The real root modules for some quivers.

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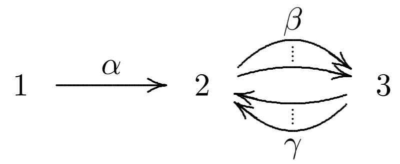
Let Q be a finite quiver with veretx set I and let $\Lambda = kQ$ be its path algebra. The quivers we are interested in will contain cyclic paths, but we may assume that there are no loops. For every vertex i, we denote by $S(i)$ the corresponding simple module, and we denote by $\operatorname{mod}\Lambda$ the category of finite length modules with all composition factors of the form $S(i)$ (thus the category of all locally nilpotent representations of finite length).

We denote by q on $\mathbb{Z}^I$ the quadratic form defined by Q (it only depends on the graph $\overline{Q}$ obtained from Q by deleting the orientation of the edges). For any vertex i, we denote by $\mathbf{e}_i$ the corresponding base element of $\mathbb{Z}^I$ and by σ_i the reflection of $\mathbb{Z}^I$ on the hyperplane orthogonal to $\mathbf{e}_i$. The group W generated by the reflections σ_i is called the *Weyl group* (and the elements σ_i its *generators*). An element of $\mathbb{Z}^I$ is called a *real root* provided it belongs to the W-orbits of some $\mathbf{e}_i$. Also, a non-zero element of $\mathbb{Z}^I$ is said to be *positive* if all its coefficients are non-negative, and *negative,* if all its coefficients are non-positive. It is well-known that all real roots are positive or negative.

According to Kac, for any positive real root $\mathbf{d}$, there is an indecomposable module $M(\mathbf{d})$ in $\operatorname{mod} kQ$ with $\mathbf{dim}\, M(\mathbf{d}) = \mathbf{d}$, and this module is unique up to isomorphism, we call it a *real root module.* The problem discussed here is the following: In general, the existence of these modules is known, but no constructive way in order to obtain them. Also, one may be interested in special properties of these modules: Are they tree modules? What is the structure of the endomorphism ring $\operatorname{End}(M(\mathbf{d}))$?

The following report is based on investigations of Jensen and Su [JS]. We consider the following quiver $\Delta(b, c)$:



with $b \geq 1$ arrows of the form β and $c \geq 1$ arrows of the form γ. The quadratic form is $q(d_1, d_2, d_3) = d_1^2 + d_2^2 + d_3^2 - d_1 d_2 - (b + c) d_2 d_3$. (Jensen and Su consider in [JS] only the case $b = 1 = c$, however the general case is rather similar.)

1. The Weyl group W**.** It is generated by $\sigma_1, \sigma_2, \sigma_3$ with relations $\sigma_i^2 = 1$ for $i = 1, 2, 3$, and $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, $\sigma_1\sigma_3 = \sigma_3\sigma_1$. The *length* $l(w)$ of an element $w \in W$ is t provided w can be written as a product of t generators, and t is minimal with this property.

Lemma 1. *Any element in* W *of length* t *can be written as a product of* t *generators such that neither* $\sigma_2\sigma_1\sigma_2$ *nor* $\sigma_1\sigma_3$ *occurs.*

Proof: Write $w = \sigma_{i_1} \cdots \sigma_{i_t}$ with generators σ_{i_s} for all s and such that the number of occurances of σ_1 is maximal. Then $\sigma_2\sigma_1\sigma_2$ does not occur. In addition, shift the σ_1 to the right, whenever possible. Then also $\sigma_1\sigma_3$ does not occur.

2. The real roots. They are obtained from the canonical base vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ by applying Weyl group elements, the positive real roots are of the form $w\mathbf{e}_{i_0}$, with $1 \le i_0 \le 3$. Note that $\sigma_2\mathbf{e}_1 = \sigma_1\mathbf{e}_2$, thus if $t \ge 1$, we can assume that i_0 is equal to 2 or 3.

Lemma 2. *The positive real roots different from $\mathbf{e}_1$ are of the form*

$$\mathbf{d} = \sigma_{i_t} \cdots \sigma_{i_1}\mathbf{e}_{i_0}$$

with the following properties (here, $1 \le s \le t$):

- $i_0 = 2$, *or* $i_0 = 3$.
- *If* $i_s = 1$, *then* $i_{s-1} = 2$.
- *If* $i_s = 2$, *then* $i_{s-1} = 3$.
- *If* $i_s = 3$, *then* $i_{s-1} = 1$ *or* $i_{s-1} = 2$.

We call $\mathbf{d} = \sigma_{i_t} \cdots \sigma_{i_1}\mathbf{e}_{i_0}$ a *standard presentation* of $\mathbf{d}$ provided these conditions are satisfied.

