BIREP SUMMER SCHOOL 2025
SUGGESTED TALKS

These are the suggested topics for the talks to be delivered by the participants at the BIREP Summer School on Finite Tensor Categories.

The talks with a star ($*$) are optional: the remaining talks without a star are obligatory, since they contain prerequisite material for other talks.

If you have a question or want to suggest a topic that would fit well with the program, send us an email to birep-school@math.uni-bielefeld.de.

Contents

1. Monday

1.1. Reminders on Abelian Categories. Talk 1.1 will be a reminder on Grothendieck categories, the ind-completion, and decomposition properties in locally finite abelian categories. A brief summary of many of the notions can be found in [Kra22, Glossary].

(1) **Abelian Categories**
   - Recall the notions of *abelian/Grothendieck* categories [EGNO15, 1.2.1, 1.3.1], and of left/right exact functors [EGNO15, 1.4.1, 1.6.1].
   - Recall the definitions of *projective* objects [EGNO15, 1.6.5], of *projective covers* [EGNO15, 1.6.6], and of *enough projectives* [EGNO15, 1.8.6(iii)].
   - Recall the construction of derived functors in an abelian category, in particular, define the Ext functor; see `stacks.math.columbia.edu/tag/05TA`.

(2) **Ind-completion**
   - Define the *ind-completion* of an additive category as the category of filtered colimits of representable contravatiant functors [Kra, 1.4].
   - State (without proof) that the ind-completion of an essentially small abelian coincides with category of left exact functors [Kra22, 11.1.8, 11.1.14].
   - State (without proof) that the category of left exact additive functors to abelian groups is Grothendieck [Par70, p. 234, Theorem 2].

(3) **Locally finite Abelian Categories**
   - Define what it means for an object of an abelian category to be (semi)simple [EGNO15, 1.5.1] and recall Schur's lemma [EGNO15, 1.5.2].
   - Define what it means for an object to be of finite length [EGNO15, 1.5.3], and recall the Jordan-Hölder Theorem [EGNO15, 1.5.4].
   - Define what it means for an object to be indecomposable [EGNO15, 1.5.6], and recall the Krull-Schmidt Theorem [EGNO15, 1.5.7].
   - Define *locally finite* abelian categories [EGNO15, 1.8.1], and define *finite* abelian categories to be the locally finite categories with finitely many iso-classes of simple objects and enough projectives [EGNO15, 1.6.5, 1.6.6, 1.8.6].

1.2. **Algebras and Coalgebras.** Main resource: CFG Notes [GP25]. We want introduce the notions of algebras, the their dual notion of a coalgebra. k will always be a ring in the following:

(1) **Algebras and Modules**
   - Rewrite the definition of an associative unital k-algebra via commutative diagrams [EGNO15, 7.8.1] (take $\mathcal{C} = \mathrm{Vec}$). Likewise define modules over such an algebra [EGNO15, 7.8.5], algebra and module homomorphisms, and the category of modules over an algebra [EGNO15, 7.8.6(ii)].

(2) **Coalgebras and Comodules** - Following [GP25, §2] and [EGNO15, §1.9].
   - Define an associative unital k-*coalgebra* via the dual commutative diagrams [EGNO15, 1.9.1], *comodules* over coalgebras [EGNO15, 1.9.2], coalgebra and comodule morphisms, and the category of comodules [GP25, Definitions 2.1, 2.3, 2.7, 2.9].
   - Show that the tensor product of comodule can be given the structure of a comodule [GP25, Example 2.6].
   - Show or state that for C a k-coalgebra, and W a k-module, that $W \otimes_k C$ has a natural C-comodule structure, moreover show that $\mathrm{Hom}_k(V, W) \simeq \mathrm{Hom}^C(V, W \otimes_k C)$ for V a C-comodule [GP25, Theorem 2.11].
   - Show that for C a flat k-coalgebra, the category of comodules over C is abelian [GP25, Proposition 2.13 and Theorem 2.14].
   - Show that for C a flat k-coalgebra, the category of comodules over C has enough injectives [GP25, Theorem 2.15].

(3) **Duality**
   - Let C be a k-coalgebra and A be a k-algebra. Show that $\mathrm{Hom}_k(C, A)$ has a natural unital k-algebra structure defined via the convolution product $*$. In particular, the dual of a k-coalgebra is naturally a k-algebra [GP25, Example 2.4].
   - State (without proof) that the dual of an k-algebra A can be given a coalgebra structure only the case that A is finitely generated and projective as a k-module [GP25, Remark 2.5].
   - Show that a C-comodule can always be made into a C^*-module, in particular, that $\mathbf{coMod}_C$ is always a subcategory of ${}_{C^*}\mathbf{Mod}$. In the case where C is fintely generated and projective over k, show that theses two categories are equivalent. See [GP25, Remark 2.17 and Theorem 2.18].

