The orthogonality relations

Let $\Bbbk$ be an algebraically closed field of characteristic zero, and let G be a finite group. The Artin-Wedderburn structure theorem implies that we can write $\Bbbk G = \bigoplus_{i=1}^{r} \Bbbk G e_i$ where the $e_i \in Z(\Bbbk G)$ are the central primitive idempotents. The algebra $\Bbbk G e_i$ is simple, and we denote its simple module by V_i. Also, let $\chi_i := \chi_{V_i}$ be the character of G afforded by V_i. Recall that $\chi_i(g) = Tr_{V_i}(g)$. If $W \cong \bigoplus_{j=1}^{m} W_j$ is a $\Bbbk G$-module, the character afforded by W is $\chi_W = \chi_1 + \cdots + \chi_m$.

Definition 1
$\varrho := \chi_{\Bbbk G}$ is called the regular character of G. □

Proposition 2

(i) $\varrho(g) = |G|\delta_{g1}$ for all $g \in G$, where δ denotes the Kronecker deltafunction.

(ii) $\varrho = \sum_{i=1}^{r} \chi_i(1)\chi_i$.

Proof: (i) Since any $g \in G$ acts on $\Bbbk G$ as a permutation matrix with respect to the basis of $\Bbbk G$ consisting of the elements of G, $\varrho(g)$ is equal to the number of fixed points of g on G.
(ii) By the Artin-Wedderburn structure theorem, $\Bbbk G \cong \bigoplus_{i=1}^{r} \dim_{\Bbbk}(V_i)V_i$ as $\Bbbk G$-module. □

Theorem 3
$e_i = \frac{1}{|G|}\sum_{g\in G} \chi_i(1)\chi_i(g^{-1})g$ *for all* $1 \leq i \leq r$.

Proof: Write $e_i = \sum_{g\in G} a_g g$. Then $|G|a_g = \varrho(e_i g^{-1}) = \sum_{j=1}^{r} \chi_j(1)\chi_j(e_i g^{-1})$ by Proposition 2. But e_i acts as $\delta_{ij}\mathrm{id}_{V_j}$ on V_j, hence $\chi_j(e_i g^{-1}) = \delta_{ij}\chi_i(g^{-1})$. Thus $a_g = \frac{1}{|G|}\chi_i(1)\chi_i(g^{-1})$. □

Theorem 4 (First orthogonality relation)
$\frac{1}{|G|}\sum_{g\in G} \chi_i(g)\chi_j(g^{-1}) = \delta_{ij}$ *for all* $1 \leq i, j \leq r$.

Proof: Applying the above formula for the idempotents to $\delta_{ij}e_i = e_i e_j$, one gets

$$\begin{aligned}\frac{\delta_{ij}}{|G|}\sum_{g\in G} \chi_i(1)\chi_i(g^{-1}) &= \frac{1}{|G|^2}\sum_{g,h\in G} \chi_i(1)\chi_i(g^{-1})\chi_j(1)\chi_j(h^{-1})gh \\ &= \frac{1}{|G|^2}\sum_{g,h\in G} \chi_i(1)\chi_i((gh)^{-1})\chi_j(1)\chi_j(h)g.\end{aligned}$$

A comparison of the coefficients of $g = 1$ yields

$$\frac{1}{|G|}\sum_{h\in G} \chi_i(h^{-1})\chi_j(h) = \delta_{ij}\frac{\chi_i(1)}{\chi_j(1)} = \delta_{ij}$$

as desired. □

Definition 5
The $\Bbbk$-vector space $\mathcal{C}(G) := \{f : G \to \Bbbk \mid f(x^{-1}gx) = f(g)\ \forall x, g \in G\}$ is called the space of class functions of G. For $f, f' \in \mathcal{C}(G)$, one sets $(f, f') := \frac{1}{|G|}\sum_{g\in G} f(g)f'(g^{-1})$. This defines a a symmetric bilinear form $(\cdot,\cdot)$ on $\mathcal{C}(G)$. □

Oviously, any character is a class function, and $\dim_{\Bbbk} \mathcal{C}(G) = r$, which is the number of conjugacy classes of G, as seen in the lecture on the Artin-Wedderburn structure theorem.