Proof: Let $\mathbf{d}$ be a positive real root different from $\mathbf{e}_1$. Write $\mathbf{d} = w\mathbf{e}_{i_0}$ with $i_0 \in \{1, 2, 3\}$. If $w = \sigma_{i_t} \cdots \sigma_{i_1}$, with generators σ_{i_s} for $1 \le s \le t$, than we can assume that all the roots $\sigma_{i_s} \cdots \sigma_{i_1}\mathbf{e}_i$ with $1 \le s \le t$ are positive. In addition, we can assume that w has smallest possible length.

Since $\mathbf{d} \neq \mathbf{e}_1$, we can assume that $i_0 \in \{2, 3\}$. Namely, we cannot have $i_1 = 3$, since $\sigma_3\mathbf{e}_1 = \mathbf{e}_1$ would contradict the minimal length of w and if $i_1 = 2$, then we replace $\sigma_2\mathbf{e}_1$ by $\sigma_1\mathbf{e}_2$.

According to Lemma 1, we can assume that w does not include a subword of the form $\sigma_2\sigma_1\sigma_2$ or $\sigma_1\sigma_3$.

The last condition is obvious: if $i_s = 3 = i_{s-1}$, then either $s = 1$ and $\sigma_{i_1}\mathbf{e}_{i_0} = \sigma_3\mathbf{e}_3$ is negative, or else $s > 1$ and there is a cancellation in w, in contrast to the minimality of the length of w.

Similarly, if $i_s = 1$, then i_{s-1} cannot be equal to 1, since otherwise there would be a cancellation. Also $i_{s-1} \neq 3$: for $s > 1$ this follows from the fact that w does not contain $\sigma_1\sigma_3$ as a subword. For $s = 1$, we could replace $\sigma_1\mathbf{e}_3$ by $\mathbf{e}_3$, contrary to the minimal choice of w.

Finally, assume that $i_s = 2$. If $s = 1$, then clearly $i_0 = 3$. Thus $s \ge 2$, and i_{s-1} is either 1 or 3, since otherwise there is a cancellation. Assume that $i_{s-1} = 1$, and therefore $i_{s-2} = 2$. For $s > 2$ this is impossible, since w does not contain a subword of the form $\sigma_2\sigma_1\sigma_2$. If $s = 2$, then we deal with $\sigma_{i_2}\sigma_{i_1}\mathbf{e}i_0 = \sigma_2\sigma_1\mathbf{e}_2 = \mathbf{e}_1$, this contradicts again that w is of smallest possible length. This completes the proof.

Remarks: (1) As a consequence, we see: The positive real roots different form $\mathbf{e}_1$ are of the form $w\mathbf{e}_2$ or $w\mathbf{e}_2$, where w is a subword of a word of the form

$$\sigma_1(\sigma_2\sigma_3)^{s_1}\sigma_1(\sigma_2\sigma_3)^{s_2} \cdots \sigma_1(\sigma_2\sigma_3)^{s_m},$$

with all $s_i \geq 1$.

(2) *If* $\mathbf{d} = \sigma_{i_t} \cdots \sigma_{i_1} \mathbf{e}_{i_0}$ *is a standard presentation, then the coefficients of the roots* $\sigma_{i_s} \cdots \sigma_{i_1} \mathbf{e}_{i_0}$ *with* $0 \leq s \leq t$ *are increased step by step.*

Proof: We apply σ_3 to $\mathbf{d} = (d_1, d_2, d_3)$ only in case $d_2 > d_3$, and then d_3 is replaced by $2d_2 - d_3 > d_2 > d_3$. Similarly, we apply σ_2 only in case $d_2 < d_3$, and then d_2 is replaced by $d_1 + 2d_3 - d_2 > d_1 + d_3 > d_1 + d_2 \geq d_2$. Finally, if we apply σ_1, we either apply it to $\mathbf{e}_2$, or else we have applied just before σ_2 to a vector $\mathbf{d} = (d_1, d_2, d_3)$ with $d_2 < d_3$, thus $\sigma_2 \mathbf{d} = (d_1, d_1 + 2d_3 - d_2, d_3)$ and therefore $\sigma_1 \sigma_2 \mathbf{d} = (2d_3 - d_2, d_1 + 2d_3 - d_2, d_3)$. But then $2d_3 - d_2 > d_3 > d_2 \geq d_1$ (the last inequality is valid for all positive roots).

3. The reflection constructions Σ_i**.** For every vertex i we are going to exhibit a *reflection construction* Σ_i which may be defined only on a full subcategory $\mathcal{M}(i)$ of $\operatorname{mod} \Lambda$ and the values may lie in a module category $\operatorname{mod} \Lambda'$ where the graphs of Λ and Λ' (obtained from the quivers by deleting the orientation) can be (and have been) identified. Always we want that an indecomposable module M is sent to an indecomposable module $\Sigma_i M$ and that

$$(*) \qquad \mathbf{dim}\, \Sigma_i M = \sigma_i(\mathbf{dim}\, M),$$

for M in $\mathcal{M}(i)$. The problem we are faced with is now visible: The construction process of the real root modules has to assure that we always are in the domain of applying a corresponding reflection construction.