1.3. **Hopf Algebras.** Main resource: [GP25]. We cover the basics of Hopf algebras, as well as the overflow from the last talk on coalgebras.

(1) **Bialgebras**
- Define a *bialgebra* as a k-module that is simultaneously both an algebra and a coalgebra such that the comultiplication and counit are algebra homomorphisms, and the unit and multiplication are coalgebra homomorphisms [EGNO15, 5.2.1, 5.2.2].
- Define what it means for a bialgebra to have an *antipode* [EGNO15, 5.3.2].

(2) **Hopf Algebras**
- Define a *Hopf algebra* as a bialgebra with an antipode S, via commutative diagrams. Explain that S is a left and right convolution inverse to the identity, [GP25, Definition 3.1], [EGNO15, 5.3.10].
- Define a *morphism of Hopf algebras* and the *category of Hopf algebras* over k [GP25, Definition 3.1]. State (without proof) that any map of bialgebras $f\colon H \to H'$ is a map of Hopf algebras automatically [GP25, Lemma 3.3].
- When H is finitely generated and projective explain that H^* is also a Hopf algebra, using the duality section above [GP25, Remark 3.4].
- Define $\mathbf{Rep}H \coloneqq \mathbf{coMod}_H$. For Hopf algebras which are finitely generated and projective over k, use the duality section from the previous talk to show this is equivalent to ${}_{H^*}\mathbf{Mod}$ [GP25, Remark 3.9].

(3) **Examples** - The speaker can choose from the following examples, or include examples of their own.
- *Group schemes*: Define a group scheme over k as a group object in the category of affine schemes over k, and show that the category of group schemes over k is antiequivalent to the category of commutative Hopf algebras over k. Explicitly name a few of your favourite group schemes, e.g., the bialgebra of functions from a finite group G to k, see [EGNO15, Exercise 5.2.6].
- *Enveloping algebras and quantum groups*, see [EGNO15, Example 5.5.1 and Sections 5.6 and 5.7].

1.4. **Monoidal Categories.** In Talk 1.4 we will cover the basic definitions and results on monoidal categories, properties of the unit object, examples, rigid monoidal categories, dualizable objects, group actions on a category, and group objects internal to a category.

(1) **Monoidal categories and functors**
  - Define *monoidal categories* [EGNO15, 2.1.1 2.1.2] and *monoidal subcategories* [EGNO15, 2.1.4], and describe some basic properties of the unit object, such as [EGNO15, 2.2.2, 2.2.3, 2.2.4, 2.2.5].
  - Provide some examples of monoidal categories, such as: the category of sets [EGNO15, 2.3.1]; any additive category [EGNO15, 2.3.2]; the category of all modules over a commutative ring [EGNO15, 2.3.3]; the category of A-A-bimodules for a ring A, and monoidal subcategories (quasicoherent sheaves) [EGNO15, 2.3.13]; representations of a group [EGNO15, 2.3.6]; and endomorphisms of a category [EGNO15, 2.3.12].
  - Define *monoidal functors* [EGNO15, 2.4.1] and the *morphisms* (natural transformations) between them [EGNO15, 2.4.8], and provide examples of monoidal functors, such as: forgetful functors [EGNO15, 2.5.1]; or those given by of a map of rings; see `stacks.math.columbia.edu/tag/0GP2`.
  - Define the *action* of a group on a monoidal category [EGNO15, 2.7.1].

(2) **Dual objects and rigidity**
  - Define the meaning of a left (or right) *dual* of an object, and discuss the *evaluation* and *coevaluation* maps [EGNO15, 2.10.1/2].
  - Provide examples of cases where the left and right duals are understood, such as: the category of finitely generated projective modules over an algebra [EGNO15, 2.10.16]; and the category of finite dimensional representations of a group [EGNO15, 2.10.13].
  - State (without proof) that left duals give rise to adjunctions [EGNO15, 2.10.8], and show that duals are unique [EGNO15, 2.10.5].
  - Example: left duals in $\mathrm{End}(\mathcal{C})$ are precisely left adjoints [EGNO15, 2.10.4].

1.5. **Tensor and Multitensor Categories.** In Talk 1.5 we will cover the basic definitions and properties of tensor categories and multitensor categories, and give several examples considered in later talks.

(1) **Multitensor Categories**
- Define *rigid* monoidal categories [EGNO15, 2.10.11], and define *multitensor* categories [EGNO15, 4.1.1] and *tensor* categories as certain rigid locally finite k-linear monoidal categories.
- Give examples of multitensor categories, such as the category of finite dimensional modules over a commutative k-algebra [EGNO15, 2.3.3], and give examples of tensor categories, such as the category of finite-dimensional representations of a group G [EGNO15, 4.1.2].
- State and prove the biexactness of the tensor product [EGNO15, 4.2.1].
- State (without proof) that duality preserves projectives, and so projectives are injective, in multitensor categories [EGNO15, 6.1.3].

