Zwara's Degeneration Theory.

Reference: G. Zwara: *A degeneration-like order for modules.* Arch. Math. 71 (1998), 437-444.

Definition: Call Y a *degeneration* of X provided there is an exact sequence of the form $0 \to U \to X \oplus U \to Y \to 0$ (such a sequence should be called a *Riedtmann-Zwara sequence*). The map $U \to U$ is called a corresponding *steering* map.

Some preliminary definitions and results. A commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y_1 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ Y_2 & \xrightarrow[f']{} & Z \end{array}$$

is said to be *exact* provided it is both a pushout and a pullback, thus if and only if the sequence

$$0 \to X \xrightarrow{\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]} Y_1 \oplus Y_2 \xrightarrow{[\,g' \ -f'\,]} Z \to 0$$

is exact.

(1) *The composition of two exact squares*

$$\begin{array}{ccccc} X & \longrightarrow & Y_1 & \longrightarrow & Z_1 \\ \downarrow & & \downarrow & & \downarrow \\ Y_2 & \longrightarrow & Z_2 & \longrightarrow & A \end{array}$$

yields an exact square

$$\begin{array}{ccc} X & \longrightarrow & Z_1 \\ \downarrow & & \downarrow \\ Y_2 & \longrightarrow & A \end{array}$$

(2) *For any map $a\colon U \to V$, and any module X, the following diagram is exact:*

$$\begin{array}{ccc} U & \xrightarrow{a} & V \\ \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\downarrow & & \downarrow\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \\ U \oplus X & \xrightarrow[a \oplus 1_X]{} & V \oplus X. \end{array}$$

(3) *Let*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "0"'] & Y_1 \arrow[d] \\ Y_2 \arrow[r, "f'"'] & Z \end{tikzcd}$$

be exact. Then f' is split mono.

(4) *Assume we have the following exact square*

$$\begin{tikzcd} U \arrow[r, "a"] \arrow[d, "b"'] & V \arrow[d, "b'"] \\ W \arrow[r, "a'"'] & X \end{tikzcd}$$

and b is a split monomorphism, then the sequence

$$0 \to U \xrightarrow{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]} V \oplus W \xrightarrow{[b'\ a']} X \to 0$$

splits.

Proofs. (1) and (2): Well-known (and obvious). (3): Since $\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right]$ is injective, $f\colon X \to Y_1$ is injective. Let Q be the cokernel of f. We obtain the map f' by forming the induced exact sequence of $0 \to X \xrightarrow{f} Y_1 \to Q \to 0$, using the zero map $X \to Y_1$. But such an induced exact sequence splits. (4) Assume $pb = 1_U$. Then $[0\ p]\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = 1_U$.

Lemma. (There is always a nilpotent steering map.) *If there is an exact sequence $0 \to U \to X \oplus U \to Y \to 0$, then there is an exact sequence $0 \to U' \to X \oplus U' \to Y \to 0$ such that the map $U' \to U'$ is nilpotent.*

Proof: We can decompose $U = U_1 \oplus U_2 = U_1' \oplus U_2'$ such that the given map $f\colon U \to U$ maps U_1 into U_1', U_2 into U_2' and such that the induced maps $f_1\colon U_1 \to U_1'$ belongs to the radical of the category, whereas the induced map $f_2\colon U_2 \to U_2'$ is an isomorphism. We obtain the following pair of exact squares

$$\begin{tikzcd} U_1 \arrow[r, "\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]"] \arrow[d, "f_1"'] & U_1 \oplus U_2 \arrow[r] \arrow[d, "f_1 \oplus f_2"'] & X \arrow[d] \\ U_1' \arrow[r, "\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]"'] & U_1' \oplus U_2' \arrow[r] & Y \end{tikzcd}$$

(the left square is exact according to (2)). The composition of the squares is the desired exact square (note that U_1' is isomorphic to U_1).

The relationship between degenerations and iterating self-extensions.

We say that $Y[\infty] = (Y[\infty], \psi)$ is a *Prüfer module*, provided ψ is a surjective, locally nilpotent endomorphism of the module $Y[\infty]$ with kernel Y. Given such a module, let $Y[n]$ be the kernel of ψ^n.