Let us start with the **vertices** 2 **and** 3, here we use a general procedure as exhibited in [R1]:

For any vertex i of a quiver Q with no loop at i, there is defined a functor

$$\rho_i \colon \mathcal{M}_i^i \to \mathcal{M}_{-i}^{-i}$$

which induces an equivalence

$$\rho_i \colon \mathcal{M}_i^i / \langle S(i) \rangle \to \mathcal{M}_{-i}^{-i}.$$

It is defined as follows: Given a kQ-module M, let $\operatorname{rad}_i M$ be the intersection of the kernels of maps $M \to S(i)$, thus $M/\operatorname{rad}_i M$ is the homogeneous component of type i of the top of M. Similarly, let $\operatorname{soc}_i M$ be the sum of the images of maps $S(i) \to M$, thus $\operatorname{soc}_i M$ is the homogeneous component of type i of the socle of M. Let $\rho_i(M) = \operatorname{rad}_i M/(\operatorname{soc}_i M \cap \operatorname{rad}_i M)$ (if M has no direct summand isomorphic to $S(i)$, then $\rho_i(M) = \operatorname{rad}_i M/\operatorname{soc}_i M$; the intersection term in the denominator is necessary in order that ρ_i can be applied also to the simple module $S(i)$). For a proof of the asserted equivalence as well as the required formula $(*)$, see [R1], Proposition 2.

Since the kernel of the functor ρ_i is just the ideal $\langle S(i) \rangle$ of $\mathcal{M}_i^i$ given by all maps which factor through direct sums of copies of $S(i)$, we see the following: Assume that M, M' belong to $\mathcal{M}_i^i$ (so that ρ_i is defined). Then

$$\dim \operatorname{Hom}(\rho_i M, \rho_i M') = \dim \operatorname{Hom}(M, M') - (\dim M/\operatorname{rad}_i M) \cdot (\dim \operatorname{soc}_i M').$$

In particular,

$$\dim \operatorname{End}(\rho_i M) = \dim \operatorname{End}(M) - (\dim M/\operatorname{rad}_i M)\cdot(\dim \operatorname{soc}_i M).$$

The reverse construction (forming universal extensions and coextensions) will be denoted by ρ_a^{-1}.

In our case of the quiver $Q = \Delta(b,c)$, let $\Sigma_2 = \rho_2^{-1}$, and $\Sigma_3 = \rho_3^{-1}$. Thus

$$\mathcal{M}(2) = \mathcal{M}_{-2}^{-2} \qquad \text{and} \qquad \mathcal{M}(3) = \mathcal{M}_{-3}^{-3}.$$

Now consider the **vertex** 1**.** The reflection construction Σ_1 is actually functorial and defined on all of $\operatorname{mod} k\Delta(a,b)$, but we restrict it to

$$\mathcal{M}(1) = \operatorname{mod} k\Delta(a,b) \setminus \langle S(1)\rangle$$

and it takes values in $\operatorname{mod} k\Delta(b,a)$. We start with the functor

$$\Sigma_1\colon\ \operatorname{mod} k\Delta(a,b) \to \operatorname{mod} k\Delta(b,a)$$

which is the composition of the BGP reflection functor at the source 1 (see [BGP]) followed by k-duality and renaming of arrows. It provides an equivalence

$$\Sigma_1\colon\ \operatorname{mod} k\Delta(a,b) \setminus \langle S(1)\rangle \to \operatorname{mod} k\Delta(b,a) \setminus \langle S(1)\rangle.$$

The following property is of importance:

(a) $$\Sigma_1(\mathcal{M}_{-3}^{-3})\ \subseteq\ \mathcal{M}_{-3}^{-3}.$$

Namely, $\Sigma_1 S(3) = S(3)$, thus a non-zero homomorphism $\Sigma_1 M \to S(3)$ yields under Σ_1 a non-zero homomorphism $S(3) = \Sigma_1 S(3) \to \Sigma_1^2 M = M$, and similarly, a non-zero homomorphism $S(3) \to \Sigma_1 M$ yields under Σ_1 a non-zero homomorphism $M = \Sigma_1^2 M \to \Sigma_1 S(3) = S(3)$.