(2) **Multiring categories and finite tensor categories**
- Define *multiring* and *ring* categories [EGNO15, 4.2.3], and state (without proof) that for any multiring category with left duals, the left dualisation functor is exact [EGNO15, 4.2.9].
- Define (quasi)tensor functors [EGNO15, 4.2.5].
- Consider examples of *finite tensor categories* (tensor categories that are finite abelian categories) such as: finite-dimensional representations of a finite group [EGNO15, 4.1.2]; or finite-dimensional modules over a commutative semilocal k-algebra.
- Define *quasi-Frobenius* categories. Using that duals of projectives are projective, prove that multitensor categories are quasi-Frobenius [EGNO15, 6.1.4].

2. Tuesday

2.1. **Module Categories.** In Talk 2.1 we will recall module categories over monoidal, multitensor, quasi-Frobenius and finite tensor categories, and what it means when such a module category is exact.

(1) **Module categories over monoidal categories**
- Define a (left/right) *module category over a monoidal category*, and submodule categories. Point out that a (muliti)tensor category is a module category over itself [EGNO15, 7.4.1].
- Consider a finite group G with a subgroup L and the restriction functor $\mathrm{Rep}(G) \to \mathrm{Rep}(L)$, this makes $\mathrm{Rep}(L)$ into a $\mathrm{Rep}(G)$-module. Show that $\mathbf{Vec}_G$-modules are just abelian categories with a G action [EGNO15, 7.4.9].
- Show that an action of a monoidal cateogry $\mathcal{C}$ on a $\mathcal{C}$-module category $\mathcal{M}$ is the same as a monoidal functor functor $\mathcal{C} \to \mathrm{End}(\mathcal{M})$ [EGNO15, 7.1.3].
- Define module functors and equivalences [EGNO15, 7.2.1], direct sums [EGNO15, 7.3.4, 7.3.5] and indecomposables [EGNO15, 7.3.6].

(2) **Exact module categories over multitensor categories**
- Define *module categories over multitensor categories* [EGNO15, 7.3.1], and note that the (bilinear) actions for (locally finite) $\mathcal{C}$-module categories over mutlitensor categories are the same as tensor (so monoidal) functors to a category of left-exact endofunctors [EGNO15, 7.3.3].
- Define *exact* module categories (over a multitensor category with enough projectives) [EGNO15, 7.5.1]. Consider the case over **Vec** [EGNO15, 7.5.4].
- Consider the example of group-graded vector spaces. Note that representations of the group are the same as morphisms between endofunctors of **Vec** [EGNO15, 7.12.19].

(3) **Projectives, injectives and decompositions**
- Show that exact module categories over multitensor categories have enough projectives [EGNO15, 7.6.1] and are quasi-Frobenius [EGNO15, 7.6.3, 7.6.4]
- Relating a pair of simple objects in an exact module category provided one is a subquotient of the image of the other under a tensor functor, prove this relations is reflexive, symmetric and transitive [EGNO15, 7.6.6].
- Show that the partition induced by this equivalence relation gives a decomposition of any exact module category into indecomposable module subcategories [EGNO15, 7.6.7].

2.2. **Coends, Takeuchi's theorem and Deligne's tensor product.** In Talk 2.2 we will recall coends, and then define Deligne's tensor product, using a theorem of Takeuchi.

(1) **Coends**
- Recall the definition of the *coend* of a functor, and explicitly define (using a formula) the comultiplication and counit that give a coalgebra structure on the coend [EGNO15, p. 14].
- State (without proof) that every exact faithful functor F from a k-linear abelian category to k-vector spaces defines an equivalence to the category of right comodules over the coend of F [EGNO15, 1.10.1].

(2) **Takeuchi's theorem**
- Define *pointed coalgebras* [EGNO15, 1.19.13] and give some examples.
- State and sketch the proof of the following theorem of Takeuchi: That any essentially small, locally finite, abelian category is the category of pointed comodules over a unique pointed coalgebra [EGNO15, 1.19.15, p. 15].

(3) **Deligne's tensor product**
- Define *Deligne's tensor product* using Takeuchi's theorem [EGNO15, 1.11.1], and discuss how it satisfies a universal property [EGNO15, 1.11.2].
- Show that if $\mathcal{C}$ and $\mathcal{D}$ are (multi)ring categories then so is $\mathcal{C} \boxtimes \mathcal{D}$ [EGNO15, 4.6.1]. Mention, in particular, that the Deligne tensor product of tensor categories is again a tensor category [EGNO15, 4.6.2].
- Define *bimodule categories* [EGNO15, 7.1.7], and state (without proof) that any tensor category $\mathcal{C}$ is a module category over $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{op}}$ [EGNO15, 7.4.2], and that a $(\mathcal{C}, \mathcal{D})$-bimodule category is the same thing as a $\mathcal{C} \boxtimes \mathcal{D}^{\mathrm{op}}$-module category [EGNO15, 7.4.3].