Theorem (Zwara). *Assume Y is a degeneration of X, steered by a nilpotent map ϕ with $\phi^t = 0$. Then there is a Prüfer module $Y[\infty]$ such that $Y[t+1] \simeq Y[t] \oplus X$.*

Corollary. *Assume Y is a degeneration of X and $\operatorname{Ext}^1(Y, Y) = 0$. Then X and Y are isomorphic.*

Proof of Corollary: The theorem asserts that $Y[t+1] \simeq Y[t] \oplus X$. If $\operatorname{Ext}^1(Y, Y) = 0$, then $Y[n] \simeq Y^n$ for all n. Thus $Y^{t+1} \simeq Y^t \oplus X$, thus $Y \simeq X$.

Converse of Theorem: Assume there is a Prüfer module $Y[\infty]$ such that $Y[t+1] \simeq Y[t] \oplus X$. We get the following two exact sequences

$$0 \to Y[t] \to Y[t+1] \to Y[1] \to 0,$$
$$0 \to Y[1] \to Y[t+1] \to Y[t] \to 0,$$

in the first, the map $Y[t+1] \to Y[1]$ is given by applying ψ^t, in the second the map $Y[t+1] \to Y[t]$ is given by applying ψ. In both sequences, we can replace $Y[t+1]$ by $Y[t] \oplus X$. Thus we obtain as first sequence a new Riedtmann-Zwara sequence, and as second sequence a dual Riedtmann-Zwara sequence:

$$0 \to Y[t] \to Y[t] \oplus X \to Y \to 0,$$
$$0 \to Y \to Y[t] \oplus X \to Y[t] \to 0,$$

note that both use the same steering module, namely $Y[t]$. Thus:

Reformulation. *The module Y is a degeneration of X if and only if there is a Prüfer module $Y[\infty]$ such that $Y[t+1] \simeq Y[t] \oplus X$ for some t.*

Also: *The module Y is a degeneration of X if and only if there exists a module V and an exact sequence* $0 \to Y \to V \oplus X \to V \to 0$ (A co-Riedtmann-Zwara sequence).

Proof of Theorem: Assume a monomorphism $w = \begin{bmatrix} \phi \\ g \end{bmatrix} : U \to U \oplus X$ with cokernel Y and $\phi^t = 0$ is given. Consider also the canonical embedding $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} : U \to U \oplus X$ and form the towers for this pair of monomorphisms $M_i(w, v)$ and the quotients $R_i(w, v) = M_i(w, v)/M_0(w, v)$. The latter modules are just the modules $Y[i] = R_i(m, v)$ we are looking for. As we know, there is a Prüfer module $(Y[\infty], \psi)$ with $Y[i]$ being the kernel of ψ^i.

We construct the maps w_n, v_n explicitly as follows:

$$w_n = \begin{bmatrix} \phi & \\ g & \\ & 1_{X^n} \end{bmatrix} = \begin{bmatrix} w & \\ & 1_{X^n} \end{bmatrix} : U \oplus X^n \to (U \oplus X) \oplus X^n$$

and

$$v_n = \begin{bmatrix} 1_{U\oplus X_n} \\ 0 \end{bmatrix} : U \oplus X^n \to U \oplus X^n \oplus X,$$

using the recipe (2). Thus we obtain the following sequence of exact squares:

$$\begin{tikzcd}[column sep=large]
U \arrow[r, "{\begin{bmatrix}\phi\\ g\end{bmatrix}}"] \arrow[d, "{\begin{bmatrix}1\\ 0\end{bmatrix}}"'] & U\oplus X \arrow[r, "{\begin{bmatrix}\phi & \\ g & \\ & 1\end{bmatrix}}"] \arrow[d, "{\begin{bmatrix}1 & \\ & 1\\ 0 & 0\end{bmatrix}}"] & U\oplus X\oplus X \arrow[r, "{\begin{bmatrix}\phi & & \\ g & & \\ & 1 & \\ & & 1\end{bmatrix}}"] \arrow[d, "{\begin{bmatrix}1 & & \\ & 1 & \\ & & 1\\ 0 & 0 & 0\end{bmatrix}}"] & U\oplus X\oplus X\oplus X \arrow[r] \arrow[d, "{\begin{bmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1\\ 0 & 0 & 0 & 0\end{bmatrix}}"] & {} \\
U\oplus X \arrow[r, "{\begin{bmatrix}\phi & \\ g & \\ & 1\end{bmatrix}}"'] & U\oplus X\oplus X \arrow[r, "{\begin{bmatrix}\phi & & \\ g & & \\ & 1 & \\ & & 1\end{bmatrix}}"'] & U\oplus X\oplus X\oplus X \arrow[r, "{\begin{bmatrix}\phi & & & \\ g & & & \\ & 1 & & \\ & & 1 & \\ & & & 1\end{bmatrix}}"'] & U\oplus X\oplus X\oplus X\oplus X \arrow[r] & {}
\end{tikzcd}$$