In addition, we also need to know that

(b) $$\mathcal{M}_2^2\ \subseteq\ \mathcal{M}_{-3}^{-3}, \qquad \text{and}$$
(c) $$\mathcal{M}_3^3\ \subseteq\ \mathcal{M}_{-2}^{-2}.$$

This follows from the following general result:

Lemma 3. *Assume there are arrows $i \to j$ and $j \to i$. Then*

$$\mathcal{M}_i^i\ \subseteq\ \mathcal{M}_{-j}^{-j}.$$

For a proof, we may refer to [R1], Lemma 4. Let us show one of the four arguments (the remaining ones are similar). Let M be a module with $\mathrm{Ext}^1(S(i), M) = 0$. We want to show that $\mathrm{Hom}(M, S(j)) = 0$. Thus assume there is a non-zero homomorphism $\phi\colon M \to S(j)$; note that ϕ is surjective. This map ϕ induces a map

$$\mathrm{Ext}^1(S(i), \phi)\colon\ \mathrm{Ext}^1(S(i), M) \longrightarrow \mathrm{Ext}^1(S(i), S(j)).$$

Since we are in a hereditary category, an epimorphism such as ϕ induces an epimorphism $\mathrm{Ext}^1(S(i), \phi)$. However, we assume that there is an arrow $i \to j$. Thus $\mathrm{Ext}^1(S(i), S(j)) \neq 0$ and therefore $\mathrm{Ext}^1(S(i), M) \neq 0$, a contradiction.

4. The real root modules.

Theorem (Jensen-Su). *For the quivers $\Delta(b, c)$, the real root modules $M(\mathbf{d})$ different from $S(1)$ are inductively obtained from the simple modules $S(2), S(3)$, using the reflection constructions $\Sigma_1, \Sigma_2, \Sigma_3$* (and following a standard presentation of $\mathbf{d}$.)

Proof: We have to show that the modules obtained inductively are contained in a subcategory $\mathcal{M}(i)$ whenever we have to use the construction Σ_i. There is no problem with Σ_1, since it is always defined (except for $S(1)$, but this does not matter).

Assume we have to use Σ_2. Then either we deal with the root $\mathbf{d} = \sigma_2\mathbf{e}_3$ or else with a root $\mathbf{d} = w\mathbf{b} = \sigma_2\sigma_3\mathbf{d}'$, for some positive real root $\mathbf{d}'$. By induction, the module $M(\sigma_3\mathbf{d}')$ has been constructed using Σ_3, thus it belongs to $\mathcal{M}_3^3$. Of course, also $M(\mathbf{e}_3) = S(3)$ belongs to $\mathcal{M}_3^3$. Thus, in both cases we have to apply Σ_2 to a module in $\mathcal{M}_3^3$. According to (c), we know that $\mathcal{M}_3^3 \subseteq \mathcal{M}_{-2}^{-2}$, thus we can apply the construction Σ_2 in order to obtain $M(\mathbf{d})$ (we obtain either $\Sigma_2 S(3)$ or $\Sigma_2 M(\sigma_3\mathbf{d}')$).

Finally, assume we have to apply Σ_3. If we deal with the root $\mathbf{d} = \sigma_3\mathbf{e}_2$ or with $\mathbf{d} = \sigma_3\sigma_2\mathbf{d}'$ for some positive real root $\mathbf{d}'$, then we argue as in the previous case, now using the assertion (b): $\mathcal{M}_2^2 \subseteq \mathcal{M}_{-3}^{-3}$. Thus it remains to consider the case where either $d = \sigma_3\sigma_1\mathbf{e}_2$ or $\mathbf{d} = \sigma_3\sigma_1\sigma_2\mathbf{d}''$ for some positive real root $\mathbf{d}''$. In this case, we start with the module $N = M(e_2)$ or with $N = M(\sigma_2\mathbf{d}'')$, both belonging to $\mathcal{M}_2^2$, thus N belongs to $\mathcal{M}_{-3}^{-3}$ (this is (b)), and apply to it first Σ_1, then Σ_3. Now, with N also $\Sigma_1 N$ belongs to $\mathcal{M}_{-3}^{-3}$, according to (a), thus there is no problem for applying Σ_3 to $\Sigma_1 N$. This completes the proof.

5. Coefficient quivers for $\mathbb{A}_2$. Let $J = \{0, 1, \dots, n\}$ and $I \subset J$ with $0 \in I$ and $n \notin I$. For $i \in I$, let $i^+ = \min\{i' | i < i', i' \in I \cup \{n\}$. We define an $(I \times J)$-matrix $A(I, J)$ by

$$a_{ij} = \begin{cases} 1 & \text{if } \ i \le j \le i^+ \\ 0 & \text{otherwise,} \end{cases}$$

This is a matrix of rank $|I|$.

Simliarly, consider $K \subset J$ with $0 \notin K$ and $n \in K$. For $k \in K$, let $k^- = \min\{k' | k < k', k' \in K \cup \{0\}$. We define an $(J \times K)$-matrix $B(J, K)$ by

$$b_{jk} = \begin{cases} (-1)^k & \text{if } \ k^- \le j \le k \\ 0 & \text{otherwise.} \end{cases}$$

This is a matrix of rank $|I|$.

