^\circ(\mathcal{P}_M^{op}) \longrightarrow \mathrm{Latt}(M)$$
$$V \longmapsto F(V) \quad \text{with } F(V)_i = \prod_{a \in \mathbb{Z}} V_{(i,a)}$$

where t maps $x \in F(V)_i$ to $(x_{a-1})_{a \in \mathbb{Z}}$.

<u>Theorem</u> [Roggenkamp-Wiedemann]

(a) $\pi \cong \pi \circ \sigma$

(b) On objects π preserves indecomposability and reflects projectivity and injectivity.

(c) On morphisms π preserves irreducibility.

(d) Denoting by $\Gamma_{\mathcal{P}_M}$ and Γ_Λ the AR-quivers of $\mathrm{rep}^\circ(\mathcal{P}_M^{op})$ and $CM(\Lambda)$, resp., π induces a morphism of translation quivers

$$\Gamma_{\mathcal{P}_M}/\sigma \xrightarrow{\pi_*} \Gamma_\Lambda$$

whose image is a union of connected components.

<u>Corollary</u> If $CM(\Lambda)$ has finite type, then Π_* is an isomorphism.

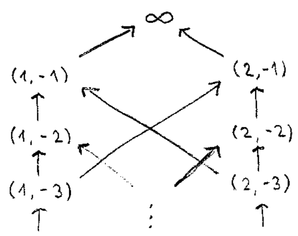
<u>Remark</u> All in all, it's easy to decide algorithmically

① whether $CM(\Lambda)$ has finite type and in this case

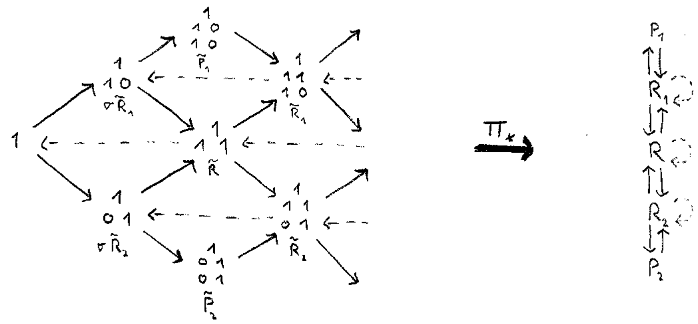
② compute Γ_Λ by knitting Γ_{P_M}.

<u>Example</u> $\Lambda = \Lambda_M$ with $M = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

① Consider the poset $\mathcal{P}^{op}_{M,-}$ obtained from $\mathcal{P}^{op}_M$ by contracting the "positive" vertices to a single vertex ∞:



② Knit the AR-quiver of $\mathrm{rep}^\circ(\mathcal{P}^{op}_{M,-})$:



<u>Remark</u> [Simson]

If k is algebraically closed, there is a notion for $CM(\Lambda)$ to be of tame type (of polynomial growth). Using the functor Π one can prove then the equivalence of:

(1) $CM(\Lambda)$ has tame type of polynomial growth.

(2) P_M contains none of "NAZAROVA-ZAVADSKIJ'S HYPERCRITICAL SEVEN" as a full subposet.

A complete classification when $CM(\Lambda)$ has tame type is unknown.

5 Commutative CM-finite type of dimension 0 and 1

Tuesday 13th 9:30 – William Crawley–Boevey (Bielefeld, Germany)

Setting.

- R commutative noetherian local ring $(R, \mathfrak{m}, k)$

5.1 Dimension 0

- $\dim(R) = 0 \Leftrightarrow R$ artinian
- All finitely generated modules are CM.

Theorem 5.1. *R has finite representation type $\Leftrightarrow$ R is a principal ideal ring.*

Proof. If $\mathfrak{m}/\mathfrak{m}^2$ has dimension 1 over k, then R is a principal ideal ring.

If $\mathfrak{m}/\mathfrak{m}^2$ has dimension ≥ 2, reduce to the case $\mathfrak{m}^2 = 0$ and $\dim(\mathfrak{m}) = 2$.

Then it looks like $k[x, y]/(x, y)^2$, i.e. it is given by the quiver with one vertex and two loops x and y subject to the relations $xy = yx = 0$ and $x^2 = y^2 = 0$, whose representation theory is essentially equivalent to the one of the Kronecker quiver. $\square$

5.2 Dimension 1

- Henceforth $\dim(R) = 1$.
- For finitely generated R-modules M:

$$\begin{aligned} & M \text{ is CM} \\ \Leftrightarrow\ & \text{there is } x \in \mathfrak{m} \text{ which is a non-zero divisor on } M, \text{ i.e. } xm = 0 \Rightarrow m = 0 \\ \Leftrightarrow\ & \operatorname{Hom}(k, M) = 0, \text{ i.e. } M \text{ doesn't have a copy of } k \text{ as a submodule} \end{aligned}$$

- If R is reduced (i.e. it has no nilpotent elements), then M is CM iff M is torsion-free.
- Total quotient ring $K = \{\text{non-zero divisors in } R\}^{-1}R$.
- $K \neq R \Leftrightarrow$ there exists a non-zero divisor which is not a unit $\Leftrightarrow$ R is CM
- $\overline{R}$ = integral closure of R in K

The following implications hold:

$$\begin{tikzcd}
 & & & \mu_R(\overline{R}) = e(R) & \\
{}_R\overline{R} \text{ fin. gen.} \arrow[rrr, equal] & & & \cdot \arrow[u, Rightarrow] & \widehat{R} \text{ CM} \arrow[dd, Leftrightarrow, "\text{cf. Matsumura}"] \\
 & & \cdot \arrow[ur, equal] \arrow[dll, Rightarrow] \arrow[drr, equal] & & \\
\widehat{R} \text{ reduced} \arrow[uu, Rightarrow] \arrow[rrr, Rightarrow, "\text{cf. stacks project 032Y}"'] & & & R \text{ reduced} \arrow[uu, equal] \arrow[r, Rightarrow, shift left] & R \text{ CM} \arrow[l, Rightarrow, shift left, "/" marking, "\text{e.g. } k[[x,y]]/(y^2)"]
\end{tikzcd}$$

("R analytically unramified")

Example 5.2.

- Non-example: $k[x,y]/(xy)$, i.e. it is given by the quiver with two loops x and y and relations $xy = yx = 0$.
- Example: $R = k[[x,y]]/(xy)$.

Definition 5.3. *A finite birational extension of R is a ring S with $R \subseteq S \subseteq K$ and ${}_RS$ finitely generated.*

Proposition 5.4. *In the situation of the definition with R, S 1-dimensional local rings:*

$$R \text{ CM-finite} \;\Rightarrow\; S \text{ CM-finite}$$

Definition 5.5. *An artinian pair is $A \hookrightarrow B$ with A and B commutative artinian rings and ${}_AB$ finitely generated.*

$$\mathrm{Rep}(A \hookrightarrow B) \;=\; \Big\{ {}_AV \hookrightarrow {}_BW \text{ of f.g. } A\text{-modules with } {}_BW \text{ proj. and } BV = W \Big\}$$

$$\begin{pmatrix} V \\ W \end{pmatrix} \text{ is a } \begin{pmatrix} A & 0 \\ B & B \end{pmatrix}\text{-module and } B \otimes_A V \to W \;\; B\text{-module homomorphism.}$$

Definition 5.6. *Let R be a CM ring and let S be a finite birational extension of R. Then the conductor C of R in S is the largest subset of R which is an ideal in S, i.e.*

$$C \;=\; \{r \in R : Sr \subseteq R\}\,.$$

Conductor square, a pullback diagram:

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d, two heads] & S \arrow[d, two heads, "\pi"] \\ A = R/C \arrow[r, hook] & S/C = B \end{tikzcd}$$

Example 5.7.

$$R \;=\; k[[x,y]/(xy) \;=\; \{(a,b) \in k[[x]] \oplus k[[y]] : a_0 = b_0\} \;\subseteq\; k[[x]] \oplus k[[y]]$$

$$K \;=\; k((x)) \oplus k((y))$$

$$\overline{R} \;=\; k[[x]] \oplus k[[y]]$$

$$C \;=\; xk[[x]] \oplus yk[[y]]$$

$$A = R/C = k \hookrightarrow B = \overline{R}/C = k \oplus k$$

$$\begin{pmatrix} A & 0 \\ B & B \end{pmatrix} \;=\; \begin{pmatrix} k & 0 \\ k^2 & k^2 \end{pmatrix}$$

$$\begin{tikzcd}[row sep=small] & \bullet \\ \bullet \arrow[ur] \arrow[dr] & \\ & \bullet \end{tikzcd}$$

$$\begin{tikzcd}[row sep=small] & k & & 0 & & k \\ k \arrow[ur] \arrow[dr] & & k \arrow[ur] \arrow[dr] & & k \arrow[ur] \arrow[dr] & \\ & k & & k & & 0 \end{tikzcd}$$

$$\text{NOT:}\quad \begin{tikzcd}[row sep=small] & k & & 0 \\ 0 \arrow[ur] \arrow[dr] & & k \arrow[ur] \arrow[dr] & \\ & 0 & & 0 \end{tikzcd}$$

- M CM R-module
- $M \to S \otimes_R M/\text{torsion} =: SM$
- $\widehat{R}$ reduced (then: R CM-finite $\Leftrightarrow$ $\mathrm{Rep}(A \hookrightarrow B)$ is representation-finite)
- $S = \overline{R}$

Theorem 5.8. *Let $(A \hookrightarrow B)$ and $(A, \mathfrak{m}, k)$ be as before. Then:*

$$(A \hookrightarrow B) \textit{ representation-finite } \Rightarrow \begin{array}{ll} \textit{(dr1)} & \dim_k(B/\mathfrak{m}B) \leq 3 \\ \textit{(dr2)} & \dim_k((\mathfrak{m}B + A)/(\mathfrak{m}^2 B + A)) \leq 1 \end{array}$$

If B is a principal ideal ring and if either $B/\operatorname{rad}(B)$ is separable over k or B is reduced, then $\Leftarrow$ holds.

Theorem 5.9. *$\widehat{R}$ reduced. Then:*

$$R \textit{ CM-finite } \Leftrightarrow \begin{array}{ll} \textit{(dr1)} & \mu_R(\overline{R}) \leq 3 \\ \textit{(dr2)} & \mu_R((\mathfrak{m}\overline{R} + R)/R) \leq 1 \end{array}$$

5.3 Simple plane curve singularities

- $k[[x, y]]/(f)$ with k algebraically closed of characteristic 0

Simple.

$$\big|\{\text{proper ideals } I \text{ in } k[[x, y]] \text{ with } f \text{ in } I^2\}\big| < \infty$$

f must be one of:

$$\begin{array}{lll} A_n & x^2 + y^{n+1} & \text{with } n \geq 1 \\ D_n & x^2 y + y^{n-1} & \text{with } n \geq 4 \\ E_6 & x^3 + y^4 & \\ E_7 & x^3 + xy^3 & \\ E_8 & x^3 + y^5 & \end{array}$$

Theorem 5.10 (Greuel–Knörrer)**.** *Let R be the complete local ring of a reduced curve singularity. Then:*

(i) *R CM-finite $\Leftrightarrow$ R finite birational extension of a simple plane curve singularity*

(ii) *R Gorenstein: R CM-finite $\Leftrightarrow$ R simple plane curve singularity*

6 Auslander-Reiten theory for lattices I

Tuesday 13^{th} 11:00 – Kunda Kambaso (Aachen, Germany)

Setting.

- R commutative noetherian with $\dim(R) = d$
- $\dim(X) = \dim(R/\operatorname{Ann}(X))$ for $X \in \operatorname{mod}(R)$
- $\operatorname{depth}(X) = \inf\{i \geq 0 : \operatorname{Ext}^i_R(R/\operatorname{rad}(R), X) \neq 0\}$
- $\operatorname{depth}(X) \leq \dim(R)$
- $\operatorname{CM}_i(R) = \{X \in \operatorname{mod}(R) : X \neq 0 \text{ and } \operatorname{depth}(X) = i = \dim(X)\}$
- $\operatorname{CM}(R) = \operatorname{CM}_d(R)$
- R <u>Gorenstein</u> $:\Leftrightarrow \operatorname{inj.dim}(R) < \infty$
- R <u>equidimensional</u> $:\Leftrightarrow \dim(R_{\mathfrak{m}}) = \dim(R)$
- R equidimensional
- Λ a noetherian R-algebra
- A Λ-module M is <u>CM</u> iff it is finitely generated and CM as an R-module.
- $\operatorname{CM}(\Lambda) = \{M \in \operatorname{mod}(\Lambda) : M \in \operatorname{CM}(R)\}$

Definition 6.1. *$M \in \operatorname{mod}(\Lambda)$ is called a <u>lattice</u> if $M \in \operatorname{CM}(\Lambda)$ and for non-maximal* $\mathfrak{p}$

(i) $M_{\mathfrak{p}}$ is a $\Lambda_{\mathfrak{p}}$-projective module,

(ii) $\operatorname{Hom}_R(M, R)_{\mathfrak{p}}$ is a $\Lambda^{\mathrm{op}}_{\mathfrak{p}}$-projective module.

Denote by $\mathcal{L}(\Lambda)$ the category of lattices, a subcategory of $\operatorname{noeth}(\Lambda)$.

Definition 6.2. *Λ is an R-order if $\Lambda \in \operatorname{CM}(\Lambda)$.*

Example 6.3.

(a) R field (0-dimensional Gorenstein ring):

R-orders are noetherian R-algebras and $\mathcal{L}(\Lambda) = \operatorname{CM}(\Lambda) = \operatorname{artin}(\Lambda)$.

(b) R equidimensional Gorenstein ring:

R is an R-order and R-lattices are all CM R-modules such that $M_{\mathfrak{p}}$ is $R_{\mathfrak{p}}$-free.

Let Λ be an R-order and $d = \dim(R)$.

Properties.

1. If Λ is an R-order, then so is Λ^{op}.
2. $M \in \mathcal{L}(\Lambda) \Rightarrow M$ is CM and $\mathrm{Hom}_R(M, R)$ is in $\mathcal{L}(\Lambda^{\mathrm{op}})$.
3. $\mathrm{Hom}_R(-, R) \colon \mathcal{L}(\Lambda) \to \mathcal{L}(\Lambda^{\mathrm{op}})$ is a duality.

Remark 6.4. $\mathcal{J}(\Lambda)$ the full subcategory of $\mathrm{noeth}(\Lambda)$ consisting of all M such that

(1) $M_{\mathfrak{p}}$ is $\Lambda_{\mathfrak{p}}$-projective for all non-maximal $\mathfrak{p}$,

(2) $\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$ for all $i = 1, \ldots, d$.

$\underline{\mathcal{J}}(\Lambda)$ is the full subcategory of $\underline{\mathrm{noeth}}(\Lambda)$ with objects in $\mathcal{J}(\Lambda)$.

There is a functor $\Omega : \mathrm{noeth}(\Lambda) \to \mathrm{noeth}(\Lambda)$, $M \mapsto \ker(P(M) \to M)$.
In general, Ω^0 the identity, $\Omega^{i+1} = \Omega \circ \Omega^i$, then $\Omega^d(M)$ is in $\mathcal{L}(\Lambda)$.
Ω^d induces a functor $\underline{\mathcal{J}}(\Lambda) \to \mathcal{L}(\Lambda)$, which is fully faithful.

Theorem 6.5. $\Omega^d \colon \underline{\mathcal{J}}(\Lambda) \to \underline{\mathcal{L}}(\Lambda)$ *is an equivalence.*

Theorem 6.6. *The duality* $\mathrm{Tr} \colon \underline{\mathrm{noeth}}(\Lambda) \to \underline{\mathrm{noeth}}(\Lambda^{\mathrm{op}})$ *induces the duality*

$$\mathrm{Tr} \colon \underline{\mathcal{L}}(\Lambda) \to \underline{\mathcal{J}}(\Lambda^{\mathrm{op}})\,.$$

Remark 6.7. We get $\mathrm{Tr}_{\mathcal{L}} : \underline{\mathcal{L}}(\Lambda) \to \underline{\mathcal{L}}(\Lambda^{\mathrm{op}})$ from $\underline{\mathcal{L}}(\Lambda) \xrightarrow{\mathrm{Tr}} \underline{\mathcal{J}}(\Lambda^{\mathrm{op}}) \xrightarrow{\Omega^d} \underline{\mathcal{L}}(\Lambda^{\mathrm{op}})$.

Proposition 6.8. $\mathrm{Tr}_{\mathcal{L}} : \underline{\mathcal{L}}(\Lambda) \to \underline{\mathcal{L}}(\Lambda^{\mathrm{op}})$ *and* $\mathrm{Tr}_{\mathcal{L}} : \underline{\mathcal{L}}(\Lambda^{\mathrm{op}}) \to \underline{\mathcal{L}}(\Lambda)$ *are inverse dualities.*

Definition 6.9.

(a) $\cdots \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to \cdots$ *in* $\mathcal{L}(\Lambda)$ *is* *<u>exact</u>* *if it is exact as a sequence of* Λ*-modules and* $\mathrm{im}(f_i)$ *in* $\mathcal{L}(\Lambda)$.

(b) C *in* $\mathcal{L}(\Lambda)$ *is* *<u>projective</u>* *if all exact* $0 \to A \to B \to C \to 0$ *split.*

$\mathcal{I}(\mathcal{L}(\Lambda))$ denotes the full subcategory of $\mathcal{L}(\Lambda)$ whose objects are injectives.
$\mathcal{I}(A, C)$ is the R-submodule of $\mathrm{Hom}_R(A, C)$ of morphisms factoring through $\mathcal{I}(\mathcal{L}(\Lambda))$.
Define $\overline{\mathcal{L}}(\Lambda)$ with $\overline{\mathrm{Hom}}_\Lambda(A, C) = \mathrm{Hom}_\Lambda(A, C)/\mathcal{I}(A, C)$.

Proposition 6.10.

(a) $\mathrm{Hom}_R(-, R) \colon \mathcal{L}(\Lambda) \to \mathcal{L}(\Lambda^{\mathrm{op}})$ *induces* $\underline{\mathcal{L}}(\Lambda) \to \overline{\mathcal{L}}(\Lambda^{\mathrm{op}})$.

(b) $\underline{\mathcal{L}}(\Lambda) \xrightarrow{\mathrm{Tr}_{\mathcal{L}}} \underline{\mathcal{L}}(\Lambda^{\mathrm{op}}) \xrightarrow{\mathrm{Hom}_R(-,R)} \overline{\mathcal{L}}(\Lambda)$ *is an equivalence of categories.*

Proposition 6.11. *Let* X, C *be in* $\mathcal{L}(\Lambda)$. *Then*

$$\mathrm{Ext}^1_\Lambda(C, X) \;\cong\; \mathrm{Hom}_R(\underline{\mathrm{Hom}}_\Lambda(\mathrm{Tr}_{\mathcal{L}}\, X^*, C), I_d)$$

is functorial in X *and* C. *We get*

$$\mathrm{Ext}^1_\Lambda(C, \mathrm{Hom}_R(\mathrm{Tr}_{\mathcal{L}}\, X, R)) \cong \mathrm{Hom}_R(\underline{\mathrm{Hom}}_\Lambda(X, C), I_d)\,.$$

Consequently:

Proposition 6.12. *Let C, X be in $\mathcal{L}(\Lambda)$ and $n \in \mathbb{Z}_{>0}$. Then there is*

$$\operatorname{Ext}^1_\Lambda(C, \operatorname{Hom}_R(\operatorname{Tr}_\mathcal{L} X, R^n)) \cong \operatorname{Hom}_R(\underline{\operatorname{Hom}}_\Lambda(X, C), I^n_d).$$

Let

$$x: 0 \longrightarrow \operatorname{Hom}_R(\operatorname{Tr}_\mathcal{L} X, R^n) \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$$

be an exact sequence in $\mathcal{L}(\Lambda)$ and $\nu : \underline{\operatorname{Hom}}_\Lambda(X, C) \to I^n_d$.

Theorem 6.13. *Let H be an R-submodule of (X, C) containing $\mathcal{P}(X, C)$ with $H/\mathcal{P}(X, C) = \ker(\nu)$. Then:*

(a)
$$\begin{aligned} & h\colon L \to C \text{ in } \mathcal{L}(\Lambda) \text{ can be written as } ft = h \text{ for some } t\colon L \to B \\ \Leftrightarrow \quad & \operatorname{im}(-, h)(X)\colon \operatorname{Hom}(X, L) \to \operatorname{Hom}(X, C) \subseteq H. \end{aligned}$$

(b) f is right X-determined in $\mathcal{L}(\Lambda)$ and $\operatorname{im}(-, f)(X)$ is a maximal $\operatorname{End}(X)^{\mathrm{op}}$-submodule.

Let $f\colon B \to C$. For all $f'\colon B' \to C$, then f' factors through f and for all $\phi\colon X \to B'$, $f'\phi$ factors through f:

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[dr] & B' \arrow[r, "f'"] \arrow[d, dashed] & C \arrow[d] \\ & X \arrow[r, "f"] & C \end{tikzcd}$$

(c) $\operatorname{im}(-, f)(X) = H \Leftrightarrow H$ is a Σ-submodule of (X, C) where $\Sigma = \operatorname{End}(X)^{\mathrm{op}}$.

Use $A \cong \operatorname{Hom}_R(\operatorname{Tr}_\mathcal{L}(\operatorname{Tr}_\mathcal{L} A)^*, R)$ to see that f is right $(\operatorname{Tr}_\mathcal{L} A)^*$-determined.

Theorem 6.14. *Let X, C be in $\mathcal{L}(\Lambda)$ and suppose $\coprod_{i=1}^k S_i^{n_i}$ is isomorphic to the socle of $(X, C)/H$ with S_i simple non-isomorphic Σ-modules, $n_i \in \mathbb{Z}_{>0}$. Let $n = \max\{n_1, \ldots .n_k\}$. There is an exact*

$$x: 0 \longrightarrow \operatorname{Hom}_R(\operatorname{Tr}_\mathcal{L} X, R^n) \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$$

satisfying

(a) $\operatorname{im}(-, f)(X) = H$ and f is right X-determined.

7 Auslander-Reiten theory for lattices II

Tuesday 13^{th} 14:00 – Jasper van de Kreeke (Amsterdam, Netherlands)

Auslander-Reiten ...

- sequences
- translates
- quivers
- duality

in the commutative and noncommutative setting.

Yesterday: $R = k[[x]]$ and Λ an arbitrary R-order.
Now: By convention, R is a commutative noetherian local complete Gorenstein ring which is an isolated singularity ($\rightsquigarrow$ AR-sequences exist).

Example 7.1.