In particular, we have $M_n = M_n(w, v) = U \oplus X^n$.

Note that the composition $w_{n-1} \cdots w_0 \colon U \to U \oplus X^n$ is of the form $\begin{bmatrix} \phi^n \\ g_n \end{bmatrix}$ for some $g_n \colon U \to X^n$.

We also have the following sequence of exact squares:

$$\begin{tikzcd}
U = M_0 \arrow[r, "w_0"] \arrow[d] & M_1 \arrow[r, "w_1"] \arrow[d] & M_2 \arrow[r, "w_2"] \arrow[d] & M_3 \arrow[r, "w_3"] \arrow[d] & \cdots \\
0 \arrow[r] & Y[1] \arrow[r, "s_1"] & Y[2] \arrow[r, "s_2"] & Y[3] \arrow[r, "s_3"] & \cdots
\end{tikzcd}$$

where the vertical maps are of the form

$$M_n = U \oplus X^n \xrightarrow{[h_n \ \ q_n]} Y[n].$$

The composition of these exact squares yields an exact square

$$\begin{tikzcd}
U \arrow[r, "w_{n-1}\cdots w_0"] \arrow[d] & U \oplus X^n \arrow[d, "{[h_n \ \ q_n]}"] \\
0 \arrow[r] & Y[n]
\end{tikzcd}$$

Here we may insert the following observation: This sequence shows that *the module* $Y[n]$ *is a degeneration of the module* X^n.

Since the composition $w_{n-1} \cdots w_0 \colon U \to U \oplus X^n$ is of the form $\begin{bmatrix} \phi^n \\ g_n \end{bmatrix}$, and $\phi^t = 0$, it follows that h_t is a split monomorphism, see (3).

Also, we can consider the following two exact squares, with $w = \begin{bmatrix} \phi \\ g \end{bmatrix} : U \to V = U \oplus X$ (the upper square is exact, according to (2)):

$$\begin{array}{ccc} U & \xrightarrow{w} & V \\ {\scriptstyle \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]}\big\downarrow & & \big\downarrow{\scriptstyle \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]} \\ U \oplus X^t & \xrightarrow{\left[\begin{smallmatrix} w & \\ & 1 \end{smallmatrix}\right]} & V \oplus X^t \\ {\scriptstyle [h_t\ q_t]}\big\downarrow & & \big\downarrow{\scriptstyle [h_{t+1}\ q_{t+1}]} \\ Y[t] & \longrightarrow & Y[t+1] \end{array}$$

The vertical composition on the left is h_t, thus, as we have shown, a split monomorphism. This shows that the exact sequence corresponding to the composed square splits (4): This yields

$$U \oplus Y[t+1] \simeq Y[t] \oplus V = Y[t] \oplus U \oplus X.$$

Cancellation of U gives the desired isomorphism:

$$Y[t+1] \simeq Y[t] \oplus X.$$

Remark to the proof. Given the Riedtmann-Zwara sequence

$$0 \to U \xrightarrow{\left[\begin{smallmatrix} \phi \\ g \end{smallmatrix}\right]} U \oplus X \to Y \to 0,$$

we have considered the following pair of monomorphisms

$$w = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], w' = \left[\begin{smallmatrix} \phi \\ g \end{smallmatrix}\right] : U \to U \oplus X.$$

The corresponding Prüfer modules are $X^{(\infty)}$ and $Y[\infty]$, respectively. And $U_n(w, w') = U \oplus X^n$. As we know, we can assume that ϕ is nilpotent. Then all the linear combinations

$$w + \lambda w' = \left[\begin{smallmatrix} 1+\lambda\phi \\ g \end{smallmatrix}\right]$$

with $\lambda \in k$ are also split monomorphisms (with retraction $\left[\begin{smallmatrix} \eta & 0 \end{smallmatrix}\right]$, where $\eta = (1 + \lambda\phi)^{-1}$).