Lemma 4. *Let $I \subset J = \{0, 1, \dots, n\}$ with $0 \in I$ and $n \notin I$. Then*

$$A(I, J)B(J, J \setminus I) = 0.$$

Example: Let $n = 9$, let $I = \{0, 1, 4, 5\}$. Then

$$A(I,J) = \begin{bmatrix} 1 & 1 & & & & & & & & \\ & 1 & 1 & 1 & 1 & & & & & \\ & & & & 1 & 1 & & & & \\ & & & & & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad B(J, J \setminus I) = \begin{bmatrix} 1 & & & & & \\ -1 & & & & & \\ 1 & 1 & & & & \\ & -1 & -1 & & & \\ & & 1 & & & \\ & & -1 & & & \\ & & 1 & 1 & & \\ & & & -1 & -1 & \\ & & & & 1 & 1 \\ & & & & & -1 \end{bmatrix}.$$

For the proof, we need the following observation: *Let $K = J \setminus I$. For $i \in I$ and $k \in K$, the intervalls $[i, i^+]$ and $[k^-, k]$ either avoid each other, or else they intersect in a pair* $[j-1, j]$. Namely, assume they do not avoid each other, let j be maximal in the intersection. In particular, $i \le j \le i^+$ and $k^- \le j \le k$. Note that j cannot be zero: If $j = 0$, then $i = 0$ and $k^- = 0$. But then also 1 belongs to $[i, i^+] \cap [k^-, k]$.

Case 1. Assume that j belongs to I, thus $k^- < j < k$ (here we use that $j \neq 0$). Note that all the elements $\{k^+ + 1, \dots, k-1\}$ belong to I. Then $j = i^+$, since otherwise $j + 1 \in [i, i^+] \cap [k^-, k]$, contrary to the maximality of j. Thus $i < j$. If $i = j - 1$, then $k^- < j - 1$ and $[i, i + 1] = [j - 1, j] \subset [k^-, k]$. If $i < j - 1$, then $i < j - 1 < i^+$ shows that $j - 1$ belongs to K, thus $j - 1 = k^-$ and $[j - 1, j] = [i, i^+] \cap [k^-, k]$.

Case 2. Now j belongs to K, thus $i < j$. The only elements in $[k^-, k]$ which belong to K are k^- and k. If $j = k^-$, then $j + 1 \in [k^-, k]$. Also $j < i^+$, thus j is not maximal in $[i, i^+] \cap [k^-, k]$, a contradiction. This shows that $j = k$. If $k^- = j - 1]$, then $[j - 1, j] = [k^-, k] \subset [i, i^+]$. If $k^- < j - 1]$, then $j - 1$ belongs to I and $j < i^+$. Thus again $[j - 1, j] = [i, i^+] \cap [k^-, k]$.

Remark: A non-zero intersection $[i, i^+] \cap [k^-, k]$ arises in different ways: we may have $[i, i^+] \subset [k^-, k]$ (in the example above, this arises for $i = 0$ and $k = 2$), or $[k^-, k] \subset [i, i^+]$ (in the example: $i = 1, k = 3$), as $[i, i^+] \cap [k^-, k] = [k^-, i^+]$ (in the example: $i = 1, k = 2$). or finally as $[i, i^+] \cap [k^-, k] = [i, k]$ (in the example: $i = 5, k = 6$).

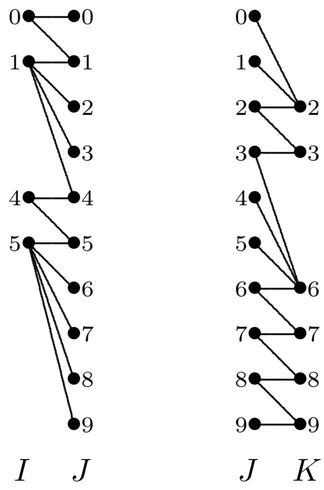
Now we are able to provide the proof of Lemma 4: If we multiply the row of $A(I, J)$ with index $i \in I$ with the column of $B(J, K)$ with index $k \in K$, we obtain

$$\sum_t a_{it} b_{tk} = \sum_{t \in [i, i^+] \cap [k^-, k]} a_{it} b_{tk}$$

All the summands are zero in case $[i, i^+] \cap [k^-, k]$ is empty. Otherwise, there is some j with $[i, i^+] \cap [k^-, k] = [j - 1, j]$ and then

$$\sum_t a_{it} b_{tk} = a_{i,j-1} b_{j-1,k} + a_{ij} b_{jk} = (-1)^{j-1} + (-1)^j = 0.$$

It is of interest for us, to draw the coefficient quivers both of $A(I,J)$ and $B(J,K)$, where $K = J \setminus I$:



(the arrows go from right to left).