2.3. **Reconstruction Theory.** In Talk 2.3 we will cover an important result known as the reconstruction theorem for Hopf algebras.

(1) **Fiber Functors**
  - Define (quasi) *fiber functors* [EGNO15, 5.1.1], and note how forgetful functors are examples [EGNO15, 5.1.2]. Explain how a fiber functor $\mathcal{C} \to \mathbf{Vec}$ is the same as a $\mathcal{C}$-module action on **Vec** [EGNO15, 7.4.6].
  - For a finite ring category equipped with a fiber functor F, use Deligne's tensor product to define the biagebra structure on $\mathrm{End}(F)$ [EGNO15, 1.11.1, 5.2.1].
  - Prove that there is a mutually inverse bijection between finite ring categories with a fiber functor and finite dimensional bialgebras [EGNO15, 5.2.3].
  - If the finite ring category has left duals, mention the extra structure on $\mathrm{End}(F)$, given by a morphism $S\colon \mathrm{End}(F) \to \mathrm{End}(F)$. Prove this morphism satisfies the antipode map diagram [EGNO15, 5.3.1].
  - Prove the reconstruction theorem: There is a mutually inverse bijection between finite tensor categories with a fiber functor and finite dimensional Hopf algebras [EGNO15, 5.3.12].

(2) **Quasi-fiber functors**
  - Define *normalized* quasi-fiber functors [EGNO15, 5.12.1].
  - Discuss how, given a normalized quasi-fiber functor (F, J), the comultiplication defined for the algebra $\mathrm{End}(F)$ (as for the case of fiber functors) need not satisfy the coassociativity axioms to be a bialgebra [EGNO15, p. 110].
  - Define *quasi-bialgebras* [EGNO15, 5.12.4, 5.12.5], and state (without proof) the reconstruction theorem in this setting [EGNO15, 5.12.6, 5.12.7].
  - Define *quasi-Hopf* algebras [EGNO15, 5.13.2], and state (without proof) the reconstruction theorem: a bijection from finite tensor categories with a quasi-fiber functor to finite-dimensional quasi-Hopf algebras [EGNO15, 5.3.17].

2.4. $\mathbb{Z}_+$**-rings.** In Talk 2.4 we will cover the definition of the Frobenius-Perron dimension and some related results which will be of interest later.

(1) $\mathbb{Z}_+$**-rings**
- Define $\mathbb{Z}_+$-*bases* and $\mathbb{Z}_+$-*rings* [EGNO15, 3.1.1, 3.1.3], and provide examples, such as: the ring of $n \times n$ matrices over $\mathbb{Z}$; the group algebra $\mathbb{Z}G$ for a finite group G, and its centre $C(\mathbb{Z}G)$; and the ring of complex representations of G [EGNO15, 3.1.9].
- State (without proof) the *Brouwer fixed point theorem for simplexes*, see for example [Iva09] (there are many references).
- State the *Frobenius-Perron* theorem [EGNO15, 3.2.1], and prove the first part [EGNO15, 3.2.1(1)].

(2) **Frobenius-Perron Dimension**
- Define *transitive* $\mathbb{Z}_+$-rings [EGNO15, 3.3.1], and decide which of the examples of $\mathbb{Z}_+$-rings (previously considered above) are transitive.
- Introduce the *Frobenius-Perron dimension* $\mathrm{FPdim}\colon A \to \mathbb{C}$ for A a unital transitive $\mathbb{Z}_+$-ring of finite rank [EGNO15, 3.3.3].
- Consider the transitive $\mathbb{Z}_+$-based ring whose basis is given by a finite group equipped with a generator following [ENO05, Example 8.19], and mention the value of FPdim in this case.

(3) **Image and invariance of the FP dimension**
- Show there exists a regular element $R \in A \otimes \mathbb{C}$ which is unique with the property that $XR = \mathrm{FPdim}(X)R$ and $RY = \mathrm{FPdim}(Y)R$, and use this to prove that, under mild conditions, $\mathrm{FPdim}(X)$ is the largest eigenvalue for a matrix corresponding to X [EGNO15, 3.3.6(2,4)].
- Prove that FPdim is invariant under basis change [EGNO15, 3.3.9].
- Prove that FPdim takes values in the algebraic integers [EGNO15, 3.3.4].

2.5. **FP dimension for Finite Tensor Categories.** In Talk 2.5 we will cover the Frobenius-Perron dimension of a mutlitensor category by means of the regular object in a module over its Grothendieck ring, and discuss how integral finite-tensor categories are the same as representation categories of finite-dimensional quasi-Hopf algebras.

(1) **Grothendieck groups and rings**
   - Define the *Grothendieck group* $\mathrm{Gr}(\mathcal{C})$ of a locally finite abelian category $\mathcal{C}$ by isoclasses of simple objects [EGNO15, 1.5.8]. Provide some examples of when the Grothendieck group is known [EGNO15, 1.5.9, 1.5.10].
   - State (without proof) that for a multiring category $\mathcal{C}$ the tensor product induces a multiplication on the Grothendieck group $\mathrm{Gr}(\mathcal{C})$ [EGNO15, 4.5.1], turning it into a $\mathbb{Z}_+$-ring [EGNO15, 4.5.5].
   - Show that if the module category is indecomposable, then its Grothendieck group is irreducible [EGNO15, 7.7.2].

