HOPF MODULES AND INTEGRALS: THE SPACE OF INTEGRALS

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Throughout, H denotes a finite dimensional Hopf algebra over a field k. As usual, the comultiplication, the counit and the antipode of H are denoted Δ, ε and η, respectively. Recall that

$$\int_H^\ell := \{x \in H \ ; \ hx = \varepsilon(h)x \quad \forall\, h \in H\} \quad \text{and} \quad \int_H^r := \{x \in H \ ; \ xh = \varepsilon(h)x \quad \forall\, h \in H\}$$

are the subspaces of left and right integrals of H, respectively. The object of this lecture is the ensuing

Theorem ([3])**.** *The following statements hold:*
(1) $\dim_k \int_H^r = 1$.
(2) *The antipode η is bijective.*
(3) $\eta(\int_H^r) = \int_H^\ell$.

The main idea of the proof is to endow H^* with the structure of a Hopf module and use the fundamental theorem [2] to show $\dim_k \int_{H^*}^r = 1$. Since H^* is also a Hopf algebra, the asserted result follows.

The multiplication and comultiplication on H^* are given by the following formulae:

$$(\varphi\psi)(h) := \sum_{(h)} \varphi(h_{(1)})\psi(h_{(2)}) \qquad \forall\, \varphi, \psi \in H^*,\ h \in H$$

and

$$\Delta(\varphi) = \sum_{(\varphi)} \varphi_{(1)} \otimes \varphi_{(2)} \ \Leftrightarrow\ \varphi(hh') = \sum_{(h)} \varphi_{(1)}(h)\varphi_{(2)}(h') \qquad \forall\, h, h' \in H.$$

These rules are obtained by dualizing those for H. For instance, the multiplication m_{H^*} is the composite

$$m_{H^*} : H^* \otimes_k H^* \longrightarrow (H \otimes_k H)^* \xrightarrow{\Delta_H^*} H^*.$$

The counit and the antipode of H^* are defined via

$$\varepsilon^*(\varphi) = \varphi(1) \ \text{ and } \ \eta^*(\varphi) = \varphi \circ \eta \quad \forall\, \varphi \in H^*,$$

respectively. In a similar fashion, the vector space H^* obtains the structure of a Hopf module for H by postulating

$$(h.\varphi)(x) := \varphi(\eta(h)x) \qquad \forall\, h, x \in H,\ \varphi \in H^*$$

as well as

$$\nabla(\varphi) = \sum_{(\varphi)} \varphi_{(0)} \otimes \varphi_{(1)} \ \Leftrightarrow\ \varphi\psi = \sum_{(\varphi)} \psi(\varphi_{(0)})\varphi_{(1)} \qquad \forall\, \psi \in H^*$$

for every $\varphi \in H^*$. Taking these structures for granted, we can prove our Theorem.

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Proof. By the fundamental theorem of Hopf modules (cf. [2]), the multiplication induces an isomorphism

$$\Phi : H \otimes_k (H^*)^{\mathrm{co}H} \longrightarrow H^* \ \ ; \ \ h \otimes \varphi \mapsto h.\varphi.$$

Given $\varphi \in (H^*)^{\mathrm{co}H}$, we have $\nabla(\varphi) = 1 \otimes \varphi$, so that $\varphi\psi = \psi(1)\varphi$ for all $\psi \in H^*$. Consequently, $(H^*)^{\mathrm{co}H} \subset \int^r_{H^*}$. The reverse inclusion follows analogously. Since $\dim_k H = \dim_k H^*$, we obtain $\dim_k \int^r_{H^*} = 1$. Replacing H by H^*, while observing $(H^*)^* \cong H$, yields (1).

Let $h \in \ker \eta$. Pick $\varphi_0 \in \int^r_{H^*} \setminus \{0\}$. Then we have $\Phi(h \otimes \varphi_0) = 0$, so that $h = 0$. As a result, η is injective and hence bijective.

Assertion (3) now follows from direct computation, using the fact that η is an anti-homomorphism of associative algebras. □

Examples. (1) Suppose that $H = kG$ is the group algebra of a finite group. Then $x := \sum_{g\in G} g$ is a two-sided integral of kG.

(2) In general, integrals of Hopf algebras are not easy to find. Suppose that $\mathrm{char}(k) = p > 0$ and let $\mathfrak{g} = kt \oplus kx$ be the two-dimensional non-abelian restricted Lie algebra with restricted enveloping algebra $U_0(\mathfrak{g})$. Thus, $U_0(\mathfrak{g})$ is generated by t and x subject to the relations $t^p = t, x^p = 0, tx-xt = x$. The generators are primitive elements (that is, they satisfy $\Delta(y) = y \otimes 1 + 1 \otimes y$) and hence are annihilated by ε. Moreover, $\eta(t) = -t$ and $\eta(x) = -x$. Then

$$(t^{p-1} - 1)x^{p-1} \in \int^\ell_{U_0(\mathfrak{g})}$$

is a non-zero (!) left integral and $x^{p-1}(t^{p-1} - 1)$ is a right integral. Since

$$(t^{p-1} - 1)x^{p-1}t = (t^{p-1} - 1)x^{p-1}$$

the left integral is not a right integral.