Definition: We say that a matrix with rank equal to the number of columns is *in standard inclusion form* provided it is a direct sum of matrices of the form $B(J,K)$ as well as of copies of $[1]\colon k \to k$ and of empty matrices $0 \to k$.

Proposition 1 (Jensen-Su). *Let $f\colon U \to V$ be an injective vector space homomorphism. Assume there is given a basis $\mathcal{U}$ of U and a basis $\mathcal{V}$ of V such that the corresponding matrix representation of f in standard inclusion form. Let $g\colon V \to W$ be a cokernel of f and consider the dual map $g^*\colon W^* \to V^*$. Let $\mathcal{V}^*$ be the dual basis of $\mathcal{V}$. Then there is a basis of $\mathcal{W}$ of W, with dual basis $\mathcal{W}^*$ such that the matrix representation of g^* with respect to $\mathcal{W}^*$ and V^* is again in standard inclusion form.*

The proof of Proposition 1 by Jensen-Su uses induction (see [JS] Proposition 6.2).

Remark. Let us characterize the shape of the coefficient quiver of a matrix of the form $A(I,J)$: It is a tree obtained from the bipartite $\mathbb{A}$-quiver with vertex set

$$I \times \{0\} \ \cup \ (I \cup \{n\}) \times \{1\}$$

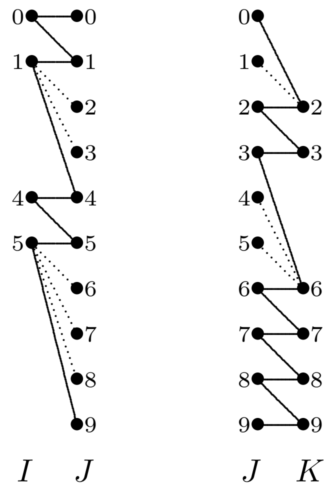
and arrows $(i,0) \leftarrow (i,1)$ and $(i,0) \leftarrow (i^+,1)$ by adding additional leaves in vertices in $I \times \{0\}$. There is a similar description for the coefficient quiver of $B(J,K)$, here the $\mathbb{A}$-subquiver has the vertex set $\mathbb{A}$-quiver with vertex set

$$(\{0\} \cup K) \times \{0\} \ \cup \ K \times \{1\},$$

and the additional leaves are attached to vertices in $K \times \{1\}$.

In the example above, let us exhibit the bipartite $\mathbb{A}$-quivers for $A(I,J)$ as well as of

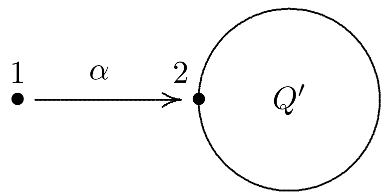
$B(J, K)$ by marking the additional leaves as dotted edges:



6. The real root representations of $\Delta(a, b)$ are tree modules (in the sense of [R2]; for $b = 1 = c$, see [JS], Theorem 6.3). This will be shown by induction, using the reflection constructions $\Sigma_1, \Sigma_2, \Sigma_3$.

Recipe for Σ_1. Here we use Proposition 1 as follows:

Let M be a representation of a quiver Q of the form



Let $\mathcal{M}(1) = \operatorname{mod} kQ \setminus \langle S(1) \rangle$ and define Σ_1 as the composition of the BGP reflection functor at the source 1 followed by k-duality (note that $\Sigma_1 M$ is a representation of the quiver Q'' obtained from Q by reversing all the arrows in Q'). A representation M of Q (or of Q'') will be said to be a *tree module with α in standard inclusion form,* provided there are bases of the vector spaces M_i such that both the corresponding coefficient quiver is a tree as well as M_α is in standard inclusion form.

Lemma 5. *Assume that M in $\mathcal{M}(1)$ is a tree module with α in standard inclusion form. Then also $\Sigma_1 M$ is a tree module with α in standard inclusion form.*

Proof. For every vertex i of Q, there is given a basis $\mathcal{B}_i$ of M_i such that the coefficient quiver Γ of M with respect to these bases is a tree and such that the matrix for M_α is in standard inclusion form. For every $i \neq 0$, let $\mathcal{B}_i^*$ be the dual basis of M_i^*. Let $g\colon M_2 \to W$ be the cokernel of M_α. According to proposition 1 there is a basis $\mathcal{W}$ of W with dual basis $\mathcal{W}^*$ such that the matrix representation of g with respect to the bases $\mathcal{B}_2^*$ and $\mathcal{W}^*$ is in standard inclusion form. It remains to show that the coefficient quiver Γ^* of $\Sigma_1 M$ with respect to the bases $\mathcal{W}^*$ and $\mathcal{B}_i^*$ with $i \neq 1$ is a tree. First we show that the Γ^* is connected. By assumption, any two elements of $\mathcal{B}$ are connected by a path in Γ. Now we have deleted the paths given by the matrix M_α. However, two vertices in $\mathcal{B}_1$ are connected by a path corresponding to the matrix M_α if and only if the corresponding vertices in $\mathcal{B}_1^*$ are connected by a path corresponding to g. Second, note that we have not created cycles, since the new connections which are established are given by unique paths.