(2) **FP-dimension of a category**
   - For a finite abelian category $\mathcal{C}$ define $K_0(\mathcal{C})$ like the Grothendieck group [EGNO15, p. 11], but where simples are replaced by projective indecomposables, and define the *Cartan matrix* [EGNO15, 1.5.8, 1.8.14].
   - Prove that, for a multitensor category $\mathcal{C}$, the group $K_0(\mathcal{C})$ is a module over the ring $\mathrm{Gr}(\mathcal{C})$ [EGNO15, 6.1.1].
   - For a finite tensor category $\mathcal{C}$ define the *regular object* $R_{\mathcal{C}} \in K_0(\mathcal{C})$ and the *Frobenius-Perron* dimension $\mathrm{FPdim}(\mathcal{C}) = \mathrm{FPdim}(R_{\mathcal{C}})$ [EGNO15, 6.1.6, 6.1.7].
   - Consider examples of where this dimension is known, such as for semisimple multitensor categories [EGNO15, 6.1.8], or the category of representations of a finite-dimensional quasi-Hopf algebra [EGNO15, 6.1.9].

(3) **Main Theorem**
   - Define *fusion rings* [EGNO15, 3.1.7], *integral* fusion rings [EGNO15, 3.5.5], and *integral* finite tensor categories [EGNO15, 6.1.13].
   - Use the reconstruction theory, developed in Talk 2.3, to prove that a finite tensor category is integral if and only if it is the representation category of a finite-dimensional quasi-Hopf algebra [EGNO15, 6.1.14].

3. Wednesday

3.1. **Morita equivalence and some constructions.** In this talk, we will recall an equivalence relation on tensor categories known as *categorical Morita equivalence*; we also review some basic constructions with module categories that are of particular interest. This talk covers Chapter 7.11-7.14 in [EGNO15]. See also [Ost03].

(1) **Morita equivalence**
   - Define the category of *right exact* $\mathcal{C}$-module functors between module categories over a multitensor category $\mathcal{C}$ [EGNO15, p. 154]. In particular, discuss on [EGNO15, Proposition 7.11.1].
   - For an exact module category $\mathcal{M}$ over a multitensor category $\mathcal{C}$, define the dual category of $\mathcal{C}$ with respect to $\mathcal{M}$ [EO03, Definition 7.12.2].
   - Explain why the dual category is a finite multitensor category and when it is indeed a tensor category [EGNO15, 7.12.6].
   - Define Morita equivalence for tensor categories [EGNO15, 7.12.17] and give an sketch of the proof that it is actually an equivalence relation [EGNO15, 7.12.18].

(2) **The center**
   - Define the *center* of a monoidal category [EGNO15, Definition 7.13.1].
   - Explain why the center of a finite multitensor category is finite [EGNO15, Proposition 7.13.8].

(3) **The Quantum Double**
   - Let H a finite dimensional Hopf algebra, and $F\colon \mathbf{Rep}H \to \mathbf{Vec}$ the associated fiber functor such that $H = \mathrm{End}(F)$ (which we have proven exists via reconstruction theory in Talk 2.3). Then the composition $F \circ i = F'$ where i is the inclusion of the center, gives us a new fiber functor F'. Put $D = \mathrm{End}\, F'$. This is called the *quantum double* $D(H)$ of H [EGNO15, 7.14.1].
   - Describe the structure of $D(H)$ very explicitly as the quotient of the free product $H * H^{*\mathrm{cop}}$ [EGNO15, p. 164]
   - From this show that $Z(\mathbf{Rep}H)$ is equivalent to the category of finite dimensional $D(H)$-modules [EGNO15, 7.14.6].
   - Let G be a finite group and H be the Hopf algebra of k valued functions on G. So the dual of H, with opposite comultiplication, is kG. Describe the quantum double $D(H)$. This is done explicitly in [DPR92].

4. Thursday

4.1. **Braided Categories.** In this talk we will cover the basic definitions of a braided category, as well as some first properties and examples of braided categories coming from Hopf algebras.