- $R = k[[x]]$
- Kleinian singularities in all dimensions (A_1 surface singularity: $R = \mathbb{C}[[x^2, y^2, xy]]$)

Theorem 7.2 (Auslander '86)**.** *A local complete Gorenstein ring R has AR-sequences for all non-free indecomposable modules $M \in \mathrm{CM}(R)$ iff R is an isolated singularity.*

7.1 AR-sequences and AR-translates

Definition 7.3. *Let $M \in \mathrm{mod}(R)$ and let $P_1 \to P_0 \to M \to 0$ be a minimal projective presentation for M. Then $\mathrm{Tr}(M) := \mathrm{coker}(P_0^* \to P_1^*)$.*

Definition 7.4. *Let $M \in \mathrm{mod}(R)$ and $0 \to N \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ be exact with P_i finitely generated projective, then N is an n-th syzygy for M.*

The following is uniquely defined up to isomorphism

$$\mathrm{redsyz}^n(M) := \text{“}N \textit{ minus its free summands”}.$$

Definition 7.5. *Let $M \in \mathrm{CM}(R)$ be indecomposable. Then a short exact sequence*

$$0 \to N \to E \to M \to 0$$

with N indecomposable is an AR-sequence for M if $E \to M$ is not a split surjection and every $X \to M$ which is not a split surjection factors through $E \to M$.

$$\begin{tikzcd} & X \arrow[dl, dashed, "\exists"'] \arrow[d] \\ E \arrow[r] & M \end{tikzcd}$$

Theorem 7.6. *For any non-free indecomposable $M \in \mathrm{CM}(R)$ there exists a unique AR-sequence. In fact $N \cong \tau(M) := \mathrm{Hom}(\mathrm{redsyz}^n(\mathrm{Tr}(M)), R)$ where $n = \dim(R)$.*

Example 7.7. The theory is trivial for a regular ring, e.g. $\mathbb{C}[[x]]$.

Exercise 7.8. Check that $\mathrm{Tr}(\mathrm{Tr}(M)) \cong M$ for non-projective indecomposable M.

Remark 7.9. If $\dim(R) = 2$, $\tau(M) \cong M$ for non-projective indecomposable $M \in \mathrm{CM}(R)$. (Look at $0 \to M^* \to P_0^* \to P_1^* \to \mathrm{Tr}(M) \to 0$.)

7.2 AR-quivers

Definition 7.10. *Let $M = \bigoplus M_i$ and $N = \bigoplus N_i$ be decompositions into indecomposables of $M, N \in \mathrm{CM}(R)$. Define*

$$\begin{aligned} \mathrm{rad}(M,N) &:= \{(\varphi_{ij}) : \textit{each } \varphi_{ij} : M_j \to N_i \textit{ not an isomorphism}\}\,, \\ \mathrm{rad}^2(M,N) &:= \{f \circ g : g \in \mathrm{rad}(M,X), f \in \mathrm{rad}(X,N)\}\,, \\ \mathrm{Irr}(M,N) &:= \mathrm{rad}(M,N)/\,\mathrm{rad}^2(M,N)\,. \end{aligned}$$

Exercise 7.11. Check that $\mathrm{rad}(M,N)$ and $\mathrm{rad}^2(M,N)$ are R-modules. In fact $\mathrm{Irr}(M,N)$ becomes a k-vector space where $k = R/\mathfrak{m}$.

Definition 7.12. *Assume k is algebraically closed. Then the AR-quiver Q_R has*

- *vertices M for all indecomposable CM modules M,*
- $\dim_k \mathrm{Irr}(M,N)$ *many arrows from M to N,*
- *remembers the AR-translates.*

Theorem 7.13. *The AR-quiver of a 2-dimensional Kleinian singularity (over $\mathbb{C}$) is the McKay double quiver.*

Example 7.14. A_1 case ($\mathbb{C}^2/\mathbb{Z}_2$): We have $R = \mathbb{C}[[x^2, y^2, xy]]$. Then R and $M = Rx + Ry$ (= power series in odd degrees) are the 2 indecomposable CM modules and

$$\begin{aligned} \mathrm{Hom}_R(R,R) &= R \\ \mathrm{Hom}_R(R,M) &= M \\ \mathrm{Hom}_R(M,R) &= M \\ \mathrm{Hom}_R(M,M) &= R \\ \mathrm{rad}(R,R) &= \{x \in R \text{ with constant term } 0\} \\ \mathrm{rad}^2(R,R) &= \{x \in R \text{ starting } x^2,\, y^2,\, xy \text{ onward}\} \\ \mathrm{Irr}(R,R) &= 0 \\ \mathrm{Irr}(R,M) &= \mathbb{C}x + \mathbb{C}y \\ \mathrm{Irr}(M,R) &= \mathbb{C}x + \mathbb{C}y \\ \mathrm{Irr}(M,M) &= 0 \end{aligned}$$

So the AR-quiver is

$$\begin{tikzcd} R \arrow[r, shift left=2] \arrow[r, shift left=1] & M \arrow[l, shift left=1] \arrow[l, shift left=2] \arrow[loop right, dashed] \end{tikzcd} .$$

Exercise 7.15. Check that $\operatorname{redsyz}^2(M) = M$ using $0 \to \ker \to R^{\oplus 2} \to R^{\oplus 2} \to M \to 0$.

7.3 AR-duality

Theorem 7.16. *We have*

$$\begin{aligned} & \operatorname{Hom}_R(\tau^{-1}(Y), X)/\{\textit{maps factoring through some } R^{\oplus n}\} \\ = \; & \operatorname{Hom}_R(Y, \tau(X))/\{\textit{maps factoring through some } R^{\oplus n}\} \\ = \; & \operatorname{Ext}^1(X, Y)^* \,. \end{aligned}$$

Example 7.17. "cluster category": $D^b(\operatorname{mod}(A_4))/\tau \cong \underline{\mathrm{CM}}$ (2-dimensional A_n)

Exercise 7.18. Using this equality check AR-duality.

7.4 Noncommutative case

Let R be a local complete noetherian commutative Gorenstein ring.
Let Λ be an R-order (i.e. $\Lambda \in \mathrm{CM}(R)$). Then we have $\mathrm{CM}(\Lambda)$.

Fact 7.19. $\mathrm{CM}(\Lambda)$ *finite type* $\Rightarrow$ Λ *isolated singularity, i.e.* $\operatorname{gl.dim}(\Lambda \otimes_R R_{\mathfrak{p}}) = \operatorname{gl.dim}(R_{\mathfrak{p}})$.

Fact 7.20. *AR-sequences exist* $\Leftrightarrow$ Λ *isolated singularity*

Fact 7.21. *AR-duality for isolated singularities* Λ*:*

$$\begin{aligned} & \operatorname{Hom}_\Lambda(\tau^{-1}(Y), X)/\{\textit{maps factoring through some } \Lambda^{\oplus n}\} \\ = \; & \operatorname{Hom}_\Lambda(Y, \tau(X))/\{\textit{maps factoring through some } (\Lambda^*)^{\oplus n}\} \,. \end{aligned}$$

8 Auslander-Buchweitz approximations

Tuesday 13th 15:15 – Manuel Flores Galicia (Bielefeld, Germany)

(following notes by Ryo Kanda)

Setting.

- R Iwanaga-Gorenstein (left and right noetherian and $\operatorname{inj.dim}({}_RR)$ and $\operatorname{inj.dim}(R_R)$ finite, actually then $\operatorname{inj.dim}({}_RR) = \operatorname{inj.dim}(R_R)$)
- For $M \in \operatorname{Mod}(R)$ there exists a short exact sequence $0 \to K \to N \to M \to 0$ where $N \in \operatorname{CM}(R)$ and K has finite projective dimension.

8.1 Approximations and cotorsion pairs

- $\mathcal{B}$ additive category
- $\mathcal{X} \subseteq \mathcal{B}$ closed under finite sums and direct summands and extensions

Definition 8.1. *A morphism $f\colon X \to M$ with $X \in \mathcal{X}$ is a* <u>*right $\mathcal{X}$-approximation*</u> *of M iff for every $f'\colon X' \to M$ with $X' \in \mathcal{X}$ the map $\operatorname{Hom}(\mathcal{X}, X) \to \operatorname{Hom}(X', M)$ is surjective:*

$$\begin{tikzcd} X' \arrow[rr, "f'"] \arrow[dr, dashed, "\exists"'] & & M \\ & X \arrow[ur, "f"'] & \end{tikzcd}$$

$\mathcal{X}$ is said to be <u>*contravariantly finite*</u> *in $\mathcal{B}$ if every $B \in \mathcal{B}$ has a right $\mathcal{X}$-approximation.*

Dually, define <u>*left $\mathcal{X}$-approximations*</u> *and* <u>*covariantly finite*</u>*.*

Proposition 8.2. *$\mathcal{X} \subseteq \mathcal{B}$ and $0 \to Y \to X \xrightarrow{f} M \to 0$ a short exact sequence with $X \in \mathcal{X}$. If $\operatorname{Ext}^1(\mathcal{X}, Y) = 0$, then f is a right $\mathcal{X}$-approximation.*

Definition 8.3. *$\mathcal{B} \subseteq \mathcal{A}$ with $\mathcal{A}$ abelian, $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{B}$ additive.*

We say that $(\mathcal{X}, \mathcal{Y})$ is a <u>*cotorsion pair in $\mathcal{B}$ with special approximations*</u> *if:*

(a) $\operatorname{Ext}^i_{\mathcal{A}}(X, Y) = 0$ *for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$, $i > 0$,*

(b) for all $B \in \mathcal{B}$ there are

$$0 \to Y_B \to X_B \xrightarrow{f} B \to 0 \qquad \text{and} \qquad 0 \to B \xrightarrow{g} Y^B \to X^B \to 0$$

with $Y_B, Y^B \in \mathcal{Y}$ and $X_B, X^B \in \mathcal{X}$.

Proposition 8.4. *Proposition 8.2 $\Rightarrow$ f is a right $\mathcal{X}$-approximation of B and g is a left $\mathcal{Y}$-approximation of B.*

Definition 8.5. *For $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{A}$ let*

$$\begin{aligned} \mathcal{X}^{\perp} &:= \{M \in \mathcal{A} : \operatorname{Ext}^{>0}_{\mathcal{A}}(\mathcal{X}, M) = 0\}, \\ {}^{\perp}\mathcal{Y} &:= \{M \in \mathcal{A} : \operatorname{Ext}^{>0}_{\mathcal{A}}(M, \mathcal{Y}) = 0\}. \end{aligned}$$

Definition 8.6. *$\omega \subseteq \mathcal{X}$ is a* <u>*cogenerator*</u> *of $\mathcal{X}$ if for all $X \in \mathcal{X}$ there exists a short exact sequence $0 \to X \to W \to Y \to 0$ with $W \in \omega$ and $Y \in \mathcal{X}$.*

ω is an <u>*injective cogenerator*</u> *if $\operatorname{Ext}^{>0}(\mathcal{X}, \omega) = 0$.*

Recall 8.7. *$\omega \subseteq \mathcal{A}$ additive. Then $\mathcal{A}/\omega$ has the same objects as $\mathcal{A}$ and morphisms*

$$\operatorname{Hom}_{\mathcal{A}/\omega}(M, N) \; = \; \operatorname{Hom}_{\mathcal{A}}(M, N) \Big/ \left\{ M \xrightarrow{f} N : f \text{ factors through some } W \in \omega \right\}.$$

Proposition 8.8. *Let $(\mathcal{X}, \mathcal{Y})$ be a cotorsion pair in $\mathcal{B}$ and $\omega = \mathcal{X} \cap \mathcal{Y}$. Then:*

(1) $\mathcal{Y} = \mathcal{X}^{\perp} \cap \mathcal{B} = \mathcal{X}^{\perp_1} \cap \mathcal{B}$ and $\mathcal{X} = {}^{\perp}\mathcal{Y} \cap \mathcal{B} = {}^{\perp_1}\mathcal{Y} \cap \mathcal{B}$ and $\omega = \mathcal{X} \cap \mathcal{X}^{\perp} = \mathcal{Y} \cap {}^{\perp}\mathcal{Y}$,

(2) ω is an injective cogenerator of $\mathcal{X}$,

(3) for all $f \colon X \to Y$ with $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ we have $f = 0$ in $\mathcal{A}/\omega$,

(4) X_B, Y^B are unique up to isomorphism in $\mathcal{A}/\omega$.

Proof. (1) $\mathcal{Y} \subseteq \mathcal{X}^{\perp} \cap \mathcal{B} \subseteq \mathcal{X}^{\perp_1} \cap \mathcal{B}$.

If $B \in \mathcal{X}^{\perp_1} \cap \mathcal{B}$, then $0 \to B \to Y^B \to X^B \to 0$ splits, so B is a direct summand of Y^B, so $B \in \mathcal{Y}$. Thus $\mathcal{Y} = \mathcal{X}^{\perp_1} \cap \mathcal{B}$ and $\omega = \mathcal{X} \cap \mathcal{Y} = \mathcal{X} \cap \mathcal{X}^{\perp_1} \cap \mathcal{B} = \mathcal{X} \cap \mathcal{X}^{\perp}$.

(2) For all $X \in \mathcal{X}$ there is $0 \to X \to Y^X \to X^X \to 0$ with $Y^X \in \mathcal{Y}$ and $X^X \in \mathcal{X}$. Since $\mathcal{X}$ is closed under extensions we get $Y^X \in \mathcal{X}$, so $Y^X \in \omega$.

Since $\omega \subseteq \mathcal{X}^{\perp}$ by (1) we get $\operatorname{Ext}^{>0}(\mathcal{X}, \omega) = 0$. □

8.2 Auslander-Buchweitz approximations

- $\mathcal{A}$ abelian category
- $\mathcal{X} \subseteq \mathcal{A}$ additive, closed under extensions and kernels of epimorphisms
- $\omega \subseteq \mathcal{X}$ additive, injective cogenerator of $\mathcal{X}$
- $\widehat{X} := \{M \in \mathcal{A} : \text{there is } n \text{ and an exact } 0 \to X_n \to \cdots \to X_0 \to M \to 0 \text{ with } X_i \in \mathcal{X}\}$

Theorem 8.9 (Auslander, Buchweitz)**.** *Under the above assumptions:*

(1) $(\mathcal{X}, \widehat{\omega}) \subseteq \widehat{\mathcal{X}}$ is a cotorsion pair,

(2) $\widehat{\mathcal{X}} = \{M \in \mathcal{A} : \exists\, 0 \to Y_M \to X_M \xrightarrow{p} M \to 0 \text{ with } Y_M \in \widehat{\omega} \text{ and } X_M \in \mathcal{X}\}$ and
$\widehat{\mathcal{X}} = \{M \in \mathcal{A} : \exists\, 0 \to M \xrightarrow{j} Y_M \to X_M \to 0 \text{ with } Y_M \in \widehat{\omega} \text{ and } X_M \in \mathcal{X}\}$,

(3) $\omega = \mathcal{X} \cap \mathcal{X}^{\perp}$.

Definition 8.10. *The morphism $p\colon X_M \to M$ is called an Auslander-Buchweitz approximation of M (or CM approximation) and $j\colon M \to Y_M$ is called an $\widehat{\omega}$-hull of M.*

Proof. Let $M \in \widehat{\mathcal{X}}$. Then there is an exact $0 \to X_n \to \cdots \xrightarrow{d_0} X_0 \to M \to 0$ with $X_i \in \mathcal{X}$.

If $n = 0$ take $0 \to 0 \to M \to M \to 0$ where $0 \in \widehat{\omega}$ and $M \in \mathcal{X}$. Now $\omega \subseteq \mathcal{X}$ cogenerating gives $0 \to M \to W \to X \to 0$ with $W \in \omega$ and $X \in \mathcal{X}$.

If $n \geq 1$, set $K = \operatorname{im}(d_0)$ and consider $0 \to K \to X_0 \to M \to 0$ and $0 \to X_n \to \cdots \to X_1 \to K \to 0$. By induction hypothesis there is $0 \to K \to Y^K \to X^K \to 0$ with $Y^K \in \widehat{\omega}$. Consider

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & Y^K \arrow[r] \arrow[d] & Y^K \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X_0 \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & X^K \arrow[r] & 0 \\
 & M \arrow[r, equal] \arrow[d] & M \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

The third column yields a right $\mathcal{X}$-approximation of M. □

8.3 Examples

1.

- R Iwanaga-Gorenstein
- $\mathcal{A} = \operatorname{mod}(R)$
- $\mathcal{X} = \operatorname{CM}(R) = {}^{\perp}R = \{M \in \operatorname{mod}(R) : \operatorname{Ext}^{>0}(M, R) = 0\}$
- $\omega = \operatorname{proj}(R)$ finitely generated projective R-modules
- $$\begin{tikzcd}
\operatorname{CM}(R) \arrow[r, shift left, "\sim"'] & \operatorname{CM}(R^{\mathrm{op}}) \arrow[l, shift left] \\
\operatorname{proj}(R) \arrow[u, hook, "\subseteq"] \arrow[r, shift left, "\sim"'] & \operatorname{proj}(R^{\mathrm{op}}) \arrow[u, hook, "\subseteq"] \arrow[l, shift left]
\end{tikzcd}$$
- $\Rightarrow M \cong M^{**}$ for all $M \in \operatorname{CM}(R)$, i.e. M is reflexive.
- $\operatorname{proj}(R) \subseteq \operatorname{CM}(R)$ is cogenerating.
- Let $M \in \operatorname{CM}(R)$. Then

$$0 \longrightarrow \Omega(M^*) \longrightarrow P_0' \longrightarrow M^* \longrightarrow 0$$

where $M^*, \Omega(M^*) \in \mathrm{CM}(R^{\mathrm{op}})$, $P_0' \in \mathrm{proj}(R^{\mathrm{op}})$ and

$$0 \longrightarrow M^{**} \longrightarrow (P_0')^* \longrightarrow (\Omega(M^*))^* \longrightarrow 0$$

where $(\Omega(M^*))^* \in \mathrm{CM}(R)$, $(P_0')^* \in \mathrm{proj}(R)$.

- $\mathcal{X} = \mathrm{CM}(R)$
- $\widehat{\mathcal{X}} = \mathrm{mod}(R)$:

  Let $M \in \mathrm{mod}(R)$. Since $n = \mathrm{inj.dim}(R_R) < \infty$, we have

  $$\mathrm{Ext}^{>n}(M, R) = \mathrm{Ext}^{>n-1}(\Omega(M), R) = \cdots = \mathrm{Ext}^{>0}(\Omega^n(M), R) = 0\,,$$

  so $\Omega^n(M) \in \mathrm{CM}(R)$. So we have

  $$0 \to \Omega^n(M) \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

  with $\Omega^n(M) \in \mathrm{CM}(R)$ and $P_i \in \mathrm{proj}(R) \subseteq \mathrm{CM}(R)$.
- $\widehat{\omega} = \widehat{\mathrm{proj}(R)} = \{M \in \mathrm{mod}(R) : \mathrm{proj.dim}(M) < \infty\} =: \mathcal{P}^{<\infty}$
- $\Rightarrow (\mathrm{CM}(R), \mathcal{P}^{<\infty}) \subseteq \mathrm{mod}(R)$ is a cotorsion pair.
- $\omega = \mathrm{CM}(R) \cap \mathcal{P}^{<\infty} = \mathrm{proj}(R)$

2.

- R commutative noetherian local Cohen-Macaulay ring with canonical module ω_R (i.e. $\omega_R \in \mathrm{mod}(R)$ with $\mathrm{Ext}^{>0}(\omega_R, \omega_R) = 0$, $\mathrm{inj.dim}(\omega_R) < \infty$, $R \xrightarrow{\sim} \mathrm{End}(\omega_R)$)
- $\mathrm{CM}(R) := \{M \in \mathrm{mod}(R) : \mathrm{Ext}^{>0}(M, \omega_R) = 0\}$
- $\Rightarrow (\mathrm{CM}(R), \mathcal{I}^{<\infty})$ is a cotorsion pair in $\mathrm{mod}(R)$ and $\mathrm{CM}(R) \cap \mathcal{I}^{<\infty} = \mathrm{add}(\omega_R)$.

9 Algebraic McKay correspondence

Wednesday 14^{th} 8:30 – Sarah Kelleher (Glasgow, United Kingdom)

Goal.

- k field
- $G \leq \mathrm{GL}(n,k)$ finite with $|G|$ invertible in k and G having no pseudoreflections
- S polynomial ring or power series ring with G acting linearly
- $R = S^G$
- The natural morphism $\gamma\colon S\#G \to \mathrm{End}_R(S)$, $\gamma(s\cdot\sigma)(t) = s\sigma(t)$, is an isomorphism.

Example 9.1. $n = 2$, $S = \mathbb{C}[[x,y]]$.

Definition 9.2. *Invariant ring $R = S^G$ with $s \in R$ iff $\sigma(s) = s$ for all $\sigma \in G$.*

Example 9.3. $G = \frac{1}{3}(1,2) := \left\langle \begin{pmatrix} \varepsilon_3 & 0 \\ 0 & \varepsilon_3^2 \end{pmatrix} \right\rangle$ acting by $x \mapsto \varepsilon_3^2 x$, $y \mapsto \varepsilon_3 y$.

Then $x^3 \mapsto x^3$, $y^3 \mapsto y^3$, $xy \mapsto xy$. So $R = \mathbb{C}[[x^3,y^3,xy]] \cong \mathbb{C}[[a,b,c]]/(ab - c^3)$.

Definition 9.4. *Skew group ring $S\#G$, group homomorphism $\varphi\colon G \to \mathrm{Aut}(S)$, then*

$$S\#G := \left\{ \sum_{g\in G} a_g g : a_g \in S, g \in G \right\}$$

with multiplication $ag \cdot bh = a\varphi(g)(b)gh$ for $a,b \in S$, $g,h \in G$.

Example 9.5. $S\#G = S \otimes_{\mathbb{C}} \mathbb{C}G$, $(a\otimes g)(b \otimes h) = (a \cdot g(b)) \otimes gh$.

