Pillars.

Claus Michael Ringel

Let Λ be an artin algebra, and $\operatorname{mod}\Lambda$ the category of Λ-modules of finite length.

1. Proposition. *Let M be an infinite length Λ-module with a Gabriel-Roiter filtration $M_1 \subset M_2 \subset M_3 \subset \cdots$ such that all the modules M_i are take-off modules. Let ω_i be the Gabriel-Roiter measure of M_i. Then, for every i there exists a module N in $\mathcal{A}(\omega_i)$ such that M has a submodule which is a direct sum of countably many copies of N.*

Let $M' = \bigoplus_{s\in\mathbb{N}} N_s$ be a submodule of M, where N_s is isomorphic to N. Then, for any $r \in \mathbb{N}$, the module $\bigoplus_{s=1}^{r} N_s$ embeds not only into M but already into the modules M_j with j large. Thus, we see:

Corollary. *Let M be an infinite length Λ-module with a Gabriel-Roiter filtration $M_1 \subset M_2 \subset M_3 \subset \cdots$ such that all the modules M_i are take-off modules. Let ω_i be the Gabriel-Roiter measure of M_i. Then, for any pair i, r of natural numbers, there is some j such that M_j contains a submodule which is a direct sum of r copies of N, with $N \in \mathcal{A}(\omega_i)$.*

Proof of the Proposition: Since $\mathcal{A}(\omega_i)$ is finite, it is sufficient to show that there exist a submodule of M which is an infinite direct sum of modules in $\mathcal{A}(\omega_i)$. Assume we have already shown the existence of a submodule X_n which is a direct sum of n modules in $\mathcal{A}(\omega_i)$. Choose $U \subseteq M$ such that $U \cap X_n = 0$ and maximal with this property (such a module exists, using Zorn's lemma). Note that the canonical map $X_n \to M/U$ is an essential monomorphism, thus M/U is of finite length.

The module U cannot be of finite type. Namely, if we assume that U is of finite type, then the fact that M/U is of finite length implies that M is not indecomposable, a contradiction.

It follows that U has indecomposable submodules of arbitrarily large length*. Now U belongs to $\mathcal{A}(\leq\omega)$. Since U is not of finite type, we see that U is not in $\mathcal{A}(\leq\omega_j)$, for any j. Thus, it follows that U has a submodule U' in $\mathcal{A}(\omega_i)$. Since $X_n \cap U' = 0$, we can consider $X_{n+1} = X_n \oplus U'$. This is a direct sum of $n+1$ modules in $\mathcal{A}(\omega_i)$.

2. Pillars. The case of uniform submodules of a module M seems to be of special interest. We call a uniform submodule V of M of largest possible length a *pillar* of M. Note that if V is a pillar of M and v is its length, then the Gabriel-Roiter measure $\mu(M)$ of M is $\mu(M) = [1, v] \cup I'$, where I' is contained in $[v+2, \infty)$. The length of a pillar of M will be called the *pillar length* of M.

Corollary. *Let M be an infinite length Λ-module with a Gabriel-Roiter filtration consisting of take-off modules. Let v be the pillar length of M. Then there is a uniform module N of length v such that M has a submodule which is a direct sum of countably many copies of N.*

If V is a pillar of the module M, the *pillar number* $M : V$ is the maximal number r such that V^r embeds into M. Pillar numbers allow to bound the dimension of the factor ring $\operatorname{End}(M)/\operatorname{rad}\operatorname{End}(M)$ for M indecomposable.

Lemma. *If V is a pillar of M with pillar number r, say with $U \subseteq M$ where U is isomorphic to V^r, then the socle of U is invariant under* $\operatorname{End}(V)$*; thus the restriction $\phi \mapsto \phi|\operatorname{soc} U$ is an algebra homomorphism* $\operatorname{End}(M) \to \operatorname{End}(S^r)$.

Proof: Let ϕ be an endomorphism of M and assume there is an element $x \in \operatorname{soc} U$ with $\phi(x) \notin \operatorname{soc} U$. Write $U = \bigoplus U_i$ with U_i isomorphic to V. Then $x = \sum x_i$ with $x_i \in \operatorname{soc} U_i$ and there is such an i with $\phi(x_i) \notin \operatorname{soc} U$. But then $U \cap \phi(U_i) = 0$, and $\phi(U_i)$ is isomorphic to V, thus we can increase r, a contradiction.

3. Example: Consider the Kronecker algebra, or the group algebra of the Klein four group. For the absolutely indecomposable regular modules and the preinjective modules, the pillars are of length 2 and the pillar number is 1. For the remaining regular modules and for the preprojective modules, the pillars are simple modules, and the pillar number can be arbitrarily large.

References.

[R1] Ringel: The Gabriel-Roiter measure. Bull. Sci. math. 129 (2005), 726-748.

[R2] Ringel: Foundation of the Representation Theory of Artin Algebras, Using the Gabriel-Roiter Measure. To appear in the Proceedings of the Queretaro Workshop 2004. Contemporary Mathematics. Amer.Math.Soc.

* Here we use the following strengthening of Brauer-Thrall 1: *If M is a module which is not of finite type, then M contains indecomposable submodule of arbitarily large finite length.*