SEPARATED QUIVERS AND REPRESENTATION TYPE

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Much of the early work in the representation theory of algebras focused on two classes of algebras, namely hereditary algebras and group algebras of finite groups. The purpose of this lecture is to illustrate that basic results concerning the former class yield useful information for arbitrary associative algebras. Our elementary approach employs functors that are sufficiently well-behaved to preserve the representation type of an algebra. There are many results concerning such functors in the literature, for instance [6].

Throughout, we will be working over an algebraically closed field k. Given a finite dimensional k-algebra Λ, the determination of its representation type is a fundamental problem. For hereditary algebras, the beautiful solution has spawned investigations of the deep connections with Lie theory:

Theorem 1 ([4, 2])**.** *Let Q be a finite connected quiver without loops or oriented cycles. The path algebra $k[Q]$ is of finite representation type if and only if the underlying graph $\bar{Q}$ of Q is a Dynkin diagram A_n, D_n, or $E_{6,7,8}$.* □

Shortly after Gabriel's determination of the representation-finite hereditary algebras, Donovan-Freislich and Nazarova independently classified the tame hereditary algebras:

Theorem 2 ([3, 7])**.** *A connected path algebra $k[Q]$ is tame if and only if the underlying graph $\bar{Q}$ of Q is a Euclidean diagram $\tilde{A}_n$, $\tilde{D}_n$, or $\tilde{E}_{6,7,8}$.* □

The foregoing results can be used to obtain information on the Gabriel quivers of arbitrary algebras of finite or tame representation type. Here tame algebras are understood to be representation-infinite. Factor algebras of tame algebras are thus tame or representation-finite.

In the sequel, $k[T]$ denotes the polynomial ring with indeterminate T. We let $[M]$ be the isoclass of the module M.

Lemma 3. *Suppose that for each $d > 0$ there exist $(\Lambda, k[T])$-bimodules $X_1, \ldots, X_{s(d)}$ that are finitely generated over $k[T]$ and such that all but finitely many isoclasses of d-dimensional indecomposable Λ-modules are of the form $[X_i \otimes_{k[T]} k_\lambda]$ for some $i \in \{1, \ldots, s(d)\}$ and some algebra homomorphism $\lambda : k[T] \longrightarrow k$. Then Λ is tame or representation-finite.*

Proof. For $d > 0$ we let $T(X_i)$ be the torsion submodule of the $k[T]$-module X_i. Then $T(X_i)$ is a sub-bimodule of X_i, and there exists $f_i \in k[T] \setminus \{0\}$ such that $xf_i = 0$ for all $x \in T(X_i)$. Given $\lambda : K[T] \longrightarrow k$, we consider the exact sequence

$$T(X_i) \otimes_{k[T]} k_\lambda \xrightarrow{\iota_\lambda} X_i \otimes_{k[T]} k_\lambda \xrightarrow{\pi_\lambda} (X_i/T(X_i)) \otimes_{k[T]} k_\lambda \longrightarrow (0)$$

If $\lambda : k[T] \longrightarrow k$ satisfies $\lambda(\prod_{i=1}^n f_i) \neq 0$, then we have

$$x \otimes 1 = xf_i \otimes \lambda(f_i)^{-1} = 0 \qquad \forall\ x \in T(X_i),$$

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so that

$$X_i \otimes_{k[T]} k_\lambda \cong (X_i/T(X_i)) \otimes_{k[T]} k_\lambda.$$

Since the finitely generated torsion-free $k[T]$-module $X_i/T(X_i)$ is free, it follows that $X_1/T(X_1), \ldots, X_{s(d)}/T(X_{s(d)})$ are parametrizing modules and Λ is tame or representation-finite. □

Definition. Let Q be a quiver with vertex set $Q_0 = \{1, \ldots, n\}$. The *separated quiver* Q_s of Q has $2n$ vertices $\{1, \ldots, n, 1', \ldots, n'\}$ and an arrow $\ell \to m'$ for every arrow $\ell \to m$ of Q.