Theorem 9.6. *$G \leq \mathrm{SL}(n,\mathbb{C})$, $S = \mathbb{C}[[x_1,\dots,x_n]]$, $R = S^G$. Then:*

$$S\#G \cong \mathrm{End}_R\left(\bigoplus\nolimits_{p \in \mathrm{Irr}(G)} ((S\otimes p)^G)^{\dim(p)} \right)$$

Definition 9.7. *$\sigma \in \mathrm{GL}(n,k)$ is a pseudoreflection if $\mathrm{rank}(\sigma - 1) \leq 1$ for all $\sigma \neq \mathrm{id}$.*

Example 9.8. $G = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$

$$\begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & \varepsilon \end{pmatrix}$$

Example 9.9. There are 3 one-dimensional representations p_0, p_1, p_2.

g acts on e_i with e_0, e_1, e_2 by weight ε_3^i.

$$\begin{aligned} M_0 &= (\mathbb{C}[[x,y]] \otimes p_0)^G = R \\ M_1 &= (\mathbb{C}[[x,y]] \otimes p_1)^G \\ M_2 &= (\mathbb{C}[[x,y]] \otimes p_2)^G \end{aligned}$$

So

$$S\#G \cong \operatorname{End}_R(R \oplus M_1 \oplus M_2)\,.$$

McKay quiver

Definition 9.10. *G a finite group acting on a fixed space V $(= \mathbb{C}^2)$.*

Then the McKay quiver of V is the directed graph with vertices $V_0, \ldots, V_d$ (non-isomorphic representations of G) and arrows $V_i \to V_j$ with multiplicity

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}G}(V_i, V_j \otimes V)\,.$$

Example 9.11.

$$p_0 = \left\langle \begin{pmatrix} \varepsilon_3 & 0 \\ 0 & \varepsilon_3^2 \end{pmatrix} \right\rangle$$

$$p_1 \oplus p_2$$

$$\begin{aligned} p_0 \otimes V &= p_0 \oplus p_2 \\ p_1 \otimes V &= p_0 \oplus p_2 \\ p_2 \otimes V &= p_0 \oplus p_2 \end{aligned}$$

So the McKay quiver is:

$$\begin{tikzcd} & p_0 \arrow[dl, bend right] \arrow[dr, bend right] & \\ p_1 \arrow[ur, bend right] \arrow[rr, bend right] & & p_2 \arrow[ul, bend right] \arrow[ll, bend right] \end{tikzcd}$$

Proposition 9.12. *R normal surface. Then* $\mathrm{CM}(R) \cong \operatorname{add}({}_R S)$.

Fact 9.13. *If $M \in \operatorname{mod}(R)$, then*

$$\operatorname{Hom}_R(M, -)\colon \operatorname{mod}(R) \to \operatorname{mod}(\operatorname{End}_R(M))$$

induces $\operatorname{add}(M) \cong \operatorname{proj}(\operatorname{End}_R(M))$.

projectivization $\rightsquigarrow$

$$\mathrm{CM}(R) \cong \operatorname{proj}(\mathbb{C}[[x,y]]\#G)$$

Lemma 9.14. *Let $G \leq \mathrm{GL}(V)$ be a finite subgroup.*

Then $\mathbb{C}[V]\#G$ is Morita equivalent to the McKay quiver with relations.

$$\operatorname{proj}(S\#G) \cong \mathrm{CM}(R) \cong \operatorname{add}({}_R S) \cong \operatorname{proj}(\operatorname{End}_R(S))$$

Theorem 9.15. *The AR-quiver of* $\mathrm{CM}(R)$ *is the McKay quiver of* G.

Sketch of proof.

(1) No pseudoreflections.

$\rightsquigarrow$ $R \to S$ is unramified, $R_{\mathfrak{p}} \to S_{\mathfrak{p}}$ with $\mathfrak{p}$ height one prime ideal.

(2) Define a right-splitting from $\mathrm{End}_R(S) \to S\#G$ for γ and show this is a surjection.

(3) Everything is torsion-free, R-modules have rank $|G|^2$.

$\rightsquigarrow$ γ isomorphism

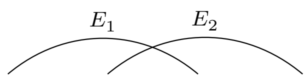
$\square$

Definition 9.16. $\{E_i\}$ *of exceptional* $\mathbb{P}^1$*'s in minimal resolution* $X \to \mathrm{Spec}(R)$.

The dual graph is as follows:

- *Draw a dot for each* E_i.
- *If two* E_i *intersect, connect the dots.*

Example 9.17. $X \to \mathrm{Spec}(R) = \mathbb{C}^2/G$.



$\rightsquigarrow$ dual graph $\bullet \to \bullet$

Theorem 9.18. *There are correspondences:*

$$\{\textit{dual graph}\} \longleftrightarrow \{\textit{McKay quiver}\} \longleftrightarrow \{\textit{AR-quiver}\}$$

- $M \mapsto D$ *by killing trivial representations and merging arrows*
- $D \mapsto M$ *by adding vertices and doubling arrows*

$$\begin{tikzcd} & p_0 \arrow[dl, shift right] \arrow[dr, shift left] & \\ p_1 \arrow[ur, shift right] \arrow[rr, shift left] & & p_2 \arrow[ul, shift left] \arrow[ll, shift left] \end{tikzcd} \qquad \leftrightarrow \qquad \bullet \to \bullet$$

10 Knörrer's periodicity and hypersurface singularities

Wednesday 14^{th} 9:45 – Shiquan Ruan (Bielefeld, Germany)

10.1 Matrix factorizations

Notations.

- $(S, \mathfrak{n}, k)$ regular local ring
- $R = S/(f)$ and $\mathfrak{m} = \mathfrak{n}/(f)$ with $0 \neq f \in \mathfrak{n}^2$
- $\dim(R) = d = \dim(S) - 1$

Definition 10.1. *A* <u>*matrix factorization*</u> *of f in S is a pair (φ, ψ)*

$$G \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F$$

where F, G are free S-modules of the same rank n such that

$$\varphi\psi \;=\; fI_n \;=\; \begin{pmatrix} f & & \\ & \ddots & \\ & & f \end{pmatrix}.$$

A <u>*homomorphism*</u> *between (φ, ψ) and (φ', ψ') is a pair of $(\alpha, \beta) \in \mathrm{mod}(S)$ such that the following diagram commutes:*

$$\begin{tikzcd} G \arrow[r, shift left, "\varphi"] \arrow[d, "\alpha"'] & F \arrow[l, shift left, "\psi"] \arrow[d, "\beta"] \\ G' \arrow[r, shift left, "\varphi'"] & F' \arrow[l, shift left, "\psi'"] \end{tikzcd}$$

We obtain the category $\mathrm{MF}_S(f)$ of matrix factorizations of f with direct sums

$$(\varphi, \psi) \oplus (\varphi', \psi') \;=\; \left(\begin{pmatrix} \varphi & \\ & \varphi' \end{pmatrix}, \begin{pmatrix} \psi & \\ & \psi' \end{pmatrix}\right).$$

Remark 10.2.

(1) φ, ψ are both injective.

(2) $\varphi\psi = fI \Leftrightarrow \psi\varphi = fI$

$\beta\varphi = \varphi'\alpha \Leftrightarrow \alpha\psi = \psi'\beta$

(3) $(\varphi, \psi) \in \mathrm{MF}_S(f) \Leftrightarrow (\psi, \varphi) \in \mathrm{MF}_S(f)$

(4) $(1, f), (f, 1) \in \mathrm{MF}_S(f)$

For any $(\varphi, \psi) \in \mathrm{MF}_S(f)$ write $(\varphi, \psi) = (\varphi', \psi') \oplus (1, f)^{\oplus p} \oplus (f, 1)^{\oplus q}$ and call (φ', ψ') reduced (any entry in φ' and ψ' is not a unit).

Example 10.3.

- $S = k\{x,y\}$, $f = x^2 + y^4 = (x+iy^2)(x-iy^2)$
- $(1,f),(f,1) \in \mathrm{MF}_S(f)$
- $i \in S$:

$$S \underset{x-iy^2}{\overset{x+iy^2}{\rightleftarrows}} S$$

- With $A = \begin{pmatrix} x & y \\ y^3 & -x \end{pmatrix} = fI$ and $B = \begin{pmatrix} x & y \\ y^3 & -x \end{pmatrix}$:

$$S^2 \underset{B}{\overset{A}{\rightleftarrows}} S^2$$

Proposition 10.4. *Assume* $R = S/(f)$.

(i) For all $M \in \mathrm{CM}(R)$ *there is* $(\varphi,\psi) \in \mathrm{MF}_S(f)$ *such that* $\operatorname{coker}(\varphi) \cong M$.

(ii) For all $(\varphi,\psi) \in \mathrm{MF}_S(f)$ *we have* $\operatorname{coker}(\varphi) \in \mathrm{CM}(R)$.

Proof. (i) For $M \in \mathrm{CM}(R)$ we have by Auslander-Buchsbaum

$$\operatorname{proj.dim}({}_SM) = \operatorname{depth}(S) - \operatorname{depth}({}_SM) = (d+1) - d = 1\,.$$

$$0 \longrightarrow S^{(n)} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} S^{(n)} \xrightarrow{\pi} M \longrightarrow 0$$

So $fM = 0$, so $M \in \operatorname{tor}(S)$.

$\pi(fx) = f\pi(x) \in fM = 0 \Rightarrow fx \in \ker(\pi) = \operatorname{im}(\varphi) \Rightarrow fx = \varphi(y)$ for a unique y

Define $\psi\colon x \mapsto y$ (satisfying $fx = \varphi(y) = \varphi\psi(x) \Rightarrow fI = \varphi\psi \Rightarrow \psi\varphi = fI$).

$\Rightarrow (\varphi,\psi) \in \mathrm{MF}_S(f)$

(ii) $(\varphi,\psi) \in \mathrm{MF}_S(f) \Rightarrow \; G \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F \;$ with $\varphi\psi = fI_n = \psi\varphi$

The sequence

$$\cdots \longrightarrow \overline{F} \xrightarrow{\overline{\psi}} \overline{G} \xrightarrow{\overline{\varphi}} \overline{F} \xrightarrow{\overline{\psi}} \cdots$$

in $\mathrm{mod}(R)$ is exact. ... $\Rightarrow \operatorname{coker}(\varphi) \in \mathrm{CM}(R)$ □

Define $\operatorname{coker}((\varphi,\psi)) := \operatorname{coker}(\varphi)$.

Remark 10.5. $\operatorname{coker}((1,f)) = 0$ and $\operatorname{coker}((f,1)) = S/(f) = R$.

$$\begin{tikzcd} G \arrow[r, shift left, "\varphi"] \arrow[d, "\alpha"'] & F \arrow[l, shift left, "\psi"] \arrow[r] \arrow[d, "\beta"] & \operatorname{coker}(\varphi) \arrow[d, dashed] \\ G' \arrow[r, shift left, "\varphi'"] & F' \arrow[l, shift left, "\psi'"] \arrow[r] & \operatorname{coker}(\varphi') \end{tikzcd}$$

$\rightsquigarrow$ additive functor $\mathrm{MF}_S(f) \to \mathrm{MF}_S(f)$, $(\varphi,\psi) \mapsto \operatorname{coker}(\varphi)$

Theorem 10.6 (Eisenbud)**.** $R = S/(f)$. *Then*

$$\mathrm{coker}\colon \mathrm{MF}_S(f)/\{(1,f)\} \xrightarrow{\sim} \mathrm{CM}(R)$$

and between the category of <u>*reduced matrix factorizations*</u> *and stable CM modules*

$$\mathrm{coker}\colon \mathrm{MF}_S(f)/\{(1,f),(f,1)\} \xrightarrow{\sim} \mathrm{CM}(R)/\{R\} = \underline{\mathrm{CM}}(R)\,.$$

10.2 Double branch covering

Definition 10.7. *The* <u>*double branch covering*</u> *of* $R = S/(f)$ *is*

$$R^\sharp \;=\; S[[z]]/(f+z^2)\,.$$

Remark 10.8.

- There is a surjection $R^\sharp \to R$ killing the class of z.
- $R^\sharp$ is a free S-module generated by $\overline{1}$ and $\overline{z}$ (S is complete).

Definition 10.9.

- *For each* $M \in \mathrm{CM}(R)$ *set* $M^\sharp := \mathrm{syz}_1^{R^\sharp}(M)$.
- *For each* $N \in \mathrm{CM}(R^\sharp)$ *set* $N^\flat := N/zN$.

$$\begin{tikzcd}
S[[z]] \arrow[d, two heads] & & S \arrow[ll, hook'] \arrow[d, two heads] \\
R^\sharp = S[[z]]/(f+z^2) \arrow[rr, "/z"] & & R = S/(f)
\end{tikzcd}$$

Lemma 10.10. *Let* $G \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F$ *be in* $\mathrm{MF}_S(f)$ *and* $M = \mathrm{coker}(\varphi)$. *Let* $\pi\colon \widetilde{F} \twoheadrightarrow \overline{F} \twoheadrightarrow M$.

(i) There exists an exact sequence of $R^\sharp$*-modules*

$$\begin{tikzcd}
\widetilde{F}\oplus\widetilde{G} \arrow[r, "A"] & \widetilde{G}\oplus\widetilde{F} \arrow[rr, "(\widetilde{\varphi}, zI)"] \arrow[dr] & & \widetilde{F} \arrow[r, "\pi"] & M \arrow[r] & 0 \\
& & M^\sharp \arrow[ur] & & &
\end{tikzcd}$$

where $A = \begin{pmatrix} \widetilde{\psi} & -zI \\ zI & \widetilde{\varphi} \end{pmatrix}$.

(ii) $\left(\begin{pmatrix} \psi & -zI \\ zI & \varphi \end{pmatrix}, \begin{pmatrix} \varphi & zI \\ -zI & \psi \end{pmatrix}\right) \in \mathrm{MF}_{S[[z]]}(f+z^2)$

(iii) $M^\sharp \cong \mathrm{coker}\left(\begin{pmatrix} \psi & -zI \\ zI & \varphi \end{pmatrix}\right)$

(iv) ${}_{R^\sharp}M^\sharp$ *stable* $\Leftrightarrow$ ${}_RM$ *stable, in which case* $\mathrm{syz}_1^{R^\sharp}(M^\sharp) \cong M^\sharp$.

Proposition 10.11. *Let $M \in \mathrm{CM}(R)$. Then $(M^\sharp)^\flat \cong M \oplus \mathrm{syz}_1^R(M)$.*

Proof. ... □

Dually:

Proposition 10.12. *Let $N \in \mathrm{CM}(R^\sharp)$ and $\mathrm{char}(k) \neq 2$ Then $(N^\flat)^\sharp \cong N \oplus \mathrm{syz}_1^{R^\sharp}(N)$.*

Corollary 10.13. *$M \in \mathrm{CM}(R)$ indecomposable and stable. Then:*

(i) $M^\sharp$ is a direct summand of either one or two indecomposable $R^\sharp$-modules.

(ii) M is a direct summand of $N^\flat$ for some indecomposable non-free $R^\sharp$-module.

Theorem 10.14 (Knörrer's Theorem)**.** *Let $R = S/(f)$ and $\mathrm{char}(k) \neq 2$. Then:*

$$R^\sharp \text{ CM-finite} \Leftrightarrow R \text{ is CM-finite}$$

Example 10.15.

- $R_{n,d} = k[[x, z_1, \dots, z_d]]/(x^{n+1} + z_1^2 + \dots + z_d^2)$ with $n \geq 1$, $d \geq 0$ is CM-finite.
  (since $R_{n,0} = k[[x]]/(x^{n+1})$ is CM-finite)
- $i \in k$, $R' = k\{x_1, \dots, x_t, y_1, \dots, y_t\}/(x_1y_1 + \dots + x_ty_t)$ is CM-finite.
  (Write $x_jy_j = u_j^2 + v_j^2$ where $x_j = u_j + iv_j$ and $y_j = u_j - iv_j$. Then $R' \cong R_{1,2d+1}$.)

10.3 Knörrer's periodicity

$$\begin{tikzcd} R^\sharp \arrow[r, bend right, "\flat"'] & R \arrow[l, bend right, "\sharp"'] \end{tikzcd}$$

Proposition 10.16. *Assume $\mathrm{char}(k) \neq 2$.*

(1) $M \in \mathrm{CM}(R)$ indecomposable non-free:

$$\begin{aligned} & M^\sharp \text{ decomposable} \\ \Leftrightarrow\ & M \cong \mathrm{syz}_1^R(M) \\ \Rightarrow\ & M^\sharp \cong N \oplus \mathrm{syz}_1^{R^\sharp}(N) \text{ with indecomposable } N \ncong \mathrm{syz}_1^{R^\#}(N) \end{aligned}$$

(2) $N \in \mathrm{CM}(R^\sharp)$ indecomposable non-free:

$$\begin{aligned} & N^\flat \text{ decomposable} \\ \Leftrightarrow\ & N \cong \mathrm{syz}_1^{R^\sharp}(N) \\ \Rightarrow\ & N^\flat \cong M \oplus \mathrm{syz}_1^R(M) \text{ with indecomposable } M \ncong \mathrm{syz}_1^R(M) \end{aligned}$$

Definition 10.17. *Set $R^{\sharp\sharp} = S\{u, v\}/(f + uv) \cong S\{z_1, z_2\}/(f + z_1^2 + z_2^2)$.*
For $M \in \mathrm{CM}(R)$ corresponding to $G \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F$ in $\mathrm{MF}_S(f)$ define

$$M^X = \mathrm{coker}\left(\left(\begin{smallmatrix} \varphi & -vI \\ uI & \psi \end{smallmatrix}\right), \left(\begin{smallmatrix} \psi & vI \\ -uI & \varphi \end{smallmatrix}\right)\right).$$

Theorem 10.18 (Knörrer)**.** *$M \mapsto M^X$ defines a bijection between isoclasses of indecomposable non-free CM modules over R and $R^\sharp$.*

Proposition 10.19.

(i) $M^{\sharp\sharp} \cong M^X \oplus \operatorname{syz}_1^{R^{\sharp\sharp}}(M^X)$

(ii) $(M^X)^{\flat\flat} \cong M \oplus \operatorname{syz}_1^R(M)$

(iii) $\left(\operatorname{syz}_1^R(M)\right)^X \cong \operatorname{syz}_1^{R^{\sharp\sharp}}(M^X)$

11 What is (should be) a noncommutative resolution of singularities? – I

...and why should it have to do with MCM modules?

Wednesday 14^{th} 11:00 – Graham Leuschke (Syracuse, United States)

See also Graham's notes!

Goal. Global dominations for algebra over algebraic geometry.
Can we completely remove the geometry from resolution of singularities?

Recall 11.1. *A resolution of singularities of an algebraic variety X is a morphism*

$$\pi \colon \widetilde{X} \to X$$

with

(1) $\widetilde{X}$ is smooth (nonsingular),

(2) π is proper (e.g. projective or finite),

(3) π is birational (induces an isomorphism on function fields).

The dictionary "algebra $\leftrightarrow$ geometry" reverses arrows, so we might want to consider a ring homomorphism $R \xrightarrow{\varphi} S$ "resolving" the singularities of R. It should satisfy:

(1) S is regular / nonsingular,

(2) S is a finitely generated R-module (probably stronger than necessary),

(3) R and S share a quotient field: $\mathrm{Quot}(R) \otimes_R S = \mathrm{Quot}(R)$.

Problem. These don't exist, e.g. $R = \mathbb{C}[[x,y,z]]/(x^3+y^2+z^2)$ has no such algebras S.

Let's allow $S = \Lambda$ to be a noncommutative ring and require ($R \to \Lambda$ sends $R \to Z(\Lambda)$)

(1) Λ has finite global dimension,

(2) Λ is a finitely generated R-module,

(3) $\mathrm{Quot}(R) \otimes_R \Lambda$ is Morita equivalent to $\mathrm{Quot}(R)$:

$$\mathrm{Quot}(R) \otimes_R \Lambda \cong M_n(\mathrm{Quot}(R)).$$

Weakest Possible Definition. A (weak) noncommutative resolution of singularities of a ring R is a module-finite R-algebra Λ of finite global dimension with

$$\mathrm{Quot}(R) \otimes_R \Lambda \cong M_n(\mathrm{Quot}(R)) \qquad \text{"birational"}.$$

Example 11.2 (McKay Correspondence)**.**

- $S = k[[x_1, \dots, x_d]]$

- $G \subseteq \mathrm{GL}_d(k)$ finite with $|G| \in k^\times$
- $R = S^G$.
- Technical assumption: no pseudoreflections.
- R is a complete local Cohen-Macaulay (Hochster-Roberts Theorem) normal domain.
- R is Gorenstein iff $G \subseteq \mathrm{SL}_d(k)$ (since no primitive roots).
- As an R-module S is finitely generated (and MCM).
- (By the way: They have different fraction fields.)

Take the skew group ring $S\#G$.

- It has finite global dimension!
- It is finitely generated free as an S-module, hence finitely generated (and MCM) as R-module,
- It is birational: we know
$$S\#G \cong \mathrm{End}_R(S)$$
and passing to $\mathrm{Quot}(R)$
$$\mathrm{End}_R(S) \otimes_R \mathrm{Quot}(R) = \mathrm{End}_{\mathrm{Quot}(R)}(\mathrm{Quot}(R)^{|G|}) = M_{|G|}(\mathrm{Quot}(R)).$$

Remark 11.3. $\mathrm{Hom}_{S\#G}(-,-) \cong \mathrm{Hom}_S(-,-)^G$ and $(-)^G$ is exact ($|G| \in k^\times$), so
$$\mathrm{Ext}^i_{S\#G}(-,-) \cong \mathrm{Ext}^i_S(-,-)^G,$$
so $\mathrm{gl.dim}(S\#G) = \dim(S) = d$ (the smallest possible finite global dimension for an S-algebra).

Example 11.4 (Finite Cohen-Macaulay Type)**.**
Let $(R, \mathfrak{m})$ be a CM local ring of finite CM-representation type.
E.g. $k[t^2, t^n]$ (with n odd), $k[t^3, t^4, t^5]$, $k[x,y]/(xy)$ (A_1 singularity).
Let $M_1, \dots, M_r$ be the indecomposable MCM R-modules.
$$G = \bigoplus_{i=1}^r M_i \quad \text{a } \underline{\text{CM-generator}} \quad \rightsquigarrow \quad \mathrm{CM}(R) = \mathrm{add}(G)$$

Set
$$\Lambda = \mathrm{End}_R(G)$$
an Auslander algebra for R.

Fact (Iyama, Leuschke, Quarles 2005; Auslander 1980's)**.**
$$\mathrm{gl.dim}(\Lambda) \leq \max\{\dim(R), 2\} < \infty.$$

- It is birational over R (same proof),

- It is module-finite over R.

So Λ is a weak noncommutative resolution of singularities.

Remark 11.5. More precisely, Iyama proves Λ has one simple module for each M_i and

$$\operatorname{proj.dim}({}_\Lambda S_i) = \begin{cases} 2 & \text{if } M_i \not\cong R, \\ 1 & \text{if } M_i \cong R. \end{cases}$$

So $\operatorname{gl.dim}(\Lambda) = d$ when $d \geq 2$ but Λ is not homologically homogeneous (simples have same projective dimension) if $d \geq 3$.

The best case is $d = 2$.

Theorem 11.6 (Auslander 1986)**.** *The* 2*-dimensional complete local* $\mathbb{C}$*-algebras of finite CM type are precisely the invariant rings* $\mathbb{C}[[u, v]]^G$ *for* $G \subseteq \mathrm{GL}_2(\mathbb{C})$.

So in this case Example 11.4 = Example 11.2.

Complaints (from a commutative algebraist)

For noncommutative rings, finite global dimension is not strong enough for most purposes. Particular issues:

(a) No Auslander-Buchsbaum formula for $\operatorname{proj.dim}({}_\Lambda M)$. In fact, we don't even know the finitistic dimension conjecture.