Let M be a representation of $\Delta(b,c)$. We denote by $\operatorname{Kr} M$ the restriction of M to the full subquiver Q' given by vertices 2 and 3; it is a submodule of M and the factor module $M/\operatorname{Kr} M$ can be identified with M_1 (considered as a representation of $\Delta(b,c)$ extended by zeros, thus as a direct sum of copies of $S(1)$).

Recipe for Σ_2 and Σ_3. Since these reflection constructions are provided by universal extensions from above and from below, we see as in [R2] that with M a tree module in $\mathcal{M}(i)$ also $\Sigma_i M$ is a tree module ($i = 1, 2$). (But observe that this argument yields a tree structure for a given base field and one cannot be sure that the construction is independent of the characteristic of the field!) In case $b = c = 1$, Jensen-Su [JS] provide a tree presentation which works for every field.

The construction Σ_3 only depends on the restriction $\operatorname{Kr} M$ of M the full subquiver Q' with vertices $2, 3$. In contrast, the construction Σ_2 also takes into account the vector space at the vertex 1. In order to provide a clear picture for Σ_2, we need the following preliminary result.

Lemma 6.

$$\mathcal{M}_3^3 \subseteq \{M \mid \operatorname{Hom}(S(2), \operatorname{Kr} M) = 0 = \operatorname{Hom}(\operatorname{Kr} M, S(2))\}.$$

Clearly, any homomorphism $S(2) \to M$ factors through the submodule $\operatorname{Kr} M$ of M, thus $\operatorname{Hom}(S(2), \operatorname{Kr} M)$ can be identified with $\operatorname{Hom}(S(2), M)$. According to Lemma 3, we know that any module M in $\mathcal{M}_3^3$ satisfies $\operatorname{Hom}(S(2), M) = 0$, thus also $\operatorname{Hom}(S(2), \operatorname{Kr} M) = 0$. On the other hand, the restriction map $\operatorname{Hom}(M, S(2)) \to \operatorname{Hom}(\operatorname{Kr} M, S(2))$ is injective, thus we see that Lemma 6 is stronger than the assertion obtained from Lemma 3.

Let M be an indecomposable representation of $\Delta(b,c)$ with $\operatorname{Hom}(\operatorname{Kr} M, S(2)) \neq 0$. As we have seen in the proof of Lemma 3, $\operatorname{Ext}^1(S(3), \operatorname{Kr} M) \neq 0$. Let

$$0 \to \operatorname{Kr} M \xrightarrow{f} N \to S(3) \to 0$$

be a non-split extension. The inclsuion map $\operatorname{Kr} M \to M$ yields an induced exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Kr} M & \xrightarrow{f} & N & \longrightarrow & S(3) & \longrightarrow & 0 \\ & & {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} & & \| & & \\ 0 & \longrightarrow & M & \xrightarrow{f'} & N' & \longrightarrow & S(3) & \longrightarrow & 0 \end{array}$$

Now assume the induced exact sequence splits, thus there is $h\colon N' \to M$ with $hf' = 1_M$. This yields a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Kr} M & \xrightarrow{f} & N & \longrightarrow & S(3) & \longrightarrow & 0 \\ & & \| & & {\scriptstyle hu'}\downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Kr} M & \xrightarrow{u} & M & \longrightarrow & M_1 & \longrightarrow & 0 \end{array}$$

The map $S(3) \to M_1$ has to be zero, since M_1 is a direct sum of copies of $S(1)$, thus hu' factors through u, and this implies that f is a split monomorphism, a contradiction.

Corollary. *Let $M \in \mathcal{M}_3^3$. The exact sequence $0 \to \operatorname{Kr} M \to M \to M_1 \to 0$ induces an exact sequence*

$$0 \to \operatorname{Ext}^1(M_1, S(2)) \to \operatorname{Ext}^1(M, S(2)) \to \operatorname{Ext}^1(\operatorname{Kr} M, S(2)) \to 0.$$

Proof: This is part of the long exact sequence

$$\operatorname{Hom}(\operatorname{Kr} M, S(2)) \to \operatorname{Ext}^1(M_1, S(2)) \to \operatorname{Ext}^1(M, S(2)) \to \operatorname{Ext}^1(\operatorname{Kr} M, S(2)) \to 0,$$

and Lemma 6 asserts that $\operatorname{Hom}(\operatorname{Kr} M, S(2)) = 0$.