Note that Q_s is a bipartite quiver, with $\{1, \ldots, n\}$ and $\{1', \ldots, n'\}$ being the sources and sinks of Q_s, respectively.

We only formulate a necessary criterion for tameness, leaving the easy modification for finite representation type to the reader.

Theorem. *Let Λ be a finite dimensional k-algebra. If Λ is tame, then the separated quiver $(Q_\Lambda)_s$ of the Gabriel quiver Q_Λ is a union of Dynkin diagrams of types A, D, E or Euclidean diagrams of types $\tilde{A}$, $\tilde{D}$, $\tilde{E}$.*

Let J be the Jacobson radical of Λ. Then the algebra $\Lambda' := \Lambda/J^2$ is representation-finite or tame, has Jacobson radical $J' = J/J^2$ and Gabriel quiver $Q_{\Lambda'} = Q_\Lambda$. We now study the representation type of radical square zero algebras by comparing them to radical square zero hereditary algebras. Accordingly, we henceforth assume that $J^2 = (0)$.

Let Σ be the triangular matrix algebra

$$\Sigma := \begin{pmatrix} \Lambda/J & 0 \\ J & \Lambda/J \end{pmatrix},$$

whose projective indecomposable modules are of the form

$$\begin{pmatrix} \mathrm{Top}(P) \\ JP \end{pmatrix} \; ; \; \begin{pmatrix} 0 \\ S \end{pmatrix},$$

where P and S run through the projective indecomposable Λ-modules and the simple Λ-modules, respectively. As $\mathrm{Rad}(\Sigma) = \begin{pmatrix} 0 & 0 \\ J & 0 \end{pmatrix}$, this implies:

Lemma 4. *The algebra Σ is hereditary, and $Q_\Sigma = (Q_\Lambda)_s$.* □

The aforementioned comparison is based on the functor

$$F : \mathrm{mod}\,\Lambda \longrightarrow \mathrm{mod}\,\Sigma \;\; ; \;\; M \mapsto \begin{pmatrix} M/JM \\ JM \end{pmatrix},$$

which takes d-dimensional modules to d-dimensional modules. Evidently, F is defined for any Λ-module. If X is a $(\Lambda, k[T])$-bimodule that is finitely generated over $k[T]$, then JX is a sub-bimodule, which is finitely generated over the noetherian ring $k[T]$.

Lemma 5. *Let X be a $(\Lambda, k[T])$-bimodule that is finitely generated and free when considered as a $k[T]$-module. Then we have isomorphisms*

$$F(X \otimes_{k[T]} k_\lambda) \cong F(X) \otimes_{k[T]} k_\lambda$$

of Σ-modules for all but finitely many algebra homomorphisms $\lambda : k[T] \longrightarrow k$.

Proof. We consider the canonical sequence

$$(0) \longrightarrow JX \longrightarrow X \longrightarrow X/JX \longrightarrow (0)$$

of bimodules. Given an algebra homomorphism $\lambda : k[T] \longrightarrow k$, we obtain an exact sequence

$$JX \otimes_{k[T]} k_\lambda \xrightarrow{\iota_\lambda} X \otimes_{k[T]} k_\lambda \xrightarrow{\pi_\lambda} (X/JX) \otimes_{k[T]} k_\lambda \longrightarrow (0)$$

of Λ-modules. Observing $\operatorname{im} \iota_\lambda = J(X \otimes_{k[T]} k_\lambda)$, we see that π_λ induces an isomorphism

$$\bar{\pi}_\lambda : (X \otimes_{k[T]} k_\lambda)/J(X \otimes_{k[T]} k_\lambda) \longrightarrow (X/JX) \otimes_{k[T]} k_\lambda.$$

Since X is a finitely generated free module over the principal ideal domain $k[T]$, general theory (cf. [5, §3.7]) provides a basis $\{x_1, \ldots, x_n\}$ of X and $f_1, \ldots, f_m \in k[T]$ $(m \leq n)$ such that $\{x_1 f_1, \ldots, x_m f_m\}$ is a basis of JX over $k[T]$. It follows that ι_λ is injective if and only if $\lambda(\prod_{i=1}^m f_i) \neq 0$. Hence ι_λ is injective for all but finitely many λ.