(b) We don't have analogs of the implications

$$\text{regular} \Rightarrow \text{Gorenstein} \Rightarrow \text{CM}$$

for noncommutative rings.

(c) Finite global dimension doesn't localize well.

Strengthen the definitions to address (a), (b), (c).

(c) Say Λ is nonsingular if $\operatorname{gl.dim}(\Lambda_\mathfrak{p}) = \dim(R_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$.
(Biao defined it on Monday for orders as $\operatorname{gl.dim}(\Lambda) = \dim(R)$.)

From now on: R is a Cohen-Macaulay normal domain, for simplicity, and assume that R has a canonical module ω_R. Most important things:

i) R is Gorenstein $\Leftrightarrow \omega_R = R$

ii) $\operatorname{Hom}_R(-, \omega_R)$ gives a duality on $\mathrm{CM}(R)$.

iii) $\mathrm{CM}(R) = \{M \in \operatorname{mod}(R) : \operatorname{Ext}_R^{>0}(M, \omega_R) = 0\}$

(b) Λ is a <u>Gorenstein</u> R-algebra if

$$\mathrm{Hom}_R(\Lambda, \omega_R) \;=:\; \omega_\Lambda$$

is a projective (left) Λ-module. It is <u>symmetric</u> if

$$\mathrm{Hom}_R(\Lambda, R) \;\cong\; \Lambda$$

as Λ-bimodules.

When R is Gorenstein

$$\text{symmetric} \;\Rightarrow\; \text{Gorenstein}$$

but not conversely.

If R is not Gorenstein, they are independent, so we may have to impose both.

12 Buchweitz's Theorem

Thursday 15^{th} 8:30 – Simon May (Leeds, United Kingdom)

(after Happel '88)

$$\begin{tikzcd}
 & \underline{\mathrm{APC}}(S) \arrow[dl, "\Omega_0"'] \arrow[dr, "\sigma_{\le k}"] & \\
\underline{\mathrm{CM}}(S) \arrow[rr, "\sim", "\iota_S"'] & & \mathcal{D}^b_{\mathrm{sg}}(S)
\end{tikzcd}$$

Setting.

- $\mathcal{B}$ additive category, fully and extension closed embedded in an abelian category $\mathcal{A}$
- $\mathcal{S}$ set of exact sequences in $\mathcal{A}$ such that terms are in $\mathcal{B}$
- A morphism $\alpha\colon Y \to \mathbb{Z}$ in $\mathcal{B}$ is a proper epimorphism if there exists an exact sequence

$$0 \to X \to Y \xrightarrow{\alpha} Z \to 0\,.$$

- An object P in $\mathcal{B}$ is called $\mathcal{S}$-projective if for all proper epimorphism $\alpha\colon Y \to Z$ and $f\colon P \to Z$ in $\mathcal{B}$ there exists $g\colon P \to Y$ such that $f = \alpha g$:

$$\begin{tikzcd}
P \arrow[d, dashed, "g"'] \arrow[dr, "f"] & \\
Y \arrow[r, "\alpha"'] & Z
\end{tikzcd}$$

- $(\mathcal{B}, \mathcal{S})$ has enough $\mathcal{S}$-projectives if for every Z in $\mathcal{B}$ there exists a proper epimorphism $\alpha\colon P \to Z$ with P an $\mathcal{S}$-projective.

Definition 12.1. *$(\mathcal{B}, \mathcal{S})$ is called a Frobenius category if it has enough $\mathcal{S}$-projectives and enough $\mathcal{S}$-injectives and they are the same.*

Let $I(X, Y)$ be the subgroup of morphisms $X \to Y$ such that they factor through an $\mathcal{S}$-projective.

Definition 12.2. *Let $(\mathcal{B}, \mathcal{S})$ be a Frobenius category, then $\underline{\mathcal{B}}$ is the stable category with*

- $\mathrm{obj}(\underline{\mathcal{B}}) = \mathrm{obj}(\mathcal{B})$,
- $\mathrm{Hom}_{\underline{\mathcal{B}}}(X, Y) = \mathrm{Hom}_{\mathcal{B}}(X, Y)/I(X, Y)$.

Triangulated structures.

- $\mathcal{B}$ additive category
- T automorphism of $\mathcal{B}$, the translation functor

- a sextuple (X, Y, Z, u, v, w)

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

- morphism:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d] & Z \arrow[r, "w"] \arrow[d] & TX \arrow[d, "Tf"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX' \end{tikzcd}$$

- A set of sextuples Δ is called a <u>triangulation</u> of $\mathcal{B}$ if the following hold:

(TR1) Every sextuple isomorphic to a triangle is a triangle.

Every morphism $u\colon X \to Y$ can be embedded into a triangle $(X, X, 0, \mathbf{1}_X, 0, 0)$.

(TR2) $(X, Y, Z, u, v, w) \in \Delta \Rightarrow (Y, Z, TX, v, w, -Tu) \in \Delta$

(TR3) If we have f, g in the diagram, we can extend to a morphism.

(TR4) octahedral axiom.

Triangulation of the stable category.

- $\mathcal{B}$ additive
- $\underline{\mathcal{B}}$ stable category

Lemma 12.3. *Let*

$$0 \longrightarrow X \longrightarrow I' \longrightarrow X' \longrightarrow 0$$

$$0 \longrightarrow X \longrightarrow I'' \longrightarrow X'' \longrightarrow 0$$

with I' and I'' $\mathcal{S}$-injective. Then X' and X'' are isomorphic.

Let $0 \to X \to I' \to Y' \to 0$. Assume there is a bijection $\gamma_X\colon [X] \to [X']$. For all objects X in $\mathcal{B}$ we choose elements

$$0 \longrightarrow X \longrightarrow I(X) \longrightarrow TX \longrightarrow 0$$

where $TX = \gamma_X(X)$.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & I(X) \arrow[r, "I(f)"] \arrow[d] & TX \arrow[d, "Tf"] \\ Y \arrow[r] & I(Y) \arrow[r] & TY \end{tikzcd}$$

T is an automorphism of $\underline{\mathcal{B}}$.

Let $(\mathcal{B}, \mathcal{S})$ be Frobenius. Define a set of sextuples in $\mathcal{B}$ via $X, Y \in \mathcal{B}$, $u\colon X \to Y$:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d] & Y \arrow[d, "v"] \\
I(X) \arrow[r] \arrow[d] & Cu \arrow[d, "w"] \\
TX \arrow[r, equal] & TX
\end{tikzcd}$$

$\rightsquigarrow$ (X, Y, Cu, u, v, w) standard in $\mathcal{B}$ is standard in $\underline{\mathcal{B}}$.

Theorem 12.4. *Let Δ be the set of all isomorphic sextuples of a standard triangle. Then Δ is a triangulation of $\underline{\mathcal{B}}$.*

Proof. Checking axioms. $\square$

Derived category.

- $\mathcal{A}$ abelian
- $\mathcal{C}(\mathcal{A})$ category of complexes

$$\cdots \longrightarrow X^{i-1} \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} X^{i+1} \longrightarrow \cdots$$

- $\mathcal{K}(\mathcal{A})$ homotopy category with

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X, Y) \;:=\; \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(X, Y)/\,\mathrm{null}(X, Y)\,.$$

- $\mathcal{D}(\mathcal{A}) := \mathcal{K}(\mathcal{A})[\text{quasi}^{-1}]$
- $\mathcal{D}^b(\mathcal{A})$ subcategory of $\mathcal{D}(\mathcal{A})$ with all complexes isomorphic to bounded complexes
- $X[1] = (X^{n+1}, -d_X^{n+1})_n$
- $X \xrightarrow{g} Y \to \mathrm{cone}(g) \to TX$

Singularity category.

- S ring
- A complex is <u>perfect</u> if in $\mathcal{D}(\mathrm{mod}(S)) =: \mathcal{D}(S)$ it is isomorphic to a finite complex of finitely generated projective S-modules.
- $\mathcal{D}^b_{\mathrm{perf}}(S)$ category of perfect complexes
- $\mathcal{D}^b_{\mathrm{sg}}(S) = \mathcal{D}^b(S)/\mathcal{D}^b_{\mathrm{perf}}(S)$

Now:

- S Gorenstein ring
- $\underline{\text{APC}}(S)$ full subcategory of $\mathcal{K}(\mathcal{A})$ of chain complexes that are isomorphic to an acyclic projective complex
- $\text{mod}(S)$ finitely generated S-modules
- $\underline{\text{mod}}(S)$ projectively stabilized category of $\text{mod}(S)$
- $\text{CM}(S)$ the full subcategory of $\text{mod}(S)$ of <u>(maximal) Cohen-Macaulay modules</u> in the sense

$$\text{CM}(S) \;=\; \{X \in \text{mod}(S) : \text{Ext}^i_S(X,S) = 0 \text{ for } i \neq 0\}$$

- $\text{CM}(S)$ is Frobenius, so $\underline{\text{CM}}(S)$ has a natural triangulated structure.

Theorem 12.5 (Buchweitz's Theorem)**.** *If we take S to be an Iwanaga-Gorenstein ring, then there is a triangulated equivalence*

$$\mathcal{D}^b_{\text{sg}}(S) \;\overset{\Delta}{\cong}\; \underline{\text{CM}}(S)\,.$$

- $\underline{\text{APC}}(S) \cong \underline{\text{CM}}(S)$:
  - Let $k \in \mathbb{Z}$.
  - Consider $\Omega_k \colon \underline{\text{APC}}(S) \to \underline{\text{CM}}(S)$, $X \to \text{coker}\left(d_X^{-k-1} \colon X^{-k-1} \to X^{-k}\right)$.
  - A module $M \cong M^{**}$ is CM iff it has a projective coresolution.
  - A complex A in $\text{APC}(S)$ is acyclic,

    $$\text{coker}(d^{-k}) \;\cong\; \text{im}(d^{1-k}) \;\cong\; \ker(d^{2-k})\,,$$

    so we get a projective coresolution

    $$0 \longrightarrow \text{coker}(d^{-k}) \hookrightarrow A_{1-k} \hookrightarrow A_{2-k} \hookrightarrow \cdots\,,$$

    so $\text{coker}(d^{-k})$ is CM.
- $\underline{\text{APC}}(S) \cong \mathcal{D}^b_{\text{sg}}(S)$:
  - For $X \in \text{APC}$ and $k \in \mathbb{Z}$:

    $$\sigma_{\leq k}(X) \;=\; \cdots \longrightarrow X^{k-1} \longrightarrow X^k \longrightarrow 0 \longrightarrow \cdots$$

13 Stably semisimple Gorenstein orders in dimension one

Thursday 15^{th} 10:00 – Wassilij Gnedin (Bochum, Germany)

(0) orders in dimension one

(1) stably semisimple

(2) Gorenstein

(0) Setup

- $k = \overline{k}$, $R = k[[x]]$, $K = k((X))$
- Λ a basic ring-indecomposable R-order in a semisimple K-algebra A
- $\rightsquigarrow \mathrm{CM}(\Lambda) = \Omega(\mathrm{mod}(\Lambda))$ has an AR-quiver.
- Λ is Gorenstein if $\omega = \mathrm{Hom}_R(\Lambda, R) \in \mathrm{proj}(\Lambda)$.
  $\rightsquigarrow \mathrm{inj}(\mathrm{CM}(\Lambda)) = \mathrm{proj}(\mathrm{CM}(\Lambda))$, so

$$\begin{tikzcd} \underline{\mathrm{CM}}(\Lambda) \arrow[loop right, "{[1]=\Omega^{-1}}"] \end{tikzcd} .$$

Example 13.1. Γ hereditary with two simples

$\rightsquigarrow \Gamma = \left[\begin{smallmatrix} R & \mathfrak{m} \\ R & R \end{smallmatrix}\right] \cong k\widehat{Q}$ where $\widehat{Q} = \begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd}$

$\rightsquigarrow \omega \cong \left[\begin{smallmatrix} R & R \\ \mathfrak{m}^{-1} & R \end{smallmatrix}\right] \xrightarrow{\sim} \left[\begin{smallmatrix} \mathfrak{m} & \mathfrak{m} \\ R & \mathfrak{m} \end{smallmatrix}\right] = \mathrm{rad}(\Gamma) \in \mathrm{proj}(\Gamma)$

Remark 13.2. Γ Gorenstein such that $\underline{\mathrm{CM}}(\Gamma) = 0 \Leftrightarrow \Gamma$ hereditary

(1) $\mathrm{rad}\,\underline{\mathrm{CM}}(\Lambda) = 0$

Lemma 13.3. *The following are equivalent:*

(a) $\underline{\mathrm{CM}}(\Lambda)$ *is semisimple, that is, for all* $L, M \in \mathrm{ind}(\underline{\mathrm{CM}}(\Lambda))$

$$\underline{\mathrm{Hom}}_\Lambda(L, M) \cong \begin{cases} k & L \cong M\,, \\ 0 & L \ncong M\,. \end{cases}$$

(b) For all $L \in \mathrm{ind}(\underline{\mathrm{CM}}(\Lambda))$

$$0 \to \Omega(L) \to P(L) \to L \to 0\,,$$

where $P(L) \to L$ *is the projective cover, is the AR sequence ending in* L.

In this case, $\mathrm{CM}(\Lambda) = \mathrm{add}(\Lambda \oplus \mathrm{rad}(\Lambda) \oplus \omega)$.

Example 13.4 (Zh. '57, GP '67)**.**

$$R \to \Lambda = \begin{bmatrix} k[[y,z]]/(yz) & (y) \\ k[[y]] & k[[y]] \end{bmatrix}, \qquad x \mapsto \begin{bmatrix} y+z & 0 \\ 0 & y \end{bmatrix}$$

$$P_2 \xrightarrow{\cdot a} P_1 \xrightarrow{\cdot z} P_1 \xrightarrow{\cdot b} P_2$$

$$\begin{tikzcd} 1 \arrow[loop left, "z"] \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \end{tikzcd}$$

$\rightsquigarrow$ AR-quiver of $\mathrm{CM}(\Lambda)$:

$$\begin{tikzcd} (z) \arrow[r, shift right] \arrow[rrr, dashed, bend left] & P_1 \arrow[l, shift right] \arrow[dr] & & P_2 \arrow[ll] \\ & & (b) \arrow[ur, hook] \arrow[ull, dashed, bend left] & \end{tikzcd}$$

To obtain the AR-quiver of $\underline{\mathrm{CM}}(\Lambda)$ remove P_1 and P_2.

(2) Rejection Lemma

Lemma 13.5 (Drozd–Kirichenko '67)**.** *Let Λ be a non-maximal order and B_1 an indecomposable projective-injective CM module.*

Then there is a unique overorder Γ_1 of Λ in A such that

$$\begin{array}{ccc} \mathrm{ind}(\mathrm{CM}(\Lambda \setminus [B_1])) & \xleftrightarrow{\;1:1\;} & \mathrm{ind}(\mathrm{CM}(\Gamma_1)) \\ {}_\Lambda N & \longleftarrow\!\!\!\shortmid & N \end{array}$$

Moreover, Γ_1 is the minimal overorder such that

$$0 \to \Lambda \to {}_\Lambda\Gamma_1 \to S_{\nu(1)} \to 0 \qquad\qquad B_1 \cong \mathbb{D}\left(e_{\nu(1)}\Lambda\right)$$

where $\mathbb{D} = \mathrm{Hom}_R(-, R)$.

Recall 13.6. $\Lambda \hookrightarrow \Gamma_1 \Rightarrow \mathrm{CM}(\Gamma_1) \hookrightarrow \mathrm{CM}(\Lambda)$

Remark 13.7. $\ell({}_\Lambda\Gamma_1 \otimes S_1) \le 2$ where $S_1 = \mathrm{top}(B_1)$.

Idea of proof. B_1 has a unique maximal overmodule $C_1 = \mathbb{D}\left(\mathrm{rad}\left(\mathbb{D}(B_1)\right)\right)$

$$\rightsquigarrow \qquad 0 \to B_1 \to C_1 \to S_{\nu(1)} \to 0\,.$$

Set $\Gamma_1 = \mathrm{End}_\Lambda(C_1 \oplus P)^{\mathrm{op}}$ where $\Lambda = B_1 \oplus P$. $\square$

Example 13.8 (Gelfand '72)**.**

$\Lambda = \begin{bmatrix} R & \mathfrak{m} & \mathfrak{m} \\ R & R & \mathfrak{m} \\ R & \mathfrak{m} & R \end{bmatrix}$ with columns corresponding to projective modules P_0, P_+, P_-.

$$\begin{tikzcd} + \arrow[r, bend left, "b"] & 0 \arrow[l, bend left, "a"] \arrow[r, bend right, "c"'] & - \arrow[l, bend right, "d"'] \end{tikzcd} \Big/ (ba - dc)$$

$\omega = \begin{bmatrix} \mathfrak{m} & \mathfrak{m} & \mathfrak{m} \\ R & \mathfrak{m} & R \\ R & R & \mathfrak{m} \end{bmatrix}$ with columns corresponding to injective modules

$$I_0 \, = \, \mathrm{rad}(P_0) \qquad I_+ \, = \, P_- \qquad I_- \, = \, P_+$$

$$0 \longrightarrow P_+ = \begin{bmatrix} \mathfrak{m} \\ R \\ \mathfrak{m} \end{bmatrix} \longrightarrow \mathrm{rad}(P_0) = \begin{bmatrix} \mathfrak{m} \\ R \\ R \end{bmatrix} \longrightarrow S = \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} \longrightarrow 0$$

$$\Lambda \xhookrightarrow{\text{rej.}} \Lambda_+ = \begin{bmatrix} R & \mathfrak{m} & \mathfrak{m} \\ R & R & \mathfrak{m} \\ R & R & R \end{bmatrix} \xhookrightarrow{\text{rej.}} \Lambda_{-+} = \begin{bmatrix} R & \mathfrak{m} & \mathfrak{m} \\ R & R & R \\ R & R & R \end{bmatrix}$$

$$\mathrm{rad}(\Lambda) \, = \, \mathrm{rad}(\Lambda_+)$$

$\widetilde{P}$ the second column of Λ_+

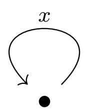
$$\begin{tikzcd} P_0 \arrow[dr, hook] & & \widetilde{P} \arrow[ll, hook'] \\ & P_- \arrow[ur, hook] & \end{tikzcd}$$

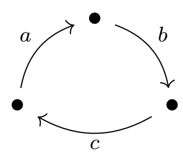
CM(Λ):

$$\begin{tikzcd} & P_+ \arrow[dr] & \\ P_0 \arrow[ur] \arrow[dr] & & \mathrm{rad}(P_0) = \widetilde{P} \arrow[ll, hook', shift right] \arrow[ll, dashed, shift left] \\ & P_- \arrow[ur, hook] & \end{tikzcd}$$

(3) Gluing some orders

- Choose cyclic quiver.





- Attach the sinks of:

$$
\begin{array}{lcc}
\mathbb{A}_2 & \begin{tikzcd} \bullet \arrow[r] & \bullet \end{tikzcd} & \varnothing \\[2ex]
\mathbb{A}_3' & \begin{tikzcd} & \bullet \\ \bullet \arrow[ur] \arrow[dr] & \\ & \bullet \end{tikzcd} & \text{“gluing”} \\[2ex]
\mathbb{A}_3'' & \begin{tikzcd} \bullet \arrow[dr] & \\ & \bullet \\ \bullet \arrow[ur] & \end{tikzcd} & \text{“blow-up”}
\end{array}
$$

$\widetilde{Q} = \widetilde{Q}^{(1)} \times \widetilde{Q}^{(2)}$

$$
\begin{tikzcd}
& & \bullet \arrow[dd, bend left, "b_+"] \\
\bullet \arrow[loop left, "x"] \arrow[urr, bend left, "a_+"] \arrow[rr, "a_-"] & & \bullet \arrow[d, "b_-"] \\
& & \bullet \arrow[ull, "c"]
\end{tikzcd}
\qquad\qquad b_+a_+ = b_-a_-
$$

$\rightsquigarrow \widehat{k\widetilde{Q}} \;\cong\; \Gamma \;=\; \begin{bmatrix} R & \mathfrak{m} & \mathfrak{m} \\ R & R & \mathfrak{m} \\ R & R & R \end{bmatrix}$

$\rightsquigarrow$ Bäckström species $\mathbb{S}_\Lambda$

The outcome.

nodal orders / quadratic orders

= Bäckström orders of type $\mathbb{A}_2$, $\mathbb{A}_3'$, or $\mathbb{A}_3''$

= Bäckström orders such that $\ell({}_\Lambda\Gamma \otimes S) \leq 2 \ \forall S$

Theorem 13.9 (Roggenkamp '85)**.** *Let Λ be Gorenstein and non-hereditary. Then the following are equivalent:*

(a) $\underline{\mathrm{CM}}(\Lambda)$ *is semisimple.*

(b) $\mathrm{CM}(\Lambda) = \mathrm{add}(\Lambda \oplus \mathrm{rad}(\Lambda))$.

(c) Λ *is Bäckström.*

(d) Λ *is nodal without* $\mathbb{A}_2$.

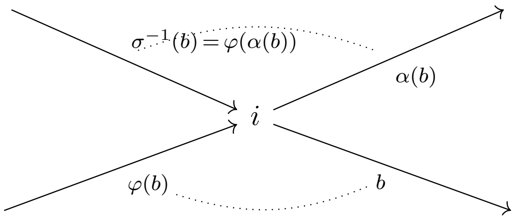
Remark 13.10. (a) $\Rightarrow$ (b) and (d) $\Rightarrow$ (c) are clear. (d) $\Rightarrow$ (b) by Rejection Lemma.

Λ nodal Gorenstein, $\Lambda \hookrightarrow \Gamma$ with $\mathrm{gl.dim}(\Gamma) = 1$ and $\mathrm{rad}(\Lambda) \subseteq \mathrm{rad}(\Gamma)$
$\Rightarrow \mathrm{CM}(\Lambda) = \mathrm{add}(\Lambda \oplus \mathrm{rad}(\Lambda))$
$\Rightarrow \mathrm{rad}(\Lambda) = \mathrm{rad}(\Gamma)$

(4) Ribbon graph orders

Λ is a <u>ribbon graph order</u> if Λ is Bäckström of type $\mathbb{A}_3'$ ("gluing").

- $\Lambda \cong \widehat{kQ}/I$ for some (Q, I) such that for all $i \in Q_0$:



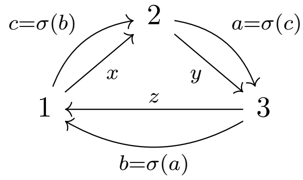
$\rightsquigarrow I = \langle b\varphi(b) : b \in Q_1 \rangle$

$\rightsquigarrow$

$$Q_1 \xrightarrow[1:1]{\sigma,\alpha,\varphi} Q_1 \qquad \varphi\alpha\sigma = \mathrm{id}\,, \alpha^2 = \mathrm{id}\,, \alpha(j) \neq j$$

$\rightsquigarrow (\sigma, \alpha, \varphi)$ "combination map" $=$ "ribbon graph" $\hookrightarrow$ surface

Example 13.11.