Transitivity of the degeneration relation.

Lemma. *Assume that there are exact sequences*

$$0 \to Y \to X \oplus U \to U \to 0, \quad 0 \to Z \to Y \oplus V \to V \to 0.$$

and such that the steering map $\phi\colon V \to V$ is nilpotent, say $\phi^t = 0$. Then there is an exact sequence

$$0 \to Z \to X \oplus W \to W \to 0$$

where W has a filtration with factors of the form U and V.

Proof. Denote by $f\colon Y \to V$ the map used in the second exact sequence. The first exact sequence yields the following induced exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, "f"'] & X \oplus U \arrow[r] \arrow[d] & U \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & V \arrow[r, "v_0"] & V_1 \arrow[r] & U \arrow[r] & 0
\end{tikzcd}$$

We compose the left square with the exact square given by the second given Riedtmann-Zwara sequence:

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, "f"'] & X \oplus U \arrow[d] \\
V \arrow[r, "\phi"'] & V \arrow[r, "v_0"'] & V_1
\end{tikzcd}$$

and obtain by composition and reflection an exact square of the form

(∗)
$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & V \arrow[d, "v_0\phi"] \\
X \oplus U \arrow[r] & V_1.
\end{tikzcd}$$

Now we form the tower for the pair $v_0, v_0\phi\colon V \to V_1$. It is of the form

(∗∗)
$$\begin{tikzcd}
V \arrow[r, "v_0"] \arrow[d, "v_0\phi"'] & V_1 \arrow[r, "v_1"] \arrow[d] & \cdots \arrow[r, "v_{t-1}"] & V_t \arrow[d] \\
V_1 \arrow[r, "v_2"] & V_2 \arrow[r, "v_2"] & \cdots \arrow[r, "v_t"] & V_{t+1}
\end{tikzcd}$$

Since $\phi^t = 0$, we know that the inclusion $v_t\colon V_t \to V_{t+1}$ splits, see the following lemma. Note that the cokernels of all the maps $v_i\colon V_i \to V_{i+1}$ are equal, and the cokernel of v_0 is U. Thus we see

$$V_{t+1} = V_t \oplus U.$$

Composing the exact square (∗) with the all the squares (∗∗), we obtain an exact sequence

$$0 \to Z \to V_t \oplus X \oplus U \to V_{t+1} \to 0.$$

Thus $W = V_{t+1} = V_t \oplus U$ is the desired module.

Remark. We can analyse the module $W = U \oplus V_t$ more carefully: V_t has V as submodule and V_t/V is a t-fold iterated self-extension of U.

Lemma. *Let $w_0\colon U_0 \to U_1$ be a monomorphism. Let $\gamma\colon U_0 \to U_0$ be a nilpotent endomorphism, say with $\gamma^t = 0.$. Form the tower $U_i = U_i(w_0; w_0\gamma)$ with $i \geq 0$ and inclusion maps $w_i\colon U_i \to U_{i+1}$. Then $w_t\colon U_t \to U_{t+1}$ splits.* (If W is the cokernel of w_0, then the cokernel of w_t is also isomorphic to W, thus $U_{t+1} \simeq U_t \oplus W$.)

Proof: The towers is formed by the following exact squares:

$$\begin{tikzcd}
U_0 \arrow[r, "w_0"] \arrow[d, "w_0\gamma"'] & U_1 \arrow[r, "w_1"] \arrow[d, "w_1'"] & \cdots \arrow[r, "w_{n-1}"] & U_n \arrow[d, "w_n'"] \\
U_1 \arrow[r, "w_2"] & U_2 \arrow[r, "w_2"] & \cdots \arrow[r, "w_n"] & U_{n+1}
\end{tikzcd}$$

Using induction, one shows that

$$w_{n-1}' \cdots w_1' w_0 \gamma = w_{n-1} \cdots w_1 w_0 \gamma^n.$$

The assertion is true for $n = 1$ (namely $w_0\gamma = w_0\gamma$). Assume it is true for some $n \geq 1$. Then

$$w_n' w_{n-1}' \cdots w_1' w_0 \gamma = w_n' w_{n-1} \cdots w_1 w_0 \gamma^n = w_n w_{n-1} \cdots w_1 w_0 \gamma \cdot \gamma^n$$

(the first equality sign is by induction, the second uses the commutativity of the squares). Thus, we see that $w_{t-1}' \cdots w_1' w_0 \gamma = 0$.