We record an important consequence of the main theorem, namely H being a Frobenius algebra. Despite the title of their article [3], the authors were apparently not aware of this fact at the time of writing[1].

Corollary 1. *Let $\pi \in \int^\ell_{H^*}$ be non-zero left integral of H^*. Then*

$$(x, y) := \pi(xy) \qquad \forall\ x, y \in H$$

defines a non-degenerate associative form on H. In particular, H is a Frobenius algebra.

Proof. Writing $(h * \varphi)(x) := \varphi(xh)$ for $h, x \in H$ and $\varphi \in H^*$, we consider the canonical homomorphism

$$\Psi : H \longrightarrow H^* \ \ ; \ \ h \mapsto h * \pi.$$

In view of our theorem, $\varphi_0 := \pi \circ \eta$ is a non-zero right integral of H^* and the map

$$\Phi : H \longrightarrow H^* \ \ ; \ \ h \mapsto h.\varphi_0$$

is an isomorphism. Direct computation shows that $\eta^{-2}(\ker \Psi) \subset \ker \Phi = (0)$. Consequently, Ψ is an isomorphism, and [1, Lemma 1] implies the result. □

[1] On page 85 of [3] they note: "The referee has pointed out to us that our main theorem implies that every finite dimensional Hopf algebra with antipode is a Frobenius algebra."

Our next application is often referred to as "Maschke's Theorem for Hopf algebras". Given two H-modules M, N, we recall that $\mathrm{Hom}_k(M,N)$ obtains the structure of an H-module via

$$(h.\varphi)(m) = \sum_{(h)} h_{(1)}\varphi(\eta(h_{(2)})m)$$

for all $h \in H$, $m \in M$, $\varphi \in \mathrm{Hom}_k(M,N)$.

Corollary 2. *The following statements are equivalent:*
(1) *H is semi-simple.*
(2) $\varepsilon(\int_H^\ell) \neq (0)$.

Proof. (1) $\Rightarrow$ (2). By assumption, the exact sequence

$$(0) \longrightarrow \ker\varepsilon \longrightarrow H \longrightarrow k \longrightarrow (0)$$

splits, so that $H = \ker\varepsilon \oplus \int_H^\ell$.

(2) $\Rightarrow$ (1). The assumption entails the splitting of the above exact sequence. As a result, the trivial H-module k is projective. Let P be a projective H-module, M be any H-module. The adjoint isomorphism

$$\mathrm{Hom}_k(P\otimes_k M, N) \cong \mathrm{Hom}_k(P, \mathrm{Hom}_k(M,N))$$

induces an isomorphism

$$\mathrm{Hom}_H(P\otimes_k M, N) \cong \mathrm{Hom}_H(P, \mathrm{Hom}_k(M,N)).$$

Consequently, $\mathrm{Hom}_H(P\otimes_k M, -)$ is exact, so that $P\otimes_k M$ is projective. Setting $P = k$, we see that $k\otimes_k M \cong M$ is projective. This shows that H is semi-simple. □

Examples. (1) Let G be a finite group and consider the integral $x := \sum_{g\in G} g \in kG$. Then $\varepsilon(x) = \mathrm{ord}(G).1$, so that kG is semi-simple if and only if $\mathrm{char}(k) \nmid \mathrm{ord}(G)$.

(2) Let $\mathfrak{g} = kt \oplus kx$ be as above. Then $\varepsilon((t^{p-1}-1)x^{p-1}) = (0)$, so that $U_0(\mathfrak{g})$ is not semi-simple. In fact, $\mathrm{Rad}(U_0(\mathfrak{g})) = U_0(\mathfrak{g})x$.

Corollary 3. *If H is semi-simple, then H is separable.*

Proof. Let K be an extension field of k. Then $H' := H\otimes_k K$ obtains the structure of a Hopf algebra by defining $\Delta' = \Delta\otimes \mathrm{id}_K$. Here we use the identification $(H\otimes_k K)\otimes_K (H\otimes_k K) \cong (H\otimes_k H)\otimes_k K$. Since the counit ε' of H' is given by $\varepsilon\bar{\otimes}\,\mathrm{id}_K$, we get

$$\int_{H'}^\ell = \int_H^\ell \otimes_k K.$$

Thus, if H is semi-simple, then

$$\varepsilon'(\int_{H'}^\ell) = \varepsilon(\int_H^\ell)K \neq (0),$$

so that H' is also semi-simple. Consequently, H is separable. □

References

[1] R. Farnsteiner. *Self-injective algebras: Comparison with Frobenius algebras*. Lecture Notes, available at http://www.mathematik.uni-bielefeld.de/~sek/selected.html

[2] ______. *Hopf modules and integrals: The fundamental theorem*. Lecture Notes, available at http://www.mathematik.uni-bielefeld.de/~sek/selected.html

[3] R. Larson and M. Sweedler. *An associative orthogonal form for Hopf algebras*. Amer. J. Math. **91** (1969), 75-94