$\sigma = (abc)(xyz)$
$\alpha = (xc)(ay)(bz)$
$\varphi = (xazcyb)$
$ax = 0 = za = \cdots$

Proposition 13.12.

$$\begin{array}{ccc} \Omega \circlearrowright \ \mathrm{ind}(\underline{\mathrm{CM}}(\Lambda)) & \xrightarrow{\quad 1:1 \quad} & Q_1 \circlearrowleft \varphi \\ \Lambda a & \longleftarrow\!\!\shortmid & a \end{array}$$

- *S_g has genus g where*

$$2 - 2g \ = \ c(\varphi) - c(\alpha) + c(\sigma)$$

In the example: $= 1 - 3 + 2 = 0 \ \Rightarrow \ g = 1.$

Summary.

$$\begin{array}{rl} & \text{projective resolutions of arrow ideals} \\ = & \text{AR-sequences in } \underline{\mathrm{CM}}(\Lambda) \\ = & \text{“Green walks around the ribbon graph”} \end{array}$$

14 What is (should be) a noncommutative resolution of singularities? – II

Thursday 15^{th} 11:15 – Graham Leuschke (Syracuse, United States)

See also Graham's notes!

Last time:

Maybe a noncommutative resolution of CM local R is an R-algebra Λ which is

- ~~of finite global dimension~~ nonsingular ($\operatorname{gl.dim}(\Lambda_{\mathfrak{p}}) = \dim(R_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$)
- birational
- module-finite

+ Gorenstein ?
+ symmetric ?

Definition 14.1. *Say Λ is an* <u>*R-order*</u> *if Λ is MCM as an R-module.*

Why?

1) Iyama–Wemyss 2010 [Auslander 1984]

The following are equivalent for an order over CM local R:

(i) Λ is nonsingular.

(ii) $\operatorname{gl.dim}(\Lambda) = \dim(R)$.

(iii) $\operatorname{gl.dim}(\Lambda) < \infty$ and Λ is a Gorenstein R-algebra.

(iv) $\operatorname{CM}(\Lambda) = \operatorname{proj}(\Lambda)$.

(Biao proved (i) $\Rightarrow$ (ii).)

So for orders, finite global dimension $\rightsquigarrow$ much better behaved than in general.

2) Stangle 2015, generalizing Iyama–Reiten 2008

Orders of finite global dimension satisfy a version of the Auslander-Buchsbaum formula:

$$\dim(R) \leq \operatorname{depth}({}_R X) + \operatorname{proj.dim}({}_\Lambda X) \leq \dim(R) + n$$

where $n = \operatorname{proj.dim}({}_\Lambda(\omega_\Lambda))$.

In particular, if Λ is a Gorenstein R-order of finite global dimension (nonsingular by (i)), then ${}_\Lambda(\omega_\Lambda)$ is projective, and we get an A-B equality on the nose [Iyama–Reiten].

3) van den Bergh 2004

If Λ is a nonsingular order, then Λ is homologically homogeneous (all simples have the same projective dimension).

Definition 14.2 (Stronger definition)**.** *A (medium-strength) noncommutative resolution of singularities of a CM local ring R is a nonsingular, birational R-order Λ.*

Back to the examples.

[i] McKay Correspondence:

$$\mathrm{End}_R(S) \cong S\#G$$

is an R-order (last time) and has global dimension $d = \dim(R)$, so is nonsingular by Iyama–Wemyss and is birational (last time).

[ii] The Auslander algebra of a ring of finite CM type might not be an order. (If $\dim(R) \geq 3$ it's not! It has simples of different projective dimensions.)

E.g. A_1 in dimension 2:

$$R = k[[x, y, z]]/(xy - z^2)$$

Then

$$\mathrm{ind}(\mathrm{CM}(R)) = \{R, I = (x, z)\}.$$

So $G = R \oplus I$,

$$\Lambda = \mathrm{End}_R(G) = \begin{pmatrix} R & I \\ I^* & \mathrm{End}_R(I) \end{pmatrix} \cong \begin{pmatrix} R & I \\ I & R \end{pmatrix}.$$

So $\Lambda \cong R^{(2)} \oplus I^{(2)}$ is an order.

E.g. A_1 in dimension 3:

$$R = k[[x, y, u, v]]/(xy - uv)$$

Then

$$\mathrm{ind}(\mathrm{CM}(R)) = \{R, \mathfrak{p} = (x, u), \mathfrak{q} = (x, v)\}$$

and

$$\Lambda = \mathrm{End}(R \oplus \mathfrak{p} \oplus \mathfrak{q}) = \begin{pmatrix} R & \mathfrak{p} & \mathfrak{q} \\ \mathfrak{q} & R & (\mathfrak{p}, \mathfrak{q}) \\ \mathfrak{p} & (\mathfrak{q}, \mathfrak{p}) & R \end{pmatrix},$$

where $\mathfrak{p} \cong \mathfrak{q}^*$ and $(\mathfrak{q}, \mathfrak{p}) = \mathrm{Hom}_R(\mathfrak{q}, \mathfrak{p})$. But

$$(\mathfrak{p}, \mathfrak{q}) = \mathrm{Hom}_R(\mathfrak{p}, \mathfrak{q}) = \left(x, u, \frac{u}{y}\right)$$

(a fractional ideal) is not MCM $\left(\frac{u}{y}v = \frac{uv}{y} = \frac{xy}{y} = x\right)$.

Connection with "classical orders" and the symmetric property:

Definition 14.3 (Auslander–Goldman 1960). *Let R be a normal domain.*

An $\widetilde{\text{order}}$ (classical order) over R is a module-finite R-algebra in a semisimple algebra D. Maximal means maximal.

Remark 14.4. Yuta (et al.) defined this when $R = k[[x]]$. We allow $\dim(R) \geq 1$.

Proposition 14.5 (Auslander–Goldman). *Let R be a normal domain and Λ an $\widetilde{\text{order}}$ in $M_n(\mathrm{Quot}(R))$. If*

(i) *Λ is nonsingular,*

(ii) $\Lambda \otimes_R \mathrm{Quot}(R) = M_n(\mathrm{Quot}(R))$,

(iii) *Λ is a symmetric R-algebra, i.e.* $\mathrm{Hom}_R(\Lambda, R) \cong {}_\Lambda\Lambda_\Lambda$,

then Λ is a maximal $\widetilde{\text{order}}$.

Remark 14.6. Yuta stated a version of this when $R = k[[x]]$. (hereditary $\Rightarrow$ maximal)

Theorem 14.7 (Auslander–Goldman). *If Λ is a maximal $\widetilde{\text{order}}$ in $M_n(\mathrm{Quot}(R))$, then*

$$\Lambda \cong \mathrm{End}_R(M)$$

for some reflexive R-module M.

Corollary 14.8 (van den Bergh 2004). *The following are equivalent for a module-finite algebra Λ over a Gorenstein normal domain R:*

(1) *Λ is a symmetric birational R-order.*

(2) *$\Lambda \cong \mathrm{End}_R(M)$ for some reflexive R-module M, is an R-order, and is homologically homogeneous.*

(3) *$\Lambda \cong \mathrm{End}_R(M)$ as above and* $\mathrm{gl.dim}(\Lambda) < \infty$.

Definition 14.9 (van den Bergh). *A noncommutative crepant resolution (NCCR) of a Gorenstein normal domain R is a symmetric birational R-order Λ.*

Equivalently, an R-order of the form $\mathrm{End}_R(M)$ with finite global dimension.

Suddenly R became Gorenstein. That is essential for the corollary.

Example 14.10.

(1) $R = k[[x, y, z, u, v]]/I$ where $I = I_2\left(\begin{smallmatrix} x & y & u \\ y & z & v \end{smallmatrix}\right)$ (scroll of type $(2, 1)$).

Then R is a 3-dimensional normal domain, not Gorenstein ($\omega = (x, y)$ not projective), but R has finite CM type [Yoshino, 16.12]:

$$\mathrm{ind}(\mathrm{CM}(R)) = \{R, \omega, \Omega^1\omega, \Omega^2\omega, (\Omega^1\omega)^\vee\}$$

By Example [2], the Auslander algebra

$$\Lambda \;=\; \mathrm{End}_R(R \oplus \omega \oplus \Omega^1\omega \oplus \Omega^2\omega \oplus (\Omega^1\omega)^\vee)$$

has global dimension 3. It is not homologically homogeneous and is not an order.

So Λ is an endomorphism ring and has finite global dimension but is not an order. So (3) $\not\Rightarrow$ (2) when the base ring is not Gorenstein.

(2) $R = k[[x^2, xy, y^2, yz, xz, z^2]] = k[[x, y, z]]^{(2)}$.

Then R is a 3-dimensional CM normal domain, not Gorenstein. It does have finite CM type [Yoshino, 16.10]:

$$\mathrm{ind}(\mathrm{CM}(R)) \;=\; \{R, \omega, \Omega^1\omega\}\,.$$

Two noncommutative resolutions:

(a) The Auslander algebra $\Lambda = \mathrm{End}_R(R \oplus \omega \oplus \Omega^1\omega)$ has global dimension $d = 3$, but has bad depth ($\mathrm{depth}(\mathrm{Hom}_R(\omega, R)) = 2 < 3$), so is not an order.

(b) McKay Correspondence $\Gamma = \mathrm{End}_R(k[[x, y, z]]) = \mathrm{End}_R(R \oplus R(x, y, z))$ and the fractional ideal

$$(x, y, z)R \;\cong\; (x^2, xy, xz)$$

is isomorphic to ω_R. So

$$\Gamma \;=\; \mathrm{End}_R(R \oplus \omega)$$

is an order of finite global dimension and (if the definition allowed non-Gorenstein R) qualifies to be an NCCR.

Point:

These two examples (Veronese and scroll) are the only two known examples of CM local rings of finite CM type in dimension ≥ 3 other than the ADE hypersurfaces.

15 Orlov's Theorem

Thursday 15^{th} 14:00 – Maximilian Hofmann (Bonn, Germany)

Setting.

- Λ noetherian graded ring:
$$\Lambda = \bigoplus_{i \geq 0} \Lambda_i$$
- $\mathrm{gr}(\Lambda)$ category of finitely generated graded Λ-modules
- $\mathrm{Hom}_\Lambda(-,-) = \mathrm{Hom}_{\mathrm{gr}(\Lambda)}(-,-)$

15.1 The category $\mathrm{qgr}(\Lambda)$

Definition 15.1. *For $M \in \mathrm{gr}(\Lambda)$, $m \in M$ is* <u>*torsion*</u> *if $m \cdot \Lambda_{\geq p} = 0$ for some $p \geq 1$.*
Denote by $\tau(M) \subseteq M$ the submodule of all torsion elements.
M is <u>*torsion*</u> *iff $\tau(M) = M$.*

$$\mathrm{tors}(\Lambda) = \{M \in \mathrm{gr}(M) : M \textit{ torsion}\}$$

Proposition 15.2. $\mathrm{tors}(\Lambda)$ *is a* <u>*Serre subcategory*</u> *of* $\mathrm{gr}(\Lambda)$*, i.e. for short exact sequences*

$$0 \to X' \to X \to X'' \to 0$$

in $\mathrm{gr}(\Lambda)$ *we have $X \in \mathrm{tors}(\Lambda)$ iff $X', X'' \in \mathrm{tors}(\Lambda)$.*
The same is true for $\mathrm{Tors}(\Lambda)$ *in* $\mathrm{Gr}(\Lambda)$ *(Serre subcategory, but also closed under $\coprod$).*

Definition 15.3. *Define the category*

$$\mathrm{qgr}(\Lambda) := \mathrm{gr}(\Lambda)/\,\mathrm{tors}(\Lambda)\,.$$

Similarly, $\mathrm{QGr}(\Lambda) := \mathrm{Gr}(\Lambda)/\,\mathrm{Tors}(\Lambda)$.

- $\mathrm{qgr}(\Lambda)$ has the same objects as $\mathrm{gr}(\Lambda)$.
- $\mathrm{qgr}(\Lambda)$ is abelian and there is an exact $\Pi\colon \mathrm{gr}(\Lambda) \to \mathrm{qgr}(\Lambda)$.
- For morphisms f in $\mathrm{gr}(\Lambda)$: Πf isomorphism $\Leftrightarrow \ker(f), \mathrm{coker}(f) \in \mathrm{tors}(\Lambda)$

Remark 15.4.

- Λ commutative noetherian graded ring
- Λ is generated in degree 1, $\Lambda_0 = k$ a field
- $X = \mathrm{Proj}(\Lambda)$

[Serre]:

$$\begin{aligned} \mathrm{QCoh}(X) &\simeq \mathrm{QGr}(\Lambda) \\ \mathrm{coh}(X) &\simeq \mathrm{qgr}(\Lambda) \end{aligned}$$

15.2 Semiorthogonal decompositions

Let $\mathcal{T}$ be a triangulated category.

Orlov's Theorem.

Definition 15.5. *Let $\mathcal{N} \subseteq \mathcal{T}$ be a full triangulated subcategory and let $I\colon \mathcal{N} \hookrightarrow \mathcal{T}$ the inclusion functor.*

We say that $\mathcal{N}$ is right admissible if I has a left adjoint.

Dually, define left admissible.

$$\begin{aligned} \mathcal{N}^{\perp} &:= \{Y \in \mathcal{T} : \mathrm{Hom}_{\mathcal{T}}(\mathcal{N}, Y) = 0\} \\ {}^{\perp}\mathcal{N} &:= \{X \in \mathcal{T} : \mathrm{Hom}_{\mathcal{T}}(X, \mathcal{N}) = 0\} \end{aligned}$$

Definition 15.6. *Let $\mathcal{N} \subseteq \mathcal{T}$ be thick and right admissible, then $\mathcal{T}$ has the SOD (semiorthogonal decomposition)*

$$\mathcal{T} = \langle \mathcal{N}^{\perp}, \mathcal{N} \rangle .$$

If $\mathcal{N} \subseteq \mathcal{T}$ is thick and left admissible, then $\mathcal{T}$ has the SOD

$$\mathcal{T} = \langle \mathcal{N}, {}^{\perp}\mathcal{N} \rangle .$$

Remark 15.7. Equivalently, an SOD is a pair $\mathcal{A}, \mathcal{B} \subseteq \mathcal{T}$ of thick subcategories with $\mathcal{A}$ left admissible and $\mathcal{B}$ right admissible and ${}^{\perp}\mathcal{A} = \mathcal{B}$ and $\mathcal{B}^{\perp} = \mathcal{A}$. We write

$$\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle .$$

For this observe:

$$\begin{aligned} \mathcal{N} \text{ right admissible} &\quad\Rightarrow\quad {}^{\perp}(\mathcal{N}^{\perp}) = \mathcal{N} \\ \mathcal{N} \text{ left admissible} &\quad\Rightarrow\quad ({}^{\perp}\mathcal{N})^{\perp} = \mathcal{N} \end{aligned}$$

Definition 15.8. *We say that $\mathcal{T}$ has an SOD*

$$\mathcal{T} = \langle \mathcal{N}_1, \ldots, \mathcal{N}_n \rangle$$

if $\mathcal{N}_i \subseteq \mathcal{T}$ are thick subcategories and there exist

$$\mathcal{T}_1 = \mathcal{N}_1 \subseteq \mathcal{T}_2 \subseteq \cdots \subseteq \mathcal{T}_n = \mathcal{T}$$

where $\mathcal{T}_i$ are left admissible in $\mathcal{T}$ and

$$\mathcal{T}_i = \langle \mathcal{T}_{i-1}, \mathcal{N}_i \rangle .$$

Example 15.9. $\langle \mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3 \rangle = \langle \langle \mathcal{N}_1, \mathcal{N}_2 \rangle, \mathcal{N}_3 \rangle$.

Warning 15.10. Orlov calls this weak semiorthogonal decomposition.

Example 15.11. Suppose $\mathcal{T}$ is k-linear.

A full exceptional collection is a sequence $(E_1, \ldots, E_n)$ with $E_i \in \mathcal{T}$

$$\operatorname{Hom}_{\mathcal{T}}(E_i, E_j[p]) \;=\; \begin{cases} k & i = j,\, p = 0, \\ 0 & i = j,\, p \neq 0, \\ 0 & i > j. \end{cases}$$

Write $\mathcal{E}_i := \operatorname{thick}(E_i)$.

Example 15.12 (Beilinson's collection)**.** $\mathcal{D}^b(\operatorname{coh}(\mathbb{P}^n)) = \langle \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n) \rangle$.

15.3 The graded singularity category and Orlov's theorem

- Λ as in § 15.1
- $\operatorname{gl.dim}(\Lambda_0) < \infty$
- grading shift on $\operatorname{gr}(\Lambda)$ via $M \mapsto M(1)$ with $M(1)_i = M_{i+1}$

Definition 15.13. *$M \in \mathcal{D}^b(\operatorname{gr}(\Lambda))$ is perfect if $M \in \operatorname{thick}\{\Lambda(e) : e \in \mathbb{Z}\} \subseteq \mathcal{D}^b(\operatorname{gr}(\Lambda))$.*

$\rightsquigarrow$ *thick triangulated subcategory* $\operatorname{perf}(\Lambda) \subseteq \mathcal{D}^b(\operatorname{gr}(\Lambda))$

Definition 15.14. *The graded singularity category is the Verdier quotient*

$$\mathcal{D}^{\mathrm{gr}}_{\mathrm{sg}}(\Lambda) \;:=\; \mathcal{D}^b(\operatorname{gr}(\Lambda)) / \operatorname{perf}(\Lambda)\,.$$

$$\underline{\operatorname{Hom}}_{\Lambda}(M, N) \;:=\; \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}(M, N(n))$$

is in $\operatorname{gr}(\Lambda)$ for all $M, N \in \operatorname{gr}(\Lambda)$.

Definition 15.15. *Λ is called (Artin-Schelter-)Gorenstein if:*

- $\operatorname{inj.dim}({}_{\Lambda}\Lambda) < \infty$ *and* $\operatorname{inj.dim}(\Lambda_{\Lambda}) < \infty$.
- *There are $n, a \in \mathbb{Z}$ such that*

  $$\mathbb{R}\underline{\operatorname{Hom}}_{\Lambda}(\Lambda_0, \Lambda) \;\simeq\; \Lambda_0[n](a)$$

  where $[n]$ is the shift in $\mathcal{D}^b(\operatorname{gr}(\Lambda))$.

The integer a is called the Gorenstein parameter of Λ.

Notation: We have two induced functors:

$$\Pi\colon \mathcal{D}^b(\mathrm{gr}(\Lambda)) \longrightarrow \mathcal{D}^b(\mathrm{qgr}(\Lambda))$$
$$q\colon \mathcal{D}^b(\mathrm{gr}(\Lambda)) \longrightarrow \mathcal{D}^{\mathrm{gr}}_{\mathrm{sg}}(\Lambda)$$

Theorem 15.16 (Orlov '09)**.** *Let $\Lambda = \bigoplus_{i\geq 0} \Lambda_i$ be a graded noetherian ring such that*

- *Λ is (AS-)Gorenstein with Gorenstein parameter a,*
- *$\mathrm{gl.dim}(\Lambda_0) < \infty$,*
- *there exists a commutative ring k such that Λ is a flat k-algebra.*

Then the following hold:

(1) If $a > 0$, there are fully faithful exact functors

$$\Phi_i\colon \mathcal{D}^{\mathrm{gr}}_{\mathrm{sg}}(\Lambda) \to \mathcal{D}^b(\mathrm{qgr}(\Lambda)) \qquad \textit{for all } i \in \mathbb{Z}$$

and SODs

$$\mathcal{D}^b(\mathrm{qgr}(\Lambda)) \;=\; \Big\langle \pi\Lambda(-i-a+1), \ldots, \pi\Lambda(-i), \Phi_i\mathcal{D}^{\mathrm{gr}}_{\mathrm{sg}}(\Lambda) \Big\rangle\,.$$

(2) If $a < 0$, there are fully faithful exact functors

$$\Psi_i\colon \mathcal{D}^b(\mathrm{qgr}(\Lambda)) \to \mathcal{D}^{\mathrm{gr}}_{\mathrm{sg}}(\Lambda) \qquad \textit{for all } i \in \mathbb{Z}$$

and SODs

$$\mathcal{D}^{\mathrm{gr}}_{\mathrm{sg}}(\Lambda) \;=\; \Big\langle q\Lambda_0(-i), \ldots, q\Lambda_0(-i+a+1), \Psi_i\mathcal{D}^b(\mathrm{qgr}(\Lambda)) \Big\rangle\,.$$

(3) If $a = 0$, then there is an exact equivalence

$$\mathcal{D}^b(\mathrm{qgr}(\Lambda)) \;\cong\; \mathcal{D}^{\mathrm{gr}}_{\mathrm{sg}}(\Lambda)\,.$$

Application:

- $\Lambda = k[x_0, \ldots, x_n]$ with $|x_i| = 1$ is (AS-)Gorenstein with $a = n+1$, $\mathrm{gl.dim}(\Lambda) < \infty$.
  $\rightsquigarrow \mathcal{D}^{\mathrm{gr}}_{\mathrm{sg}}(\Lambda) = 0$
  $\rightsquigarrow \mathcal{D}^b(\mathrm{coh}(\mathbb{P}^n)) \cong \mathcal{D}^b(\mathrm{qgr}(\Lambda)) = \langle \pi\Lambda(0), \ldots, \pi\Lambda(n)\rangle$
- $\Lambda = k[x]/(x^{n+1})$ with $|x| = 1$ is (AS-)Gorenstein with parameter $a = -n$.
  $\rightsquigarrow \mathcal{D}^{\mathrm{gr}}_{\mathrm{sg}}(\Lambda) \cong \langle qk(0), \ldots, qk(a+1)\rangle$
  $\rightsquigarrow \mathcal{D}^{\mathrm{gr}}_{\mathrm{sg}}(\Lambda) \cong \mathcal{D}^b(k\vec{A}_n)$

16 Tilting theory for Gorenstein rings in dimension one

Thursday 15^{th} 15:15 – Umamaheswaran Arunachalam (Prayagraj, India)

Umamaheswaran:

The study of maximal Cohen Macaulay (CM) modules is one of the central subjects in commutative algebra and representation theory [1,2,4–6]. A Frobenius category is an exact category in which the notion of injective objects coincide with the projective objects and there are enough injectives (or equivalently enough projectives). When the ring R is Gorenstein, the category

$$\mathrm{CM}(R) = \{X \in \mathrm{mod}(R) : \mathrm{Ext}^i_R(X, R) = 0 \text{ for all } i \geq 1\}$$

of $\mathrm{CM}(R)$-modules forms a Frobenius category and its stable category $\underline{\mathrm{CM}}(R)$ has a natural structure of a triangulated category.