(1) **Basic Definitions**
- Define a *braiding* on a monoidal category as a natural isomorphism $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ satisfying some commutative diagrams. Then define a braided monoidal category as a monoidal category equipped with such a braiding. [EGNO15, Definitions 8.1.1 and 8.1.2].
- Define what it means for a monoidal functor between braided monoidal categories to be *braided* [EGNO15, Definition 8.1.7]. Define a *symmetric* monoidal category as a braided monoidal category satisfying the condition that $c_{Y,X} \circ c_{X,Y} = 1_{X \otimes Y}$ [EGNO15, Definition 8.1.12].
- Give some basic examples of braided monoidal categories, such as: the category of sets; modules over a commutative ring; the category of representations of a Hopf algebra; graded modules over a ring R, which has as many braidings as there are units u in R, and one of these braidings is symmetric if and only if $u^2 = 1$ [EGNO15, Section 8.2].
- Explain why the center $\mathcal{Z}(\mathcal{C})$ of a monoidal category $\mathcal{C}$ (as defined in the previous talk) is a braided monoidal category [EGNO15, 8.5.1].

(2) **Quasitriangular Hopf Algebras**
- Motivate and define a *quasitriangular* Hopf algebra as a Hopf algebra H such that there exists an invertible element $R \in H \otimes H$ satisfying some special equations [EGNO15, Definition 8.3.1].
- Explain how H being quasitriangular is equivalent to the condition that the category of representations of H is braided monoidal category. In particular, that there is a bijection between braidings on $\mathbf{Rep}(H)$ and quasitriangular structures on H.
- Define a *triangular Hopf algebra* as in [EGNO15, 8.3.3], and explain how the above bijection restricts to a bijection between symmetric braidings on $\mathbf{Rep}(H)$ and triangular structures on H.

(3) **Examples and computations**
- Explain why a cocommmutative Hopf algebra H is quasitriangular by considering $R = 1 \otimes 1$ [EGNO15, Example 8.3.4].
- State, without proof, that the quantum double $D(H)$ of any finite-dimensional Hopf algebra H is quasi-triangular [EGNO15, 8.3.8].

Let G be a finite group and H be the Hopf algebra of k valued functions on G.
- Explain why, if G is noncommutative, then $\mathbf{Rep}(H)$ does not admit a braiding, and hence H does not have a quasi-triangular structure [EGNO15, 8.3.5].
- Describe the universal R-matrix of $D(H)$. See [DPR92].

4.2. **Fusion Categories.** For this talk, points (3) and (4) are interchangeable and just one of them might be presented.

(1) **Definitions and Examples**
- Recall the definition of a simple and semisimple object, and then the definition of a semisimple category [EGNO15, Definition 1.5.1].
- Define a *multifusion* category as a finite semisimple multitensor category, and similarly define a fusion category as a finite semisimple tensor category. Make explicit the conditions on the ground field k [EGNO15, 4.1.1].
- Give some basic examples of (multi)fusion categories, such as: finite dimensional vector spaces; finite dimensional representations of a Hopf algebra; finite dimensional representations of a Lie algebra; bimodules over a finite dimensional semisimple algebra [EGNO15, Section 4.1].

(2) **Ocneanu Rigidity**
- Explain why a fusion category have no non-trivial deformations. Instead of a proof, an example would be great. For instance, mention the case of a separable algebra, see [EGNO15, Section 9.1].
- Conclude that there are countably many fusion categories over k up to tensor equivalence [EGNO15, 9.1.6].

(3) **Group Theoretical Fusion Categories**
- Define a *group theoretical* fusion category [EGNO15, 5.11.1, 9.7.1].
- In order to give examples, recall the monoidal category of G–graded vector spaces and its twisted version by a cocycle of $\omega \in H^3(G, k^\times)$, see [EGNO15, Example 2.3.8], the latter is denoted by Vec_G^ω.
- Explain the structure of a module category over Vec_G^ω, see [EGNO15, Example 9.7.2].
- Discuss on the dual fusion category of a group theoretic one, and mention why group theoretic fusion categories have integral FP-dimension [EGNO15, 4.5.9, 7.16.7, 9.4.2, 9.6.2, 9.8.2, 9.9.11].

(4) **Tannakian Fusion Categories**
- Recall the definition of a symmetric braided fusion category [EGNO15, 8.1.12].
- Define a *Tannakian* fusion category as a symmetric fusion category that admits a braided tensor functor to the monoidal category of vector spaces [EGNO15, 9.9.16].
- Discuss [EGNO15, Theorem 9.9.22], and sketch a proof that any symmetric braided category is equivalent (as braided categories) to one of the form $\mathrm{Rep}(G, z)$, where the latter denotes the category of representation of a finite group G with a braiding induced from a central element z in G (see [EGNO15, Example 9.9.1]).

4.3. **Cohomological Finite Generation for Finite Group Schemes*.** This talk could take two different directions: (a) One could cover some of the history of the cohomological finite generation conjecture for finite tensor and mention some of the cases where it is known, without mathematical details aside from the relevant definitions; (b) one could specialize to the case of cohomology of finite group schemes. Some relevant points to cover during the talk:

(1) **Basic definitions:** This part makes sense independently of the direction chosen.
   - Quick remainder on extension groups in an abelian category and the Yoneda product, see [Kra22, Section 4.2] and [SA04].
   - For a finite tensor category define is graded cohomology ring. See for instance [EO03].
   - State the cohomological finite generation (CFG) conjecture.