Given λ with ι_λ injective, the desired isomorphism sends $\binom{a}{b} \in F(X \otimes_{k[T]} k_\lambda)$ to $\binom{\bar{\pi}_\lambda(a)}{\iota_\lambda^{-1}(b)} \in F(X) \otimes_{k[T]} k_\lambda$. □

Lemma 6. *Let* $\operatorname{ind}\Lambda$ *be the set of isoclasses of indecomposable* Λ*-modules. Then there exists a finite subset* $\mathcal{F} \subset \operatorname{ind}\Sigma$ *such that* $F(\operatorname{ind}\Lambda) \cup \mathcal{F} = \operatorname{ind}\Sigma$.

Proof. The assertions follow from [1, (X.2.1)] and the description of $\operatorname{mod}\Sigma$, given in [1, (II.2.2)]. □

We now turn to the proof of our Theorem:

Proof. Let $d > 0$. Since Λ is tame or representation-finite, we find parametrizing bimodules $X_1, \ldots, X_{s(d)}$ for the set $\operatorname{ind}_d\Lambda$ of isoclasses of d-dimensional indecomposable Λ-modules. Then $Y_i := F(X_i)$ $(1 \leq i \leq s(d))$ are $(\Sigma, k[T])$-bimodules that are finitely generated over $k[T]$, and Lemma 5 provides a finite subset $E_d \subset \operatorname{Alg}_k(k[T], k)$ with

$$F(X_i \otimes_{k[T]} k_\lambda) \cong Y_i \otimes_{k[T]} k_\lambda \qquad \forall\, \lambda \in \operatorname{Alg}_k(k[T], k) \setminus E_d, \quad \forall\, i \in \{1, \ldots, n\}.$$

Let $\mathcal{B}_d \subset \operatorname{ind}_d\Lambda$ be the finite set of isoclasses of indecomposable modules, that are not parametized by one of the X_i, and put $\mathcal{A}_d := \operatorname{ind}_d\Lambda \cap \{[X_i \otimes_{k[T]} k_\lambda]\ ;\ \lambda \in E_d,\ i \in \{1, \ldots, n\}\}$.

Let M be an indecomposable Λ-module such that $[M] \in \operatorname{ind}_d\Sigma \setminus (\mathcal{F} \cup F(\mathcal{A}_d) \cup F(\mathcal{B}_d))$, where $\mathcal{F}$ is the finite set from Lemma 6. Since $[M] \notin \mathcal{F}$, Lemma 6 provides a d-dimensional indecomposable Λ-module N such that $M \cong F(N)$. The isoclass of N does not belong to $\mathcal{B}_d$, so there exists an index i and an algebra homomorphism $\lambda : k[T] \longrightarrow k$ with $N \cong X_i \otimes_{k[T]} k_\lambda$. Thus, $\lambda \notin E_d$, so that

$$M \cong F(X_i \otimes_{k[T]} k_\lambda) \cong Y_i \otimes_{k[T]} k_\lambda.$$

In view of Lemma 3, the algebra Σ is tame or representation-finite. Our Theorem thus follows from a consecutive application of Lemma 4, Theorem 1 and Theorem 2. □

Corollary. *Let* S *and* T *be simple* Λ*-modules such that* $\dim_k \operatorname{Ext}^1_\Lambda(S,T) \geq 3$. *Then* Λ *is wild.*

Proof. By assumption, the separated quiver $(Q_\Lambda)_s$ contains a subquiver of the form $\bullet \Rrightarrow \bullet$. In view of our Theorem, Λ is neither tame nor representation-finite. By Drozd's Theorem, this implies that Λ is wild. □

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