Tilting theory controls triangle equivalence between derived categories of rings, and plays an important role on various areas of mathematics. Tilting theory also gives a powerful tool to study the stable categories of Gorenstein rings.
If $\dim(R) = 0$, then $\mathrm{CM}_0^{\mathbb{Z}}(R) = \mathrm{mod}^{\mathbb{Z}}(R)$ always has a tilting object.
Our main aim of this notes is to study about the following problem:

Question: Let $R = \bigoplus R_i$ be a $\mathbb{Z}$-graded Gorenstein ring such that R_0 is a field. When does the stable category $\underline{\mathrm{CM}}_0^{\mathbb{Z}}(R)$ of $\mathbb{Z}$-graded CM R-modules have a tilting object?

Umamaheswaran:

Recently, Ragnar-Olaf Buchweitz, Osamu Iyama and Kotya Yamaura gave a complete answer to the above problem when $\dim(R) = 1$.

Definition 16.1. *A graded ring is a ring that is a direct sum of abelian groups R_i such that $R_i R_j \subseteq R_{i+j}$.*

Setting.

(R1) R is a $\mathbb{Z}$-graded commutative Gorenstein ring of Krull dimension one.

(R2) $R = \bigoplus_{i \geq 0} R_i$ and $k := R_0$ is a field.

Setting.

- $\mathrm{mod}^{\mathbb{Z}}(R)$ the category of $\mathbb{Z}$-graded finitely generated R-modules
- $\mathrm{mod}_0^{\mathbb{Z}}(R)$ the category of $\mathbb{Z}$-graded finitely generated R-modules of finite length
- $\mathrm{proj}^{\mathbb{Z}}(R)$ the category of $\mathbb{Z}$-graded finitely generated projective R-modules

Remark 16.2. Clearly, $\operatorname{mod}_0^{\mathbb{Z}}(R) \subseteq \operatorname{mod}^{\mathbb{Z}}(R)$.

Consider the quotient category

$$\operatorname{qgr}(R) \;:=\; \operatorname{mod}^{\mathbb{Z}}(R)/\operatorname{mod}_0^{\mathbb{Z}}(R)\,.$$

Let $\operatorname{perf}(\operatorname{qgr}(R))$ be the thick subcategory generated by $\operatorname{proj}^{\mathbb{Z}}(R)$.

Definition 16.3.

<u>*Umamaheswaran:*</u>

Let $\mathcal{T}$ be a triangulated category with suspension functor. A full subcategory of $\mathcal{T}$ is thick if it is closed under cones, $[\pm 1]$ and direct summands. We call on object $T \in \mathcal{T}$ tilting (resp. silting) if $\operatorname{Hom}_{\mathcal{T}}(T, T[i]) = 0$ holds for all integers $i \neq 0$ (resp. $i > 0$), and smallest thick subcategory of $\mathcal{T}$ containing T is $\mathcal{T}$.

For $X \in \operatorname{mod}^{\mathbb{Z}}(R)$ and $n \in \mathbb{Z}$ let

$$X_{\geq n} \;:=\; \bigoplus_{i \geq n} X_i\,.$$

Let S be the set of all homogeneous non-zero divisors in R and

$$K \;:=\; RS^{-1} \qquad \text{the } \mathbb{Z}\text{-graded total quotient ring of } R.$$

There exists an integer $p > 0$ such that $K(p) = k$ as graded R-module.

Theorem 16.4. *Under the settings (R1) and (R2) the following are true:*

(a) $\operatorname{qgr}(R)$ *has a progenerator*

$$U \;:=\; \bigoplus_{i=1}^{p} K(i)_{\geq 0} \;=\; \bigoplus_{i=1}^{p} K(i)_{\geq i}(i)$$

and $\operatorname{perf}(\operatorname{qgr}(R))$ *has U as a tilting object.*

(b) We have an equivalence

$$\operatorname{qgr}(R) \;\cong\; \operatorname{mod}(\Lambda)$$

and a triangle equivalence

$$\operatorname{perf}(\operatorname{qgr}(R)) \;\cong\; K^b(\operatorname{proj}(\Lambda))\,.$$

(c) We have $\Lambda \cong \operatorname{End}_R^{\mathbb{Z}}(U)$ with

$$\Lambda \;=\; \begin{pmatrix} K_0 & K_{-1} & \cdots & K_{2-p} & K_{1-p} \\ K_1 & K_0 & \cdots & K_{3-p} & K_{2-p} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{p-2} & K_{p-3} & \cdots & K_0 & K_{-1} \\ K_{p-1} & K_{p-2} & \cdots & K_1 & K_0 \end{pmatrix}.$$

(d) Λ *is a finite-dimensional selfinjective* k*-algebra.*

(e) If R *is reduced, then* Λ *is a semisimple* k*-algebra.*

Otherwise, Λ *has infinite global dimension.*

Proposition 16.5.

(a) $P = \bigoplus_{i=1}^{p} K(i)$ *is a progenerator of* $\operatorname{mod}^{\mathbb{Z}}(K)$ *such that* $\operatorname{End}_R^{\mathbb{Z}}(P) \cong \Lambda$*.*

(b) There is an equivalence

$$\operatorname{Hom}_R^{\mathbb{Z}}(P,-)\colon \operatorname{mod}^{\mathbb{Z}}(K) \xrightarrow{\simeq} \operatorname{mod}(\Lambda)\,.$$

(c) $U = \bigoplus_{i=1}^{p} K(i)_{\geq 0}$ *is a progenerator in* $\operatorname{qgr}(R)$*.*

Therefore U *is a tilting object in* $\operatorname{perf}(\operatorname{qgr}(R))$*.*

(d) Λ *is a finite-dimensional selfinjective* k*-algebra.*

(e) If R *is reduced, then* Λ *is a semisimple* k*-algebra.*

Otherwise, Λ *has infinite global dimension.*

Proof of theorem.

Umamaheswaran:

Theorem follows from the following Proposition.

□

Proof of proposition.

(a) Since $\{K(i) : i \in \mathbb{Z}\}$ is a progenerator of $\operatorname{mod}^{\mathbb{Z}}(K)$ and $K(i+p) = K(i)$ for all i, it follows that P is a progenerator. Since $\operatorname{End}_R(P) = \operatorname{End}_K(P)$, we have

$$\operatorname{End}_R^{\mathbb{Z}}(P) = \operatorname{End}_K^{\mathbb{Z}}(P) \cong \Lambda\,.$$

(b) Use:

Theorem (Morita)**.** *Two rings* R *and* S *are Morita equivalent iff there is a progenerator* P *of* $\operatorname{mod}(R)$ *such that* $S \cong \operatorname{End}_R(P)$*.*

By (a) and Morita's Theorem, $\Lambda \cong \operatorname{End}_R^{\mathbb{Z}}(P)$ and then

$$\operatorname{Hom}_R^{\mathbb{Z}}(P,-)\colon \operatorname{mod}^{\mathbb{Z}}(K) \xrightarrow{\simeq} \operatorname{mod}(\Lambda)\,.$$

(c) Considering the functors

$$(-)_{\geq 0}\colon \operatorname{mod}^{\mathbb{Z}}(K) \to \operatorname{mod}^{\mathbb{Z}}(R)$$

and

$$K \otimes -\colon \operatorname{mod}^{\mathbb{Z}}(R) \to \operatorname{mod}^{\mathbb{Z}}(K)\,,$$

one can check that they induce mutually quasi-inverse equivalences

$$\operatorname{mod}^{\mathbb{Z}}(K) \;\cong\; \operatorname{qgr}(R)\,.$$

Quasi-equivalence relation: Let $F\colon \mathcal{C} \to \mathcal{D}$ be an equivalence of categories, i.e. there is a functor $G\colon \mathcal{D} \to \mathcal{C}$ (called quasi-inverse of F) such that

$$F \circ G \;\cong\; \mathrm{id}_{\mathcal{D}} \qquad \text{and} \qquad G \circ F \;\cong\; \mathrm{id}_{\mathcal{C}}\,.$$

Since $P \in \operatorname{mod}^{\mathbb{Z}}(K)$ corresponds to $U \in \operatorname{qgr}(R)$, U is a progenerator in $\operatorname{qgr}(R)$ by (a).

$\Rightarrow U$ is a tilting object in $\operatorname{perf}(\operatorname{qgr}(R))$.

(d)

Lemma. *For any $i \in \mathbb{Z}$, $K(i)_{\geq 0} \in \operatorname{mod}^{\mathbb{Z}}(R)$ holds.*

By the lemma for any $X \in \operatorname{mod}^{\mathbb{Z}}(K)$ we have $K \otimes_R \otimes X_{\geq 0} = X$.

Proposition. *K is an injective object in* $\operatorname{mod}^{\mathbb{Z}}(K)$.

Proof. Let $X \in \operatorname{mod}^{\mathbb{Z}}(K)$. Then we have $X_{\geq 0} \in \operatorname{mod}^{\mathbb{Z}}(R)$. Since $\dim(R) = 1$, we have $X_{\geq 0} \in \mathrm{CM}^{\mathbb{Z}}(R)$. Thus $\operatorname{Ext}^1_K(X, K) \cong \operatorname{Ext}^1_K(K \otimes_R X_{\geq 0}, K) \cong K \otimes \operatorname{Ext}^1_K(X_{\geq 0}, K) = 0$. □

By the proposition, P is injective in $\operatorname{mod}^{\mathbb{Z}}(K)$.
$\Rightarrow \Lambda$ is injective in $\operatorname{mod}(\Lambda)$.

(e) R reduced $\Leftrightarrow$ K reduced $\Leftrightarrow$ Any homogeneous element of K is invertible.
This is equivalent to that any object in $\operatorname{mod}^{\mathbb{Z}}(K)$ is projective.

$\Rightarrow \operatorname{gl.dim}(\operatorname{mod}^{\mathbb{Z}}(K)) = 0$. By (b), Λ is semisimple.

On the other hand, by a classical result of Eilenberg and Nakayama, a selfinjective algebra is either semisimple or of infinite global dimension.

(e) follows from (d). □

a-invariant: There exists an integer $a \in \mathbb{Z}$ such that

$$\operatorname{Ext}^1_R(k, R(a)) \;\cong\; K$$

in $\operatorname{mod}^{\mathbb{Z}}(R)$. We call a the a-invariant or the Gorenstein parameter of R.

$$\mathrm{CM}_0^{\mathbb{Z}}(R) \;:=\; \left\{X \in \operatorname{mod}^{\mathbb{Z}}(R) : X \in \mathrm{CM}_0(R) \text{ as an ungraded } R\text{-module}\right\}$$

with stable category $\underline{\mathrm{CM}}_0^{\mathbb{Z}}(R)$.

Umamaheswaran:

Notations:
It is known in representation theory that the following subcateogory

$$\mathrm{CM}_0(R) = \{X \in \mathrm{CM}(R) : X_{\mathfrak{p}} \in \operatorname{proj}(R_{\mathfrak{p}}) \;\forall\, \mathfrak{p} \in \operatorname{Spec}(R)\}\,.$$

Theorem 16.6. *Under the settings (R1) and (R2). Assume moreover that the a-invariant of R is negative. Then:*

(a) $\underline{\mathrm{CM}}_0^{\mathbb{Z}}(R)$ *has a silting object*

$$\bigoplus_{i=1}^{a+p} R(i)_{\geq 0}\,.$$

(b) We have a triangle equivalence

$$\underline{\mathrm{CM}}_0^{\mathbb{Z}}(R) \;\cong\; K^b(\mathrm{proj}(\Lambda))/\,\mathrm{thick}(P)\,,$$

where Λ is given as in Theorem 16.4 and P is the projective Λ-module corresponding to the first $-a$ rows.

(c) $\underline{\mathrm{CM}}_0^{\mathbb{Z}}(R)$ *has a tilting object $\Leftrightarrow$ R is regular.*

Umamaheswaran:

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17 Stable categories of Cohen-Macaulay modules and cluster categories

Thursday 15^{th} 17:00 – Julia Sauter (Bielefeld, Germany)

Literature:

[AIR 15]
[I1] Auslander-Reiten theory revisited
[I2] Tilting Cohen-Macaulay representations
[IY 08]

17.1 Quotient singularities

- V affine variety
- $G \subseteq \operatorname{Aut}(V)$ finite subgroup
- $\rightsquigarrow V/G$ "quotient singularity"

Here only:

- finite subgroup $G \subseteq \mathrm{GL}_d(k)$ acting on $V = k^d$
- $V = \operatorname{Spec}(k[x_1, \ldots, x_d])$
- $V/G = \operatorname{Spec}((k[x_1, \ldots, x_d])^G)$

$\rightsquigarrow$ complete rings:

Main setup:

- $S = k[[x_1, \ldots, x_d]]$
- $R = S^G$ and assume $k = \overline{k}$ with $\operatorname{char}(k) = 0$ and G has no pseudoreflections
- R Gorenstein $\Leftrightarrow G \subseteq \mathrm{SL}_d(k)$
- R isolated singularity $\Leftrightarrow \operatorname{rank}(\sigma - 1) = d \;\forall\, \sigma \neq 1$ in G

Recall in general:

- R a commutative noetherian, local Gorenstein ring with $d = \dim(R)$
  (CM with ${}_R(\omega_R) \cong {}_R R$)
- Λ an R-order
- Λ Gorenstein R-order (= Gorenstein R-algebra in the sense of Leuschke)
  $\Leftrightarrow \omega_\Lambda = \operatorname{Hom}_R(\Lambda, R) \cong \Lambda$ as left Λ-module
- Λ symmetric
  $:\Leftrightarrow \omega_\Lambda \cong \Lambda$ as Λ-Λ-bimodule $\overset{[I2]}{\Leftrightarrow}$ Λ d-Iwanaga-Gorenstein R-order

For any Iwanaga-Gorenstein ring A there is the Frobenius category

$$\operatorname{gp}(A) := {}^{0<\perp}A\,.$$

Warning 17.1. In general $\mathrm{CM}(A) \neq \mathrm{gp}(A)$.

But if Λ is an R-order which is Iwanaga-Gorenstein, then:

$$\mathrm{CM}(\Lambda) \;=\; {}^{0<\perp}\omega_\Lambda \;=\; {}^{0<\perp}\Lambda \;=\; \mathrm{gp}(\Lambda)$$

$\Leftrightarrow\; {}_\Lambda(\omega_\Lambda) \cong {}_\Lambda\Lambda \;\Leftrightarrow\; \Lambda$ Gorenstein order

Example 17.2.

1) $R = k$:

   Λ R-order $\Leftrightarrow$ Λ finite-dimensional k-algebra

   Λ Gorenstein order $\Leftrightarrow$ Λ selfinjective

2) Assuming the main setup:

   R Gorenstein order, symmetric over R

   $\Rightarrow \mathrm{CM}(R) = {}^{0<\perp}R$ Frobenius category

   $\underline{\mathrm{CM}}(R) \cong D^b_{\mathrm{sg}}(R)$ according to Buchweitz

   R also $\mathbb{Z}$-graded (S $\mathbb{Z}$-graded, $\deg(x_i) = 1$, G action by graded automorphisms)

   $\mathrm{CM}^{\mathbb{Z}}(R) := \mathrm{mod}^{\mathbb{Z}}(R) \cap \mathrm{CM}(R)$ Frobenius category

   $\underline{\mathrm{CM}}^{\mathbb{Z}}(R) \cong D^{\mathbb{Z}}_{\mathrm{sg}}(R)$ by [I2, 2.10]

Definition 17.3. *Let $\mathcal{E}$ be an exact category and $n \in \mathbb{N}_{\geq 1}$.*
Then $E \in \mathcal{E}$ is an n-cluster tilting object if

$$\mathrm{add}(E) \;=\; \bigcap_{i=1}^{n-1} \ker \mathrm{Ext}^i_{\mathcal{E}}(-, E) \;=\; \bigcap_{i=1}^{n-1} \ker \mathrm{Ext}^i_{\mathcal{E}}(E, -)\,.$$

Theorem 17.4 (IY08, Theorem 8.4)**.** *Assume the main setup.*
Then ${}_RS \in \mathrm{CM}(R)$ is a $(d-1)$-cluster tilting object iff $\mathrm{End}_R(S)$ is a NCCR by [I1,3.17].
The "quiver" of $\mathrm{add}({}_RS)$ is the McKay quiver of G with respect to $V = k^d$.
In case $d = 2$: $\mathrm{add}({}_RS) = \mathrm{CM}(R)$ (cp. Sarah's talk).

Definition 17.5. *Let $\mathcal{T}$ be a triangulated category with functorially finite subcategory $\mathcal{C}$.*
Then $\mathcal{C}$ is an n-cluster tilting subcategory iff

$$\mathcal{C} \;=\; \bigcap_{i=1}^{n-1} \mathcal{C}[-i]^{\perp} \;=\; \bigcap_{i=1}^{n-1} {}^{\perp}\mathcal{C}[i]\,.$$

Corollary 17.6. *Assume the main setup.*
$\mathrm{add}({}_RS) \subseteq \underline{\mathrm{CM}}(R)$ *is a $(d-1)$-cluster tilting subcategory.*

Definition 17.7. *Let $\mathcal{T}$ be an R-linear triangulated category with* $\operatorname{Hom}_{\mathcal{T}}(X,Y) \in \mathrm{f.l.}(R)$ *for all $X, Y \in \mathcal{T}$.*

We call an autoequivalence $\mathbb{S}\colon \mathcal{T} \to \mathcal{T}$ a <u>*Serre functor*</u> *if there is a bifunctorial isomorphism for all $X, Y \in \mathcal{T}$*

$$\operatorname{Hom}_{\mathcal{T}}(X,Y) \to D\operatorname{Hom}_{\mathcal{T}}(Y, \mathbb{S}X)$$

where $D\colon \mathrm{f.l.}(R) \to \mathrm{f.l.}(R)$ is the Matlis duality.

We call $\mathcal{T}$ an <u>*n-Calabi-Yau triangulated category*</u> *if $\mathbb{S} = [n]$ is a Serre functor where $[n]$ is the shift by n.*

Theorem 17.8 (I1, 3.21, 3.22)**.** *Assume the general setup.*

(1) Let Λ be a Gorenstein R-order that is an isolated singularity.

Then $\underline{\mathrm{CM}}(\Lambda)$ is a triangulated category with respect to $[1] = \Omega_\Lambda^{-1}$ and has the Serre functor $\Omega_\Lambda^{-1} \circ \tau$.

(2) Let Λ be as above and symmetric over R.

Then $\tau = \Omega^{2-d}$ and $[d-1]$ is a Serre functor of $\underline{\mathrm{CM}}(\Lambda)$.

So $\underline{\mathrm{CM}}(\Lambda)$ is a $(d-1)$-Calabi-Yau triangulated category.

Proof. For $X, Y \in \mathrm{CM}(\Lambda)$

$$\underline{\operatorname{Hom}}(X, \Omega^{-1}\tau Y) \;=\; \underline{\operatorname{Hom}}(\Omega X, \tau Y) \overset{(*)}{=} \operatorname{Ext}^1(X, \tau Y)$$

where (*) follows by applying $(-, \tau Y)$ to

$$0 \to \Omega X \to P \to X \to 0$$

with P projective. By AR-duality $\operatorname{Ext}^1(X, \tau Y) = D\underline{\operatorname{Hom}}(Y, X)$. □

17.2 Cluster categories

[Amiot 09, Guo 10] for finite-dimensional k-algebras $\underline{A}$ with $\operatorname{gl.dim}(\underline{A}) \leq n$ defined an n-Calabi-Yau triangulated category $\mathcal{C}_n(\underline{A})$ together with a triangle functor

$$\pi\colon D^b(\underline{A}) := D^b(\operatorname{mod}(\underline{A})) \to \mathcal{C}_n(\underline{A})$$

where $\operatorname{add}(\pi(\underline{A}))$ is an n-cluster tilting subcategory and π factors through the fully faithful

$$D^b(\underline{A})/\mathbb{S}_n \hookrightarrow \mathcal{C}_n(\underline{A})$$

where $\mathbb{S} = - \otimes_\Lambda D\Lambda$ and $\mathbb{S}_n := \mathbb{S} \circ [-n]$.

The category on the left hand side is not necessarily triangulated!

([Keller 05] investigates when it is.)

Example 17.9.

- $\underline{A} = KQ \rightsquigarrow$ [Happel]:

$$\mathcal{C}_2(KQ) \;=\; D^b(KQ)/\mathbb{S}_2$$

- $\underline{A} = KQ$ where Q is an ADE Dynkin quiver, for all $d \geq 1$:

$$\mathcal{C}_d(KQ) \ = \ D^b(KQ)/\mathbb{S}_d$$

$(\mathcal{C}_1(KQ) = D^b(KQ)/\tau)$

Question: Find a $\mathbb{Z}$-graded ... R-order Λ and a finite-dimensional ... algebra $\underline{A}$ such that there is a commutative diagram:

$$\begin{tikzcd}
\underline{\mathrm{CM}}^{\mathbb{Z}}(\Lambda) \arrow[r, "\sim"] \arrow[d] & D^b(\underline{A}) \arrow[d, "\pi"] \\
\underline{\mathrm{CM}}(\Lambda) \arrow[r, "\sim"] & \mathcal{C}_n(\underline{A})
\end{tikzcd}$$

Example 17.10. $Q = \vec{A}_n$ and $\Lambda = K[X]/(X^{n+1})$ Gorenstein order:

Then $\underline{\mathrm{CM}}(\Lambda) = \mathrm{mod}(\Lambda)$ has the AR-quiver with rightmost vertex deleted:

$$\begin{tikzcd}
K \arrow[loop above, dotted] \arrow[r, shift left] & K[T]/(T^2) \arrow[l, shift left] \arrow[loop above, dotted] \arrow[r, shift left] & {} \arrow[l, shift left] \arrow[loop above, dotted] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & {} \arrow[l, shift left] \arrow[loop above, dotted] \arrow[r, shift left] & K[T]/(T^{n+1}) \arrow[l, shift left]
\end{tikzcd}$$

See the poster of the summer school for a picture of $D^b(K\vec{A}_n) \to \mathcal{C}_1(K\vec{A}_n) = D^b(K\vec{A}_n)/\tau$.

Example 17.11.

1) Q Dynkin, $R = S^G$, G of some Dynkin type, $d = 2$:

$$\underline{\mathrm{CM}}(R) \overset{[\mathrm{IY}]}{=} \text{mesh category of the double quiver } \overline{Q} \ \cong \ \mathcal{C}_1(KQ)$$

(Knörrer's periodicity: $\underline{\mathrm{CM}}(\Lambda) \cong \underline{\mathrm{CM}}(k[[x,y,z]]/(x^{n+1} + yz)))$

2) This generalizes for G cyclic ([AIR 15]). They also have examples from dimer models.

3) [DL] for certain tiled orders (see David's talk tomorrow).

17.3 AIR construction

Setting.