(2) **For direction** (a)**:** There are many classes of finite tensor categories for which the CFG conjecture is known, so it is hard to give a full account on those; here are some references that might be useful:
   - The introduction of [FN18].
   - Cohomology of finite group schemes [Pev13].
   - Pointed finite tensor categories [LL24].
   - Duals and Drinfeld centers of finite tensor categories which satisfy the CFG conjecture [NP22], also see Corollary 4.8 in *loc. cit.*

(3) **For direction** (b)**:** CFG for finite groups schemes over a field was proved in [FS97] and generalized to finite group schemes over a commutative Noetherian base in [vdK23]. While the proof is quite involved, one can highlight the following ideas which are essentially following the introduction in [GP25].
   - CFG for finite group schemes over a Noetherian base can be translated to cohomological finite generation of GL_n via the so-called embedding lemma together with finite generation of invariants for GL_n.
   - van der Kallen proved that for CFG for GL_n reduces to bounded torsion of the cohomology ring. This relays on the existence of universal classes in cohomology [TvdK10], [FS97] and the existence of certain filtration of GL_n-algebras called good-Grosshans filtrations.
   - The cohomology of a finite group scheme over an arbitrary Noetherian base has bounded torsion, this is the content of [vdK23].

4.4. **Support Varieties*.** Main resource for this talk is [BKSS20], and the goal is to define support varieties for triangulated categories with an action of a monoidal triangulated category. This is an approach that unifies several constructions of support varieties and it applies in great generality, hence its relevance. The main ideas to cover during the talk are:

(1) **Preliminaries:** One should briefly recall the notion of monoidal triangulated category (not necessary symmetric), and then cover the following points.
   - Define a tensor action of a monoidal triangulated category on a triangulated category. See also [Ste13].
   - Explain how a monoidal triangulated category acts on itself.
   - Define the graded center of a monoidal triangulated category and central actions by graded rings on a triangulated category.
   - Explain why the graded endomorphism ring of the monoidal unit always acts centrally on the category.

(2) **Support varieties:** For this part, one can follow [BKSS20, Section 3].
   - Introduce the restrictions placed in Assumption 3.1 in [BKSS20].
   - Briefly recall the *homogeneous spectrum* of a commutative graded ring.
   - Define the support of pair of objects with respect to the homogeneous spectrum of a ring acting centrally on the category, and discus on the properties it has.
   - Define *Koszul objects* and point out and describe their support.

(3) **Complexity:** Fix a central ring action on a triangulated category as before. Then
   - Define the complexity of an object. Also define perfect objects.
   - Explain why objects with trivial complexity are perfect and why the converse also holds (see [BKSS20, Proposition 4.4]).

(4) **Examples:** Consider a finite dimensional algebra A over an algebraic closed field, and its bounded derived category on finitely generated A-modules $\mathbf{D}^b(A)$. Specialize the discussion above to central action on $\mathbf{D}^b(A)$ by the Hochschild cohomology ring $\mathrm{HH}^*(A)$ (see also [EHS$^+$03]). In particular, explain Proposition 9.6 in [BKSS20].

5. Friday

5.1. **Complexes with small homology*.** The content of talk is strongly related with a conjecture of G. Carlsson in [Car86] which says that for an elementary abelian p-group G, any finite dimensional complex of free $\mathbb{F}_pG$-modules the sum of the dimensions of the homology groups of the complex is at least 2^r. There is also a topological version of this conjecture. The algebraic version is known to be false by work of Iyengar and Walker. In [Ber24], Bergh extends the study of complexes with small homology to the context of finite tensor categories, and the goal of this talk is to introduce basic terminology and ideas to understand the what it means for a complex of projective objects in a finite tensor category to have small homology. The main ideas to cover during the talk are:

(1) **Preliminaries:** One should have for granted that at this point we all know about the CFG conjecture for finite tensor categories, but it might be worth recalling:
   - The category of complexes over an additive category.
   - The homology of a complex and recall the definition of quasi-isomorphism.
   - The homotopy category, and its bounded version. In particular, explain that whenever we consider the bounded homotopy category of projective objects in an abelian category, the homotopy equivalences are simply quasi-isomorphisms.

(2) **Complexes over finite tensor categories:** we need first to place some restrictions on our finite tensor categories, e.g., we assume that the satisfy the CFG conjecture. Then:
   - Define additive functions on finite tensor categories, and highlight examples such as the FP dimension and the length of objects.
   - Recall support varieties for objects in a finite tensor category in terms of the homogeneous spectrum of the cohomology ring.
   - Highlight Construction 3.1 in [Ber24] and explain why it is a complex of projective objects.

(3) **Examples:** Explain why we recover the examples of Carlson and Iyengar-Walker when considering the group algebra $\mathbb{F}_pG$. Also, discuss on the case that the finite tensor category is the category of representations of a finite dimensional Hopf algebra.