But we obtain w_n by forming the induced exact sequence

$$\begin{tikzcd}
0 \arrow[r] & U_0 \arrow[r, "w_0"] \arrow[d, "w_{n-1}'\cdots w_1' w_0\gamma"'] & U_1 \arrow[r] \arrow[d, "w_n'\cdots w_1'"] & U \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U_n \arrow[r, "w_n"] & U_{n+1} \arrow[r] & U \arrow[r] & 0
\end{tikzcd}$$

(now we arrange the squares of the tower vertically, and not horizontally).

Since $w_{t-1}' \cdots w_1' w_0 \gamma = 0$, we see that w_t is a split monomorphism.

We assume now that we deal with modules over an artin algebra Λ, say a module-finite k-algebra, where k is an artinian commutative ring. By $|-|$ we denote the length over the ground ring k.

Proposition. *Assume there is an exact sequence $0 \to U \to U \oplus X \xrightarrow{g} Y \to 0$. Then*

$$|\operatorname{End}(X)| \leq |\operatorname{Hom}(Y, X)| \leq |\operatorname{End}(Y)| \quad \textit{and} \quad |\operatorname{End}(X)| \leq |\operatorname{Hom}(X, Y)| \leq |\operatorname{End}(Y)|.$$

If X, Y are not isomorphic, then the sequence does not split, and we have both

$$|\operatorname{Hom}(Y, X)| < |\operatorname{End}(Y)| \quad \textit{and} \quad |\operatorname{Hom}(X, Y)| < |\operatorname{End}(Y)|,$$

and thus also $|\operatorname{End}(X)| < |\operatorname{End}(Y)|$.

Proof (I learnt it from Smalø): Apply $\operatorname{Hom}(Y,-)$, we get

$$0 \to (Y,U) \to (Y,U) \oplus (Y,X) \xrightarrow{(Y,g)} (Y,Y),$$

thus $|(Y,X)| \le |(Y,Y)|$. Also, apply $\operatorname{Hom}(-,X)$, we get

$$0 \to (Y,X) \to (U,X) \oplus (X,X) \to (U,X),$$

thus $|(Y,X)| \ge |(X,X)|$. Altogether, we have

$$|(X,X)| \le |(Y,X)| \le |(Y,Y)|.$$

This yields the first assertion.

If $|(Y,X)| = |(Y,Y)|$, then it follows that (Y,g) is surjective, thus 1_Y can be lifted: there is a map $f\colon Y \to U \oplus X$ with $gf = 1_Y$, that means: g splits. But if g splits, then $U \oplus X$ is isomorphic to $U \oplus Y$, and thus X and Y are isomorphic. This (and the dual) yield the second assertion.

Example 1. Let A be a submodule of B. Then B degenerates to $A \oplus B/A$ with Riedtmann-Zwara sequence

$$0 \to A \xrightarrow{\left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right]} B \oplus A \xrightarrow{p \oplus 1_A} B/A \oplus A,$$

where $u\colon A \to B$ is the inclusion map, $p\colon B \to B/A$ the canonical projection.

Example 2. Consider the algebra with an arrow $1 \to 2$ and a loop $\alpha\colon 2 \to 2$ with $\alpha^2 = 0$. There are two indecomposable modules $P(1)$ and M with composion factors $1, 2, 2$, with $\operatorname{top} M = 1 \oplus 2$, and M being a degeneration of $P(1)$. There is a corresponding Riedtmann-Zwara sequence

$$0 \to P(2) \to P(1) \oplus P(2) \to M \to 0.$$

Thus M is a degeneration of $P(1)$. Note that both $P(1)$ and M are indecomposable.

Further reference. For the construction and the properties of the tower of a pair of maps $w, v\colon U_0 \to U_1$ with w being a monomorphism, see Ringel: *Degenerations of modules* (in preparation). In particular, such a pair w, v gives rise to a Prüfer module $W[\infty]$, where W is the cokernel of w.