- $B = \bigoplus_{\ell \geq 0} B_\ell$ a graded noetherian k-algebra
- $\dim({}_k B_0) < \infty$
- There is an idempotent $1 \neq e = e^2 \in B_0$ such that

$$B/(e) \text{ is a finite-dimensional } k\text{-algebra and}$$

(A1*) B is bimodule d-Calabi-Yau with Gorenstein parameter 1.

- $\Rightarrow C = eBe$ Iwanaga-Gorenstein

- Be_C is a $(d-1)$-cluster tilting object in $\mathrm{CM}(C)$
- $B \cong \mathrm{End}_C(Be)$
- $A = B_0$ is a $(d-1)$-representation-infinite algebra
  (i.e. $\mathrm{gl.dim}(A) < d$ and $\mathbb{S}_{d-1}^{-i} A \in \mathrm{mod}(A)$ for all $i \geq 0$)
- $B = \Pi_d(A)$ d-preprojective algebra of A where

$$\Pi_d(A) \;:=\; T_A\big({}_A \mathrm{Ext}^{d-1}(DA, A)_A\big) \;=\; A \oplus {}_A M_A \oplus {}_A(M \otimes M)_A \oplus \cdots$$

- $\underline{A} := A/(e)$ $(d-1)$-Auslander

$$\begin{tikzcd} C \arrow[r, shift left, "\text{alg.}"] & B \arrow[l, shift left, "e(-)e"] \arrow[r, shift left, "\text{deg.}0"] & A \arrow[l, shift left, "\Pi_d(-)"] \arrow[r, "(-)/(e)"] & \underline{A} \end{tikzcd}$$

- $d = 2$, Q Dynkin and $\widetilde{Q}$ extended Dynkin:

$$R = S^G \rightleftarrows S\#G = \mathrm{End}_R(S) \sim \Pi(K\widetilde{Q}) \rightleftarrows K\widetilde{Q} \longrightarrow KQ$$

Theorem 17.12. *Let* $\mathrm{gl.dim}(\underline{A}) \leq d-1$ *and* $A \twoheadrightarrow \underline{A}$. *Then:*

$$\begin{tikzcd} \mathrm{CM}^{\mathbb{Z}}(C) \arrow[d] & D^b(\mathrm{gr}(C)) \arrow[l] & D^b(A) \arrow[l] & D^b(\underline{A}) \arrow[l] \arrow[lll, bend right, "\sim"'] \arrow[d, "\pi"] \\ \underline{\mathrm{CM}}(C) \arrow[rrr, "\sim"] & & & \mathcal{C}_{d-1}(\underline{A}) \end{tikzcd}$$

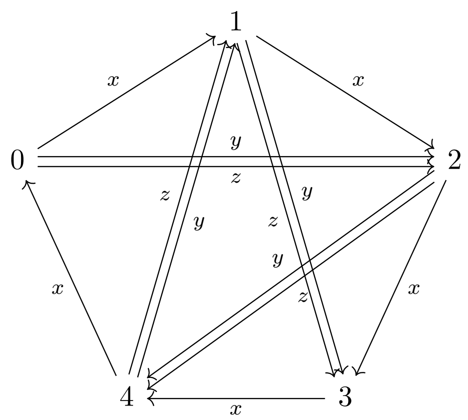
Theorem 17.13. *Assume the main setup.*
Let ζ be an n-th primitive root of unity.
Let $a_j \in \{1, \dots, n-1\}$ with $\sum_j a_j = n$ and $\gcd(a_j, n) = 1$ and

$$G = \left\langle \begin{pmatrix} \zeta^{a_1} & & \\ & \ddots & \\ & & \zeta^{a_d} \end{pmatrix} \right\rangle$$

and S $\frac{1}{n}\mathbb{Z}$-graded with $\deg(x_i) = \frac{a_i}{n}$. Then:

- $C = R = S^G = \bigoplus_{\ell \in \mathbb{Z}} S_\ell$
- $T := \bigoplus_{i=0}^{n-1} T^i \in \mathrm{CM}^{\mathbb{Z}}(R)$ *where* $T^i = \bigoplus_{\ell \in \mathbb{Z}} S_{\ell + \frac{i}{n}}$
- $B = \mathrm{End}_R(T) = S\#G$
- $A = \mathrm{End}_{\mathrm{gr}(R)}(T)$
- $\underline{B} = \mathrm{End}_{\underline{\mathrm{CM}}(R)}(T)$
- $\underline{A} = \mathrm{End}_{\underline{\mathrm{CM}}^{\mathbb{Z}}(R)}(T)$

Example 17.14. $d = 3$ and $G = \frac{1}{5}(1,2,2) = \frac{1}{n}(a_1, a_2, a_3)$:



modulo $xy = yx$, $yz = zy$, $zx = xz$ describes $\mathrm{mod}(A)$. Deleting 0 gives $\mathrm{mod}(\underline{A})$.

18 Triangulations, ice quivers and Cohen-Macaulay modules over orders

Friday 16th 8:30 – David Fernández Alvarez (Bielefeld, Germany)

You find David's handwritten notes after the following notes of his talk!

Goal. Give a survey of *Demonet–Lu*: "Ice quivers with potential associated with triangulations and Cohen-Macaulay modules over orders", *Trans. AMS 368(6), 2016, 4257–4293.*

Notations.

- k field
- $R = k[x]$
- $\mathcal{P}_n$ regular polygon of n sides and n vertices
- $Q = (Q_0, Q_1, h, t)$ finite connected quiver without loops, $Q_0 = \{1, \dots, n\}$

  $\rightsquigarrow$ kQ with multiplication $ab = \xrightarrow{a}\xrightarrow{b}$

18.1 Introduction

Representation theory: If you want to study a k-algebra A, you should study $\mathrm{mod}(A)$ (maybe with some restrictions).

Fruitful idea: Associate to $\mathrm{mod}(A)$ certain combinatorial invariants:
$\rightsquigarrow$ Auslander-Reiten quiver, exchange graph

$$\underset{\text{(hard)}}{\boxed{\text{Algebra}}} \longrightarrow \underset{\text{(hopefully easier)}}{\boxed{\text{Geometry}}}$$

However, in geometry ...

[Caldero–Chapoton–Schiffler]
[Fomin–Shapiro–Thurston
[Labardini-Fragoso]

$$\underset{\text{triangulated surface}}{\boxed{\text{Geometry}}} \longrightarrow \underset{\text{quiver with potential}}{\boxed{\text{“Combinatorics”}}} \longrightarrow \underset{\substack{\text{Jacobian algebra}\\ \mathcal{J} = kQ/\langle \partial_a W\rangle_{a\in Q_1}}}{\boxed{\text{Algebra}}}$$

in this talk: [Demonet-Lu]

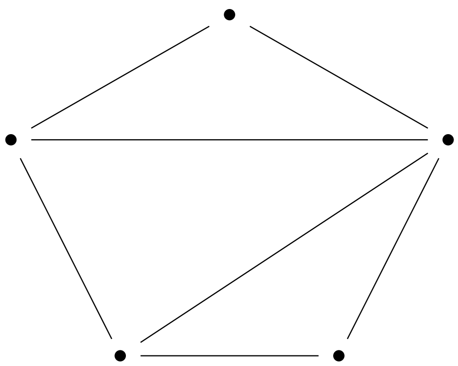
triangulated polygon $\mathcal{P}_n$ ice quiver with potential $\cdots + W_\sigma$ frozen Jacobian algebra Γ_σ

Idea: Study Γ_σ from the viewpoint of CM-representation theory; a lot of properties can be deduced from the triangulation of $\mathcal{P}_n$.

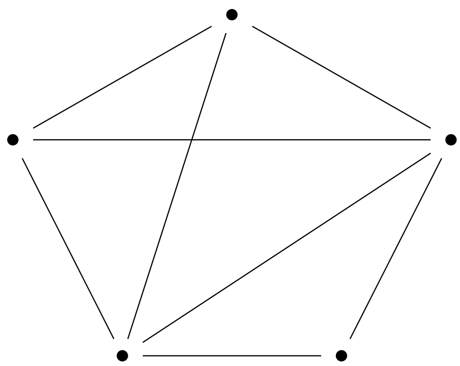
18.2 Ice quivers with potential associated to triangulations

Triangulations of polygons

A diagonal of $\mathcal{P}_n$ is a line segment connecting two vertices of $\mathcal{P}_n$ and lying in its interior.



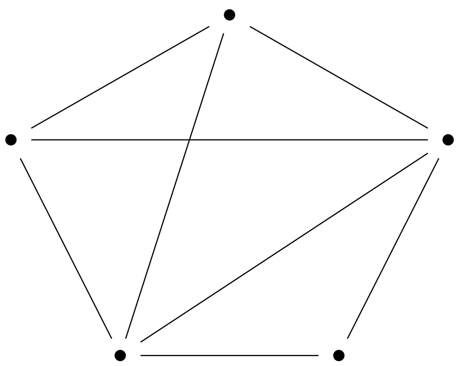
non-crossing diagonals

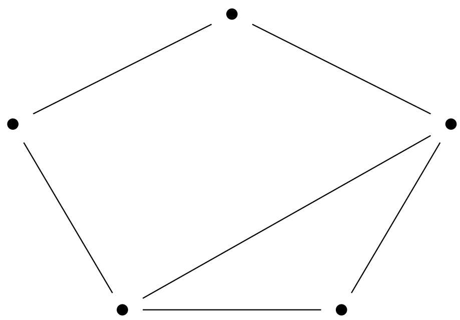


crossing diagonals

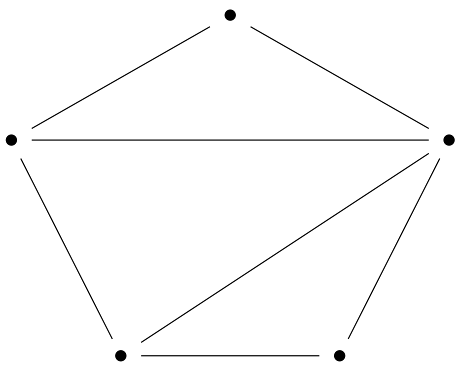
Definition 18.1. *A triangulation of $\mathcal{P}_n$ is a decomposition of $\mathcal{P}_n$ into triangles by a maximal set of non-crossing diagonals.*

not triangulations:





a triangulation:



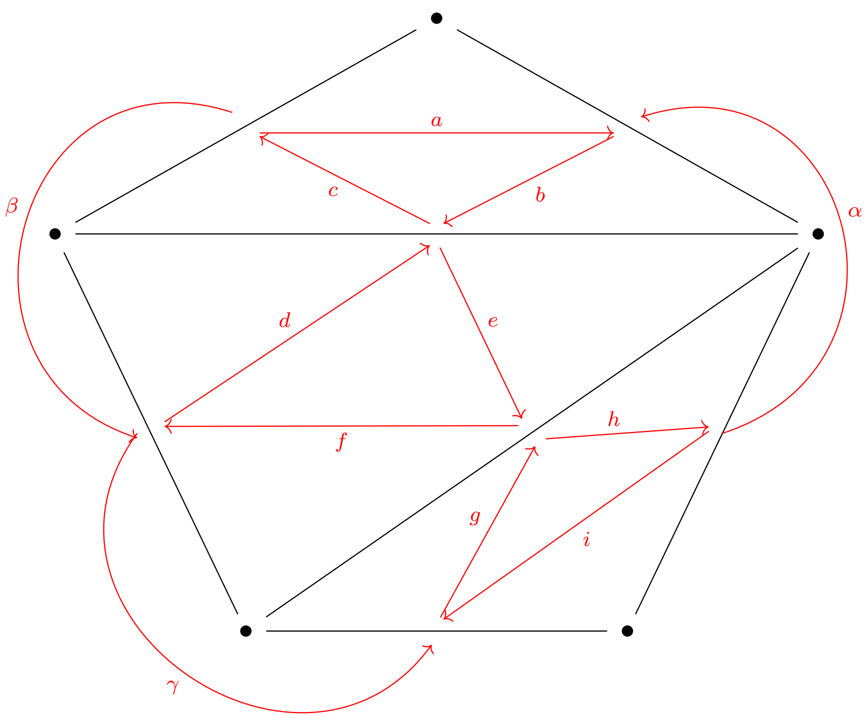
Quivers associated to triangulations

- vertices: middle points of diagonals and sides (edges)
- internal arrows: if two edges a and b are sides of a common triangle in σ there is an arrow $a \to b$ if a is a predecessor of b with respect to the anti-clockwise orientation centered at the common vertex.
- external arrows: there is $a \to b$ where a and b are incident sides at a common vertex (with at least one incident diagonal) such that a is a predecessor of b.

Algorithm

1. Draw the triangulation.
2. Tag the vertices.
3. Put the vertices of the quiver.
4. Draw internal arrows.
5. Draw external arrows.

Example 18.2.



$\rightsquigarrow$ triangulation σ

$\rightsquigarrow$ quiver Q_σ

Definition 18.3. *A minimal cycle of Q_σ is a cycle in which no arrow appears more than once and which encloses a part of the plane whose interior is connected and does not contain any arrows of σ.*

Example 18.4. Non-examples: $abc\beta\gamma gh$, $c\beta deh\alpha b$

Two types of minimal cycles:

- cyclic triangles:

  abc, def, ghi

- big cycles: internal arrows and one external arrow around a vertex of $\mathcal{P}_n$:

  αbeh, βdc, γgf

Ice quivers with potential associated to triangulations

In the previous situation:

- frozen vertices: $F = \{1, \ldots, n\} \subseteq (Q_\sigma)_0$
- frozen arrows: $(Q_\sigma)_1^F = \{a \in (Q_\sigma)_1 : h(a) \in F \text{ and } t(a) \in F\}$

Example 18.5. $F = \{1, \ldots, 5\}$ and $(Q_\sigma)_1^F = \{a, i, \alpha, \beta, \gamma\}$

Definition 18.6. *An ice quiver (associated to a triangulation σ) is the pair (Q_σ, F).*

Potentials (in general)

- Q arbitrary quiver
- kQ_i k-vector space with basis the paths of length i
- $kQ_{i,\text{cyc}} := kQ/[kQ_j, kQ_t]_{j+t=i}$ spanned by cycles in kQ_i

Definition 18.7. *An element $W \in \bigoplus_{i\geq 1} kQ_{i,\text{cyc}}$ is a potential.*

Kontsevich defined the cyclic derivative for each arrow $a \in Q_1$ as the k-linear maps

$$\bigoplus kQ_{i,\text{cyc}} \to kQ$$

defined on cycles as

$$\partial_a(a_1 \cdots a_d) \;=\; \sum_{a_i=a} a_{i+1} \cdots a_d a_1 \cdots a_{i-1}\,.$$

Example 18.8. $\partial_e(\alpha beh) = h\alpha b$.

Ice quivers with potential

We define the potential W_σ of (Q_σ, F) as

$$W_\sigma = \sum(\text{cyclic triangles}) - \sum(\text{big cycles}).$$

Definition 18.9. *An ice quiver with potential is a triple (Q_σ, F, W_σ).*

Example 18.10. $W_\sigma = abc + def + ghi - \alpha beh - \beta dc - \gamma gf$.

Frozen Jacobian algebras

Definition 18.11. *Let (Q_σ, F, W_σ) as above. We define the frozen Jacobian algebra as*

$$\Gamma_\sigma := kQ_\sigma \Big/ \langle \partial_a W_\sigma \rangle_{a \in (Q_\sigma)_1 \setminus (Q_\sigma)_1^F}.$$

18.3 Γ_σ is a tiled R-order

Theorem 18.12. *The frozen Jacobian algebra Γ_σ has the structure of a tiled R-order.*

Now set

$$e_F = \text{sum of idempotents at all frozen vertices in } Q_\sigma$$

and define the suborder

$$\Lambda_\sigma := e_F \Gamma_\sigma e_F.$$

Theorem 18.13. *The R-order Γ_σ is isomorphic to*

$$\Gamma := \begin{pmatrix} R & R & R & \cdots & R & (x^{-1}) \\ (x) & R & R & \cdots & R & R \\ (x^2) & (x) & R & \cdots & R & R \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ (x^2) & (x^2) & (x^2) & \cdots & R & R \\ (x^2) & (x^2) & (x^2) & \cdots & (x) & R \end{pmatrix}.$$

18.4 CM-modules over Λ

Theorem 18.14.

(i) For any triangulation σ and $(P_t, P_s) \in \sigma$ with $1 \le s < t \le n$ the vertex $j = (P_s, P_t)$ satisfies

$$e_F \Gamma_\sigma e_F \cong (s,t) = \begin{bmatrix} R & \cdots & R & (x) & \cdots & (x) & (x^2) & \cdots & (x^2) \end{bmatrix}$$

where there are s entries R, $t-s$ entries (x), and $n-t$ entries (x^2).

(ii) The construction in (i) induces 1:1 correspondences:

$$\{\textit{edges of } \mathcal{P}_n\} \longleftrightarrow \{\textit{ind. objs. of } \mathrm{CM}(\Lambda)\}$$

$$\{\textit{sides of } \mathcal{P}_n\} \longleftrightarrow \{\textit{ind. projs. of } \mathrm{CM}(\Lambda)\}$$

$$\{\textit{triangulations of } \mathcal{P}_n\} \longleftrightarrow \{\textit{basic cluster tilting objs. of } \mathrm{CM}(\Lambda)\}$$

18.5 Relation to cluster categories

Question: If we view the cluster algebra as a combinatorial invariant associated to the cluster category. Is the category determined by this invariant?

Using [Keller–Reiten '08]:

Theorem 18.15. *Let Λ be the R-order given above.*

(i) The stable category $\underline{\mathrm{CM}}(\Lambda)$ is 2-Calabi-Yau.

(ii) If k is perfect, then there exists a triangle equivalence $\mathcal{C}(kQ) \cong \underline{\mathrm{CM}}(\Lambda)$ for a quiver Q of type A_{n-3}.

TRIANGULATIONS, ICE QUIVERS &
COHEN-MACAULAY MODULES OVER ORDERS

DAVID FERNÁNDEZ | BIREP Summer School | 16 Aug 2019

<u>Goal</u>: Give a survey of

Demonet & Luo / Ice quivers with potential associated with triangulations and Cohen-Macaulay modules over orders.

Trans. AMS, 368 (6), 2016, 4257-4293.

Plan

1. Introduction
2. Ice quivers with potential assoc. to triang.
3. Γ_σ is a tiled R-order.
4. CM modules over Λ.
5. Relation to cluster categories

Notation

$k \equiv$ field ; $R := k[x]$.

$\mathcal{P}_n \equiv$ regular polygon of n vertices and n sides

$Q \equiv$ finite connected quiver without loops

$Q = (Q_0, Q_1, h, t) \rightarrow kQ$ w/ mult. (path algebra) $ab \equiv \overset{a}{\curvearrowright}\overset{b}{\curvearrowright}$ if $h(a) = t(b)$

$Q_0 = \{1, \dots, n\}$

$D := \mathrm{Hom}_k(-, k)$

$D_R := \mathrm{Hom}_R(-, R)$

1. Introduction

<u>Repr. th.</u>: If you want to study a k-alg. A, you should study $\mathrm{mod}\, A$ (maybe with some restrictions).

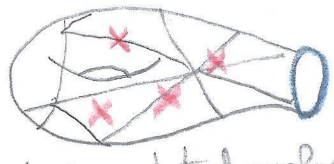
<u>Fruitful idea</u>: Associate to $\mathrm{mod}\, A$ certain combinatorial inv. E.g. → Auslander-Reiten quiver
→ Exchange graph.

$\boxed{\text{Algebra}}$ (Hard) $\longrightarrow$ $\boxed{\text{Combinatorics}}$ (hopefully easier).

However, in Geometry...

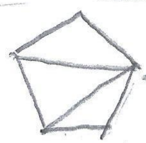
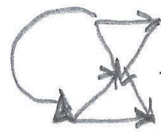
<u>Caldero-Chapoton-Schiffer</u>, <u>Fomin-Shapiro-Thurston</u>, <u>Labardini-Fragoso</u>...

$\boxed{\text{Geometry}}$ $\boxed{\text{Combinatorics}}$ $\boxed{\text{Algebra}}$


triangulated surface
$\longrightarrow$

quiver w/ potential
$+ W \longrightarrow \mathcal{J} := kQ / \langle \partial_a W \rangle_{a \in Q_1}$ Jacobian algebra

<u>In this talk</u> (<u>Demonet - Luo</u>):


triangulated polygon
$\xrightarrow{\mathrm{I}}$

ice quiver with potential
$+ W_\sigma \xrightarrow{\mathrm{II}} \Gamma_\sigma := \mathcal{J}(Q_\sigma, W_\sigma, F)$ frozen Jacobian algebra

<u>Idea</u>: Study $\mathcal{J}(Q_\sigma, W, F)$ from the viewpoint of Cohen-Macaulay repr. th.; a lot of properties can be deduced from the triangul.

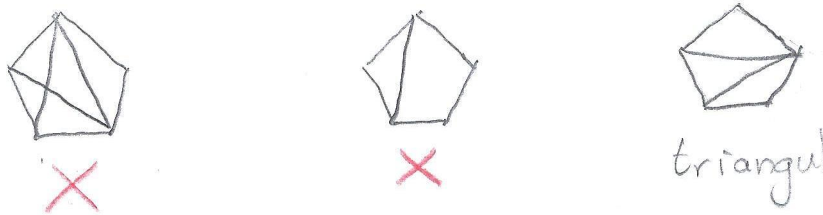
2. Ice quivers with potential assoc to triangulations

2.1. Triangulations of polygons

A diagonal of $\mathcal{P}_n$ is a line segment connecting two vertices of $\mathcal{P}_n$ and lying in its interior.

non-crossing diagonals crossing diagonals

DEF A triangulation of $\mathcal{P}_n$ is a decomposition of $\mathcal{P}_n$ into triangles by a maximal set of non-crossing diagonals.



triangulation.

2.2. Ice quivers with potential assoc. to triang.

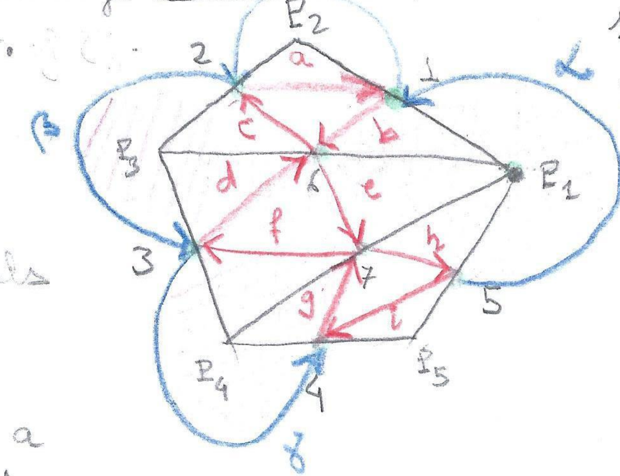
Quivers assoc to triangulations $|Q_0| = 2n-3$.

vertices: middle points of diagonals and sides ($\equiv$ edges)

internal arrows: If two edges a and b are sides of a common triangle of σ, $\exists\ a \to b$ if a is a predecessor of b wrt anti-clockwise orientation centered at the common vertex.

external arrows: $\exists\ a \to b$ where a and b are incident sides, a being a predecessor of b wrt to anti-clockwise orientation centered at the common vertex. (w/ at least one incident diagonal)

EXAMPLE



Algorithm

1. Draw the triangulation
2. Tag the vertices of $\mathcal{P}_n$ ↺
3. Put the vertices of the quiver.
4. Draw internal arrows.
5. Draw external arrows.