5.2. **tt-geometry of the stable module category of a finite group (scheme)*.** The goal of this talk is to cover the basic notions of tt-geometry which will serve as a reference/comparison for the non-commutative setting. In particular, the main example to keep in mind for this part is the stable module category of a finite group (scheme) over a field of positive characteristic. Points to cover during the talk:

(1) **Recollections on tt-geometry:** Briefly recall basic terminology from [Bal05], [Bal10b]:
   - Include the definition of an essentially small tensor triangulated category.
   - Explain the tt-structure on the stable module category of a finite group.
   - Define the Balmer spectrum via prime ideals, its topology using the support, and discuss on the universal property of this construction.

(2) **Functoriality of the Balmer spectrum:** Explain why the construction of the Balmer spectrum is functorial with respect to tt-functors. Moreover, include the following properties:
   - The Balmer spectrum of the idempotent completion of a tt-category.
   - A essentially surjective functor induces an inclusion on Balmer spectra.
   - When a tt-functor induces a surjection on Balmer spectra? see [Bal18].

(3) **Ordinary vs modular characteristic:** Explain why the tt-geometry of the stable module category in the ordinary characteristic is not so interesting. In fact, using the above tools, one can check that is also not very interesting for $D^b(kG)$, with the monoidal structure induced by $\otimes_k$.

(4) **Examples:** The idea is to explain the Balmer spectrum of the stable module category of a finite group in the modular case. This was proved in [BCR97]. Also compare with the approach in [Bal10a, Section 8] using the comparison map to Zariski spectra. Note that this uses the identification of the stable module category with the Verdier quotient $\mathbf{D}^b(kG)/\mathrm{Perf}(kG)$.

5.3. **Non-commutative tt-geometry*.** The main reference for this subsection is [NVY22]. The general idea for this talk is to define the non-commutative Balmer spectrum, explain its relevance, and point out the main differences with its counterpart in the commutative setting. If time permits, one should try to give an example. The relevant ideas to cover during the talk are: (note that point (1) is only relevant if the talk 5.2 is omitted.)

(1) **Recollections on tt-geometry:** Briefly recall basic terminology from [Bal05], [Bal10b]:
   - Definition of an essentially small tensor triangulated category.
   - Examples: For instance, the stable category of finitely generated representations of a group scheme over a field.
   - Prime ideals, and support of objects in order to define the topology in the Balmer spectrum.

(2) **Monoidal tensor categories:** Set up terminology from [NVY22, Section 2] in order to include the following points:
   - The different notions of primes that one could consider.
   - The non-commutative Balmer spectrum.

(3) **Comparison with the commutative case**
   - In the commutative setting there are two very important properties of the Balmer spectrum: first, it is not empty as soon as the category is not trivial; second, the construction is functorial with respect to tensor triangulated functors. Discuss these two properties for the two constructions in part (2). One way to motivate this is from ordinary commutative and non-commutative algebra.

(4) **Support datum**
   - The relevant part here is to define the notion of a support datum on the non-commutative Balmer spectrum and explain the universal property it has. This is covered in [NVY22, Section 4].

(5) **Examples:** While diving in full detail into examples is complicated, it would be useful to include an a discussion on one of the following examples.
   - The stable module categories of small quantum groups for Borel subalgebras [NVY22, Section 8].
   - The stable module categories of the Benson–Witherspoon Hopf algebras [NVY22, Section 9].

5.4. **The spectrum of a finite tensor category*.** The main reference for this talk is [NVY24]. The goal is to introduce the necessary terminology to present Conjecture E in *loc. cit.*, discuss on the consequences of its validity, and present examples where it is known. One should try to cover the following points:

(1) **Recollections:** Briefly recall the construction of the stable category of a finite tensor category, in particular:
   - The triangulated structure. In fact, this construction makes sense for any Frobenius exact category, so one can present it in this generality.
   - The monoidal structure. In particular, explain why it is compatible with the triangulated structure.

(2) **Graded center and categorical center:** This part will allow us to introduce different support data on a monoidal triangulated category.
   - Cohomological support for the cohomology ring and for the Tate cohomology ring of a monoidal triangulated category.
   - Define the categorical center of the (Tate) cohomology ring and the central cohomological support, and include examples.
   - The finite generation and the weak finite generation conjectures.

(3) **Comparison map and Conjectures:** This part focus on comparing the support data introduced in the previous point.
   - Define the comparison map from the Balmer spectrum to the homogeneous spectrum of the categorical center of the (Tate) cohomology ring.
   - Spell out some of the properties of this map, in particular those included in [NVY24, Theorem B].
   - Mention that for some Hopf algebras this comparison map lands in the projectivization of the homogeneous spectrum of the cohomology ring of the Hopf algebra.
   - Include [NVY24, Conjecture E], and its relation to the (weak) finite generation for finite tensor categories.

(4) **Examples:** Include one of the examples where the conjecture has been verified. See Section 9 in [NVY24].

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