For our problem of finding a tree presentation for $\Sigma_2 M$, we see the following: we start with a tree presentation of M and attach copies of $S(2)$ from above by taking a basis of $\operatorname{Ext}^1(S(2), M)$. Then we attach copies of $S(2)$ from below by taking on the one hand a basis of $\operatorname{Ext}^1(M_1, S(2))$, on the other hand a basis of $\operatorname{Ext}^1(\operatorname{Kr} M, S(2))$. The process of attaching copies of $S(2)$ from below dealing with $\operatorname{Ext}^1(M_1, S(2))$ is achieved as follows: we just attach to each base element b of M_1 a corresponding leaf at b.

Let us denote by Q' the full subquiver of $\Delta(b, c)$ with vertices $2, 3$. *Let M belong to $\mathcal{M}(2)$. Then* $\operatorname{Kr} \Sigma_2 M = \Sigma_2 \operatorname{Kr} M \oplus (M_1, 0)$. (Here, $N = (M_1, 0)$ is the representation of Q' with $N_2 = M_1$ and $N_3 = 0$.)

In case $b = c = 1$, the reflection constructions Σ_2 and Σ_3 for representations $N = \operatorname{Kr} N$ are very easy to describe: an indecomposable module $N = \operatorname{Kr} N$ is serial and the process of attaching copies of $S(2)$ or $S(3)$ from above or from below just means that we enlarge the length of it by 2: we write N as the subfactor $N = \operatorname{rad} N' / \operatorname{soc} N'$, where N' is indecomposable. Of course, such serial modules are tree modules, with coefficient quiver being linearly ordered of type $\mathbb{A}$ (in particular, the coefficient quiver is independent of the given base field).

7. Some properties of real root modules. Having constructed the real root modules $M(\mathbf{d})$ for $\Delta(a, b)$, one may use the construction in order to study properties of these modules.

Let M be a real root module, and $\operatorname{Kr} M$ its restriction to the full subquiver Q' of $\Delta(b, c)$ with vertices $2, 3$. The indecomposable direct summands N of $\operatorname{Kr} M$ are real root modules for Q', thus of odd Loewy length. if the Loewy length of such a direct summand N is equal to $2t + 1$, then $\operatorname{rad}^t M / (\operatorname{soc}^t M \cap \operatorname{rad}^t M)$ is a simple module, called the center of N.

(1) *For $M = S(1)$, the module $\operatorname{Kr} M$ is 0, otherwise non-zero. If $\operatorname{Kr} M \neq 0$,, then $\operatorname{Kr} M$ has at most one direct summand with center $S(3)$, the remaining direct summands have center $S(2)$.* Write $\mathbf{d} = w\mathbf{b}$ with $w \in W$ and $\mathbf{b} = \mathbf{e}_2$ or $\mathbf{b} = \mathbf{e}_3$. The restriction $\operatorname{Kr} M(w\mathbf{b})$ has a direct suammmand with center $S(3)$ if and only if $\mathbf{b} = \mathbf{e}_3$.

(2) *Either the image of* α *is contained in* $\operatorname{rad}\operatorname{Kr} M$*, or else* M *is generated by the subspace* M_1. Note that M is generated by M_1 if and only if either $M = S(1)$, or $M = \Sigma_1 S(2)$, or $M = \Sigma_1\Sigma_2 M''$ for some real root module $M'' \in \mathcal{M}(2)$.

8. Further properties of real root modules. Some other properties of the real root modules should be considered. In the case $b = 1 = c$, this is done in [JS].

(a) One can use the indications mentioned above concerning the change of endomorphism rings under the reflection constructions in order to describe $\operatorname{End}(M(\mathbf{d}))$ at least partly; in particular one is interested in the growth of $\dim \operatorname{End}(M(w\mathbf{b}))$ depending on the length of w (where $w\mathbf{b}$ is given by a standard presentation), see [JS], sections 5 and 7.

(b) For any positive real root $\mathbf{d}$, one may compare the module $M(\mathbf{d})$ with the other modules with dimension vector $\mathbf{d}$, in particular with those with smallest possible endomorphism ring dimension, see [JS], section 7 (for the quivers with 2 vertices, see [R1], Proposition 4).

9. Final remarks. It should be noted that the reflection functors ρ_i are very special cases of the reflection functors ρ_E introduced in [R1]: in general, one considers instead of $S(i)$ an arbitrary exceptional module E (this means: an indecomposable module without self-extensions, such modules are always real root modules), and a suitably defined subcategory $\mathcal{M}^E_E$. This then provides a partial realization of the reflection σ_E at the hyperplane in $\mathbb{Z}^I$ orthogonal to $\mathbf{dim}\, E$.

Note that the special cases ρ_i allow to construct all the real root modules in case we deal with a quiver Q with the following property ([R3]): Given an arrow $i \to j$ in the quiver, there are also arrows $j \to i$.

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