DEF

A minimal cycle of Q_σ is a cycle in which no arrow appears more than once, and which encloses a part of the plane whose interior is connected and does not contain any arrow of Q_σ.

No EXAMPLES: $\alpha bc\beta\gamma gh$ (contains f) $c\beta deh\alpha b$

Two types of minimal cycles:

* Cyclic triangles: abc def ghi
* Big cycles: internal arrows and one external arrow around a vertex of $\mathcal{P}$: αbeh βdc γgf

Ice quivers assoc. to triang

In the previous situation,

frozen vertices: $F = \{1, \ldots, m\} \subseteq (Q_\sigma)_0$

frozen arrows: $Q_1^F = \left\{ a \in (Q_\sigma)_1 \;\middle|\; \begin{array}{c} h(a) \in F \\ \text{and} \\ t(a) \in F \end{array} \right\}$

Example

$F = \{1, 2, 3, 4, 5\}$; $(Q_\sigma)_1^F := \{a, i, \alpha, \beta, \gamma\}$.

DEF

An ice quiver (assoc. to a triang. σ) is the pair (Q_σ, F).

Potentials (in general)

Let Q be an arbitrary quiver

$kQ_i \equiv k$-vector sp. w/ basis paths of length i.

$kQ_{i,cyc} := kQ_i / [kQ_i, kQ_i]$ ($\equiv$ cycles in kQ_i).

DEF

(i) An element $W \in \bigoplus_{i \geq 1} kQ_{i,cyc}$ is a potential

(ii) Two potentials are cyclically equiv if $W - W' \in [kQ, kQ]$

(iii) A quiver with potential is a pair (Q, W), where Q is a quiver without loops and W a potential with no two cyclically invariant terms.

Finally, Kontsevich defined the cyclic derivative for each arrow $a \in Q_1$ as the k-linear map

$$\bigoplus_{i \geq 1} kQ_{i,cyc} \longrightarrow kQ$$

defined on cycles by

$$\partial_a (a_1 \cdots a_d) = \sum_{a_i = a} a_{i+1} \cdots a_d a_1 \cdots a_{i-1}$$

In particular, we can compute $\{\partial_a W\}_{a \in Q_1}$.

Ice quivers with potential

Let σ be a triangulation of P_n;

(Q_σ, F) the assoc. ice quiver.

We define the potential W_σ of (Q_σ, F) as

$$W_\sigma := \sum (\text{cyclic triangles}) - \sum (\text{big cycles}).$$

DEF

An ice quiver with potential is a triple (Q_σ, W_σ, F).

Example

In the example above,

$$W_\sigma = abc + def + ghi - \alpha beh - \beta dc - \gamma gf.$$

2.3. Frozen Jacobian algebras

DEF

Let (Q_σ, W_σ, F) be an ice quiver with potential. We define the frozen Jacobian algebra as

$$\Gamma_\sigma := kQ_\sigma \Big/ \mathcal{J}(W_\sigma, F) = kQ_\sigma \Big/ \langle \partial_a W_\sigma \rangle_{a \in (Q_\sigma)_1 \setminus (Q_\sigma)_1^F}$$

Example

$$\Gamma_\sigma = kQ_\sigma \Big/ \left\langle \begin{matrix} b_1 c_1, & a_1 b_1 - b_2 a_3, & c_2 a_2 - a_3 c_1 \\ c_1 a_1 & b_2 c_2 & , \; b_3 c_3 - c_1 b_2 \\ & c_3 a_3 & \end{matrix} \right\rangle$$

DEF

We say that two paths w_1 and w_2 of Q_σ are equivalent $(w_1 \sim w_2)$ if $w_1 = w_2$ in Γ_σ.

LEMMA

Let P_{Q_σ} be the basis of the path alg. kQ_σ. The set $P_{Q_\sigma}/\sim$ is a basis of Γ_σ. Moreover, $e_i \Gamma_\sigma e_j \cap (P_{Q_\sigma}/\sim)$ is a basis of $e_i \Gamma_\sigma e_j$.

2.4. Interlude: technical ingredients

The central element C

Fact: Let $i \in (Q_\sigma)_0$. All minimal cycles of Q_σ passing through i are equal in Γ_σ. The common element will be denoted by C_i.

LEMMA A

The element $C = \sum_{i \in (Q_\sigma)_0} C_i$ is central in Γ_σ.

EXAMPLE

We have, e.g.

$C_1 = bca = beh\alpha, \quad C_2 = abc = \beta dc$

$\Rightarrow C = bca + abc + dc\beta + ghi + igh + cab + hig.$

Θ-lengths

Goal: Define a grading on kQ_σ s.t. $\mathcal{J}(W_\sigma, F)$ is a homog. ideal.

Fact: Each angle of P_n is $\frac{(n-2)\pi}{n}$, and the angles of triangles of σ are multiples of π/n.

DEF

The Θ-length of an

(i) an int. arrow is t $(1 \le t \le n)$ if the angle b/w two edges i and j of a Δ in σ is $t\pi/n$.

(ii) an ext arrow is 2.

Rmk: This extends additively to a map ℓ_θ from paths in Q_σ to $\mathbb{N}$, defining a grading on kQ_σ.

Example

$\ell_\theta(\alpha) = \ell_\theta(\beta) = \ell_\theta(\gamma) = 2$

$\ell_\theta(a) = \ell_\theta(i) = 3$

The rest of arrows have θ-length $= 1$.

LEMMA B

(i) If $w_1 \sim w_2$, then $\ell^\theta(w_1) = \ell^\theta(w_2)$ and ℓ_θ induces the str. of a graded alg. on Γ_σ.

(ii) Let $i, j \in (Q_\sigma)_0$. $(\exists)$ a path $w_0(i,j)$ from i to j with minimal θ-length s.t. for any path w from i to j,

$$w \sim w_0(i,j) C^N \text{ for some } N \in \mathbb{N}_{\geq 0}$$

3. Γ_σ is a tiled R-order

DEF

Let S be a comm. Noeth. PID ring of Krull dim 1.

(i) An S-algebra A is an S-order if it is a f.g. free S-module

(ii) For an S-order A, a left A-module M is a ($m \times l$) CM-A-module if it is f.g. as an S-module and $soc_S M = 0$ (equiv. $soc_A M = 0$). $CM(A)$ is the full exact subcateg. of mod A formed by CM-A-modules.

(iii) An S-order A is a tiled S-order if $A = ((x^{a_{ij}}))_{i,j \in \{1,\dots,n\}} \subseteq \mathbb{M}_n(R)$ for $a_{ij} \in \mathbb{Z}$ satisf. $a_{ij} + a_{jt} \geq a_{it}$, $\forall i,j,t \in \{1,\dots,n\}$

THM 1

The frozen Jacobian algebra Γ_σ has the str. of an R-order.

Sketch of the proof

Let $w_0(i,j)$ be the path from i to j constr. in Lemma B (ii)

Fact: For any two vertices $i, j \in (Q_\sigma)_0$,

$$\mathbb{N}_{\geq 0} \xrightarrow{\simeq} e_i \Gamma_\sigma e_j \cap (P_{Q_\sigma}/\sim), \; \ell \longmapsto w_0(i,j) C^\ell$$

Now, we consider the ring homom.

$$\phi : R \hookrightarrow \Gamma_\sigma, \; x^i \longmapsto C^i \quad (i \geq 0)$$

$\Rightarrow \Gamma_\sigma$ is an R-algebra.

By Lemma B and Fact, $(\exists)$ isom. of R-mod.

$$R \xrightarrow{\simeq} e_i \Gamma_\sigma e_j, \; r \longmapsto \phi(r) w_0(i,j)$$

$\Rightarrow \Gamma_\sigma = \bigoplus_{i,j \in (Q_\sigma)_0} e_i \Gamma_\sigma e_j$ is a f.g. free R-module $\Rightarrow \Gamma_\sigma$ is an R-order.

PROP C

Let $t \in (Q_\sigma)_0$. Then Γ_σ is isomorphic to the tiled R-order

$$\left(\left(x^{d^i_{t,j}} \right) \right)_{i,j \in \{1, \dots, 2n-3\}}$$

Finally,

$e_F \equiv$ sum of idempot. at all frozen vertices in Q_σ.

$\Lambda_\sigma := e_F \Gamma_\sigma e_F$ (suborder)

THM 2

The R-order Λ_σ is isomorphic to

$$\Lambda := \begin{pmatrix} R & R & R & \cdots & R & (x^{-1}) \\ (x) & R & R & \cdots & R & R \\ (x^2) & (x) & R & \cdots & R & R \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (x^2) & (x^2) & (x^2) & \cdots & R & R \\ (x^2) & (x^2) & (x^2) & \cdots & (x) & R \end{pmatrix}$$

DEF

Let $\mathcal{C}$ be a triang. or exact categ. $T \in Ob(\mathcal{C})$ is cluster tilting if

$$\operatorname{add} T = \{ Z \in \mathcal{C} \mid \operatorname{Ext}^1_{\mathcal{C}}(T, Z) = 0 \} = \{ Z \in \mathcal{C} \mid \operatorname{Ext}^1_{\mathcal{C}}(Z, T) = 0 \}$$

where add T is the set of fin. direct sums of direct summands of T

4. Cohen-Macaulay modules over Λ.

Idea ($\exists$) a close relation b/w the repr. of Λ and triang. of $\mathcal{P}_n$.

PROP D

(i) A CM-Λ-module is indecomposable if and only if it is isomorphic to

$$(i,j) = [\overbrace{R \cdots R}^{i} \; \overbrace{(x) \cdots (x)}^{j-i} \; \overbrace{(x^2) \cdots (x^2)}^{n-j}]^t$$

for some $1 \le i < j \le n$.

(ii) Any CM-Λ-module is isomorphic to $\bigoplus_{i,j} (i,j)^{l_{i,j}}$ for some non-negat. integers $l_{i,j}$. Moreover, $l_{i,j}$ are uniquely determined.

Rmk: The Krull-Schmidt-Azumaya property holds (despite of the base ring $R = k[x]$ is not local!).

THM 3

(i) For any triangulation σ and $(P_s, P_t) \in \sigma$ $(1 \le s < t \le n)$, the vertex $j = (P_s, P_t)$ satisfies that

$$e_F \Gamma_\sigma e_j \simeq (s,t) = [\overbrace{R \cdots R}^{s} \; \overbrace{(x) \cdots (x)}^{t-s} \; \overbrace{(x^2) \cdots (x^2)}^{n-t}]^t$$

(ii) The construction in (i) induces 1-1 correspond.:

$$\{\text{edges of } \mathcal{P}_n\} \longleftrightarrow \{\text{indec. obj. of } CM(\Lambda)\}/\simeq$$

$$\{\text{sides of } \mathcal{P}_n\} \longleftrightarrow \{\text{indec. proj. of } CM(\Lambda)\}/\simeq$$

$$\{\text{triangulations of } \mathcal{P}_n\} \longleftrightarrow \{\text{basic cluster tilting objects of } CM(\Lambda)\}/\simeq$$

5. Relation to cluster categories

Fomin – Zelevinski '02: a cluster algebra is a comm. $\mathbb{Q}$-algebra with a family of distinguished generators (cluster variables) grouped into overlapping subsets (the clusters) of finite cardinality, which are constructed recursively using mutations.

Idea: Categorify cluster alg (as a strategy to attack problems) $\longrightarrow$ $\boxed{\text{cluster category}}$

(i) That is, find some nice categ. (like module or triang) categ.) where we have some objects with similar properties as cluster and cluster variables.

(ii) If we view the cluster alg. as a combinatorial inv. assoc. to the cluster categ., Is the category determined by this invariant?

Auslander – Reiten translation

DEF

For an acyclic quiver Q, the cluster categ. $\mathcal{C}(kQ)$ is the orbit category $\mathcal{D}^b(kQ)/F$, where we have the functor $F = \tau^{-1}[1]$.

Rmks:

(i) $\mathcal{C}(kQ)$ have more indecomposable obj than mod kQ, and also more morphisms b/w the old objects.

(ii) (Keller '05) $\mathcal{C}(kQ)$ is Hom-finite and triangulated, and we have the functorial isom. $D\,\mathrm{Ext}^1_{\mathcal{C}(kQ)}(A,B) \simeq \mathrm{Ext}^1_{\mathcal{C}(kQ)}(B,A)$ (i.e. $\mathcal{C}(kQ)$ is 2-CY).

Regarding (ii) above: $\left\{\text{It's the stable categ. of a Frobenius category}\right.$

THM E (Keller – Reiten '08)

If k is a perfect field and $\mathcal{C}$ is an algebraic 2-CY triang. categ. with a cluster tilting object T s. t. $\mathrm{End}_{\mathcal{C}}(T) \simeq kQ$ hereditary, then ($\exists$) a triangle-equiv

$$\mathcal{C}(kQ) \longrightarrow \mathcal{C}.$$

THM 4

Let Λ be the R-order given above.

(i) The stable categ. $\underline{\mathrm{CM}}(\Lambda)$ is 2-CY.

(ii) If k is perfect, then ($\exists$) a triangle-equiv $\mathcal{C}(kQ) \simeq \underline{\mathrm{CM}}(\Lambda)$ for a quiver Q of type A_{n-3}.

Sketch of the proof of (ii)

Take the triang σ with set of diagonals $\{(P_1, P_3), (P_1, P_4), \text{—}, (P_1, P_{n-1})\}$

The full subquiver Q of Q_σ with the set of vertices $(Q_\sigma)_0 \setminus F$ is of type A_{n-3}

$$\Rightarrow \Gamma_\sigma^{op}/(e_F) \simeq (kQ)^{op}$$

Fact: Let $T_\sigma := e_F \Gamma_\sigma$. Through right multip, $\mathrm{End}_\Lambda(T_\sigma) \simeq \Gamma_\sigma^{op}$.

$$\Rightarrow \underline{\mathrm{End}}_\Lambda(T_\sigma) \simeq \Gamma_\sigma^{op}/(e_F)$$

$$\overset{\text{THM E}}{\Longrightarrow} \mathcal{C}((kQ)^{op}) \simeq \underline{\mathrm{CM}}(\Lambda).$$

19 What is (should be) a noncommutative resolution of singularities? – III

Friday 16th 10:00 – Graham Leuschke (Syracuse, United States)

See also Graham's notes!

Last time:

[van den Bergh]: An NCCR of a $\boxed{\text{Gorenstein}}$ normal domain R is an R-algebra Λ which is a

$$\underline{\text{symmetric}}\ \underline{\text{birational}}\ \text{nonsingular}\ \text{order}\ .$$

Equivalently*,

$$\Lambda \cong \operatorname{End}_R(M) \text{ for some reflexive } {}_RM \text{ with } \operatorname{gl.dim}(\Lambda) < \infty \text{ and } \Lambda \text{ MCM over } R\,.$$

*These are not equivalent if R is not Gorenstein (example last time).

The following implication fails:

$$\begin{array}{l}\text{symmetric} +\\ \text{finite gl. dim}\end{array} \Rightarrow \text{nonsingular}$$

Perhaps we can improve the situation for non-Gorenstein rings by considering totally reflexive modules rather than MCMs.

Several times this week, the distinction between $\mathrm{CM}(R)$ and $\mathrm{GP}(R)$ has come up.

Definition 19.1. *An R-module M (where R is any commutative ring) is totally reflexive (or Gorenstein projective) if*

- $M \cong M^{**}$ *(reflexive),*
- $\operatorname{Ext}_R^{>0}(M, R) = 0$*,*
- $\operatorname{Ext}_R^{>0}(M^*, R) = 0$*.*

Fact 19.2. *For a Gorenstein local ring R, this is equivalent to M being MCM.*

For CM rings, total reflexivity is stronger.

So let's consider totally reflexive R-algebras Λ.

Definition 19.3. *A strong noncommutative resolution of singularities of a Cohen-Macaulay normal domain R is an R-algebra Λ of the form $\Lambda = \operatorname{End}_R(M)$ for some reflexive ${}_RM$ with $\operatorname{gl.dim}(\Lambda) < \infty$ and ω totally reflexive as R-module.*

Observation 19.4. *If R is Gorenstein, this is just an NCCR.*

Theorem 19.5 (Stangle '15)**.** *If R has a strong NC resolution, then R is Gorenstein.*

(so "strong" means "too strong")

Proof. Enough to consider local $(R, \mathfrak{m}, k)$ and show

$$\operatorname{Ext}^i_R(k, R) = 0 \text{ for } i \gg 0\,.$$

Let Λ be a strong NC resolution. Then $\Lambda/\mathfrak{m}\Lambda$ is a k-vector space of finite dimension, so it is enough to show

$$\operatorname{Ext}^i_R(\Lambda/\mathfrak{m}\Lambda, R) = 0 \text{ for } i \gg 0\,.$$

Since Λ is totally reflexive as R-module, we know

$$\operatorname{Ext}^j_R(\Lambda, R) = 0 \text{ for } j > 0\,.$$

One can show (spectral sequence or by hand)

$$\operatorname{Ext}^i_R(\Lambda/\mathfrak{m}\Lambda, R) \;\cong\; \operatorname{Ext}^i_\Lambda(\Lambda/\mathfrak{m}\Lambda, \operatorname{Hom}_R(\Lambda, R))$$

but Λ has finite global dimension, so that vanishes for $i \gg 0$. □

That word "crepant"

Let X be a CM algebraic variety.

Let ω_X be the canonical sheaf (dualizing sheaf) of X.

If $\widetilde{X} \xrightarrow{\pi} X$ is a resolution of singularities, there is also a canonical sheaf $\omega_{\widetilde{X}}$ and in fact

$$\omega_{\widetilde{X}} \;\cong\; \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_{\widetilde{X}}, \omega_X) \qquad \big(\omega_{\widetilde{X}} \text{ is "co-induced" from } \omega_X\big)\,.$$

We could also induce ω_X up to $\widetilde{X}$

$$\pi^*\omega_X \text{ "="} \;\omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_{\widetilde{X}}\,.$$

The resolution π is crepant if

$$\pi^*\omega_X \;\cong\; \omega_{\widetilde{X}}\,.$$

The discrepancy divisor of π is the difference between $\pi^*\omega_X$ and $\omega_{\widetilde{X}}$.

$$\text{\sout{not dis}crepant} \overset{\text{[Miles Reid]}}{=} \text{crepant}$$

In the special case where X is Calabi-Yau, i.e. $\omega_x \cong \mathcal{O}_X$ we get

$$\omega_{\widetilde{X}} \;\cong\; \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_{\widetilde{X}}, \mathcal{O}_X)\,.$$

So π is crepant iff

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_{\widetilde{X}}, \mathcal{O}_X) \;\cong\; \mathcal{O}_{\widetilde{X}}$$

i.e. $\mathcal{O}_{\widetilde{X}}$ is a symmetric $\mathcal{O}_X$-algebra (sheaf).

Fact 19.6 (from Algebraic Geometry)**.** *If X over $\mathbb{C}$ has a crepant resolution of singularities, then it has at worst rational ("nice" / "mild") singularities.*

Question If a (Gorenstein) ring R has an NCCR, must $\operatorname{Spec}(R)$ have at worst rational singularities?

Answer Yes.

Theorem 19.7 (Stafford–van den Bergh)**.**

Let k be an algebraically closed field of characteristic 0 and Δ a prime affine k-algebra which is finitely generated as a module over its center $Z(\Delta)$. If Δ is a nonsingular order over $Z(\Delta)$, then $\operatorname{Spec}(Z(\Delta))$ has at worst rational singularities.

In particular, if a Gorenstein normal domain R has an NCCR Λ, then $R = Z(\Lambda)$ and so $\operatorname{Spec}(R)$ has at worst rational singularities.

What are NCCRs good for?

The minimal model program (MMP) is a strategy for carrying out a birational classification of algebraic varieties.

It consists of "moves" which are intended to improve the variety until you can't improve it further (terminal singularities).

Bondal & Orlov suggest to view the "moves" as operations / functors on the bounded derived category.

Example 19.8. Blowing up a smooth subvariety (that's one of the "moves") induces a fully faithful functor (even an SOD) on the bounded derived category.

Example 19.9. Another "move" is a flop: replace Y by Y'

$$\begin{tikzcd} Y \arrow[dr, "f"'] & & Y' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

where f and f' are both crepant resolutions of singularities of X (+ some other technical condition).

Conjecture 19.10 (Bondal–Orlov '99)**.** *If Y and Y' are related by a flop, then they are derived equivalent:*

$$D^b(\operatorname{coh}(Y)) \simeq D^b(\operatorname{coh}(Y'))$$

Theorem 19.11 (Bridgeland 2002)**.** *The BO Conjecture holds for* $\dim(Y) = 3$.

Bridgeland's proof uses Fourier-Mukai transforms.

Around the same time, Bridgeland–King–Reid ['01] described an approach to the McKay Correspondence based on Fourier-Mukai transforms.

[van den Bergh]: "An essential feature of the McKay Correspondence is the appearance of a noncommutative ring $S\#G$, the twisted group ring."

Theorem 19.12 (van den Bergh 2004)**.** *Let R be a Gorenstein normal $\mathbb{C}$-algebra and let $X = \operatorname{Spec}(R)$ and $\pi\colon \widetilde{X} \to X$ a crepant resolution of singularities. Assume the fibers of π are at most 1-dimensional (automatic if $\dim(X) \leq 3$). Then R has an NCCR Λ and*

$$D^b(\operatorname{mod}(\Lambda)) \simeq D^b(\operatorname{coh}(\widetilde{X}))\,.$$

Corollary 19.13. *The BO Conjecture holds in dimension* 3*:*

$$\begin{tikzcd} D^b(\mathrm{coh}(Y)) \arrow[dr, "\simeq"'] & & D^b(\mathrm{coh}(Y')) \arrow[dl, "\simeq"] \\ & D^b(\mathrm{mod}(\Lambda)) & \end{tikzcd}$$

One can strengthen the BO Conjecture:

Conjecture 19.14 (Iyama–Wemyss, "ncBO Conjecture")**.** *All crepant resolutions of a given variety/ring are derived equivalent, the commutative and the noncommutative ones.*

Known in dimension ≤ 3 by [Iyama–Wemyss].