Corollary 6
The irreducible characters of G, namely $\chi_1, \ldots, \chi_r$, form an orthonormal basis of $\mathcal{C}(G)$ (with respect to $(\cdot, \cdot)$). □

Corollary 7
Let V be a finite dimensional $\Bbbk G$-module. Then $V \cong \bigoplus_{i=1}^{r} (\chi_V, \chi_i) V_i$. □

Definition 8
Let $g_1, \ldots, g_r$ be representatives of the conjugacy classes of G. The matrix

$$X := (\chi_i(g_j))_{1 \leq i,j \leq r} \in \Bbbk^{r \times r}$$

is called the character table of G. Of course, X is only defined up to permutation of rows and columns. □

Theorem 9 (Second orthogonality relation)

$$\sum_{i=1}^{r} \chi_i(g)\chi_i(h^{-1}) = \begin{cases} 0 & g, h \text{ not conjugate} \\ |C_G(g)| & g \text{ conjugate to } h \end{cases}$$

for all $g, h \in G$.

Proof: Let $X^\dagger := (\chi_j(g_i^{-1}))_{1 \leq i,j \leq r}$. The first orthogonality relation asserts

$$\delta_{ij} = \frac{1}{|G|} \sum_{g \in G} \chi_i(g)\chi_j(g^{-1}) = \frac{1}{|G|} \sum_{m=1}^{r} \frac{|G|}{|C_G(g_m)|} \chi_i(g_m)\chi_j(g_m^{-1}).$$

This can be written as a matrix identity as follows.

$$E_r = X \mathrm{diag}(\frac{1}{|C_G(g_1)|}, \ldots, \frac{1}{|C_G(g_r)|}) X^\dagger = \mathrm{diag}(\frac{1}{|C_G(g_1)|}, \ldots, \frac{1}{|C_G(g_r)|}) X^\dagger X$$

Since $(X^\dagger X)_{i,j} = \sum_{m=1}^{r} \chi_m(g_i^{-1})\chi_m(g_j)$, the theorem is proved. □

Although the second orthogonality relation is an easy consequence of the first, it is quite useful in the construction of character tables.

Suppose $\Bbbk \subseteq \mathbb{C}$. Then $\chi(g^{-1}) = \overline{\chi(g)}$, since any matrix of finite order is diagonalizable with roots of unity as eigenvalues. In this situation, $(\cdot, \cdot)$ is a Hermitian form on $\mathcal{C}(G)$. The first orthogonality relation asserts that the rows of the character table of G are orthonormal with respect to the standard Hermitian form on $\Bbbk^r$, if the i^{th} column gets multiplied with $\frac{1}{\sqrt{|C_G(g_i)|}}$. The second orthogonality relation states that the columns of the character table are orthogonal.

Example: Let G be non-Abelian group of order eight. Since the centre of G is non-trivial and $G/Z(G)$ is not cyclic, $Z(G) = \langle z \rangle$ has order two and $G/Z(G) = \langle aZ(G), bZ(G) \rangle$ is the Klein four group. Hence

$$\{1\} \ \ \{z\} \ \ \{a, az\} \ \ \{b, bz\} \ \ \{ab, abz\}$$

are the conjugacy classes of G. The characters of the Klein four group yield this

	1	z	a	b	ab
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	1	1	1	-1	-1
χ_4	1	1	-1	1	-1

part of the character table of G. Since there five conjugacy classes, there is one more irreducible character χ_5 of G. The values of χ_5 are easily computed using the second orthogonality relation. The complete character table of G is:

	1	z	a	b	ab
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	1	1	1	-1	-1
χ_4	1	1	-1	1	-1
χ_5	2	-2	0	0	0

It is not to hard to determine the values of χ_5 without the orthogonality relations. Indeed, it follows from the Artin-Wedderburn structure theorem that $\chi_5(1)$ equals two. Since z acts as a scalar on the module affording χ_5, we have $\chi_5(z) = \pm 2$. However, if $\chi_5(z) = 2$, the centre of G would act trivially on any $\Bbbk G$-module, which cannot happen (it does act non-trivially on $\Bbbk G$). If V and W are $\Bbbk G$-modules, then so is $V \otimes_{\Bbbk G} W$, where G acts via $g(v \otimes w) := (gv) \otimes (gw)$. The Character afforded by $V \otimes_{\Bbbk} W$ is $\chi_V \chi_W$, the product of the characters afforded by V and W, respectively. Since there is only one simple two dimenional $\Bbbk G$-module, $\chi_i \chi_5 = \chi_5$ for $1 \leq i \leq 4$. Thus, the remaining values of χ_5 are equal to zero.

REMARK: The fact that the tensor product of two $\Bbbk G$-modules is again a $\Bbbk G$-module is due to the coalgebra structure of $\Bbbk G$.