Symmetries of flat manifolds

Let $\mathcal{A}_n \cong GL_n(\mathbb{R}) \ltimes \mathbb{R}^n$ be the group of affine maps from $\mathbb{R}^n$ onto itself, and let $\mathcal{E}_n \cong O_n(\mathbb{R}) \ltimes \mathbb{R}^n$ be the group of isometries from $\mathbb{R}^n$ onto itself. Both $\mathcal{A}_n$ and $\mathcal{E}_n$ will be considered as topological groups with their natural topologies.

Definition 1

(i) A n-dimensional crystallographic group Γ is a discrete, cocompact (i.e. the orbit space $\mathbb{R}^n / \Gamma$ is compact) subgroup of $\mathcal{E}_n$.

(ii) A Bieberbach group is a torsionfree crystallographic group.

(iii) A n-dimensional (compact) flat (Riemannian) manifold is the orbit space of a n-dimensional Bieberbach group. □

Of course, this is not the usual definition of a flat manifold. Usually, one takes a compact Riemannian manifold with curvature zero. However, every such manifold has universal cover $\mathbb{R}^n$, and the corresponding group of Deck transformations is a Bieberbach group. Thus one could define Bieberbach groups to be the fundamental groups of flat manifolds.

Examples: Obviously, $\Gamma := \mathbb{Z}^n$ is a Bieberbach group. The corresponding flat manifold is $\mathbb{R}^n/Z^n$, a flat torus.

Now consider

$$\Gamma := \langle\, (\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}),\ (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix})\,\rangle \leq \mathcal{E}_2.$$

This is easily seen to be Bieberbach group. Its orbit space is the Klein bottle. Γ and $\mathbb{Z}^2$ are the only 2-dimensional Bieberbach groups.

Definition 2

Let $\Gamma_1, \Gamma_2 \leq \mathcal{E}_n$ be Bieberbach groups with orbit spaces $X_i := \mathbb{R}^n / \Gamma_i$. A diffeomorphism $\alpha : X_1 \to X_2$ is called affine, if there exists $\tilde{\alpha} \in \mathcal{A}_n$ such that the diagram

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\tilde{\alpha}} & \mathbb{R}^n \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow{\alpha} & x_2 \end{array}$$

commutes. $\mathrm{Aff}(X_1) := \{\alpha : X_1 \to X_1 \mid \alpha\, \textit{affine}\}$ *is called the group of affinites of X_1.* □

Once again, there is some geometry hidden. A Riemannian manifolds possesses a unique connection, called the Levi-Civita connection, that is compatible with the Riemannian structure. An affine diffeomorphism of two Riemannian manifolds is one that respects the Levi-Civita connection.

Conjecture 3 (Malfait, 1998)

If X is flat manifold with $\dim(X) > 0$, *then* $\mathrm{Aff}(X)$ *is not torsionfree.* □

The task for my diploma thesis was to prove or disprove this conjecture. Fortunately, I found a counterexample, which will be presented at the end of this notes.

We now state Bieberbach's fundamental theorem on the structure of crystallographic groups. The third part is the solution to (half of) Hilbert's 18th problem.

Theorem 4 (Bieberbach, 1911-1912)
Let $\Gamma \leq \mathcal{E}_n$ be a crystallographic group.

(i) The translational part $L := \Gamma \cap \mathbb{R}^n$ is a full lattice in $\mathbb{R}^n$, i.e. $L \cong \mathbb{Z}^n$, and L contains a basis of $\mathbb{R}^n$. The holonomy group $G := \Gamma/L$ of Γ is finite.

(ii) Let $\Gamma' \leq \mathcal{E}_n$ be a crystallographic group, and let $f : \Gamma \to \Gamma'$ be an isomorphism. Then there exists $\alpha \in \mathcal{A}_n$ such that $f(\gamma) = \alpha\gamma\alpha^{-1}$ for all $\gamma \in \Gamma$, i.e. f is induced by conjugation with α.

(iii) Up to conjugation with an element of $\mathcal{A}_n$, there are only finitely many n-dimensional crystallographic groups. □

The corresponding theorem for flat manifolds is:

Theorem 5

(i) Every flat manifold is covered by a flat torus and has finite holonomy group.

(ii) If two flat manifolds have isomorphic fundamental groups, they are isomorphic.

(iii) Up to affine equivalence, there are only finitely many flat manifolds of a given dimension. □

The name holonomy group for the quotient Γ/L comes from differential geometry: a connection on a manifold X yields a linear map of tangent spaces $T_xX \to T_yX$ for a given path from x to y, called parallel transport along this path. Doing this for loops, one obtains a subgroup of $GL(T_xX)$ for each $x \in X$, called the holonomy group at x. If X is path connected, all holonomy groups are isomorphic. One can show that the holonomy group of $\mathbb{R}^n/\Gamma$ (with respect to the Levi-Civita connection) is Γ/L.

From now on, let $\Gamma \leq \mathcal{E}_n$ be a crystallographic group with translation subgroup L and holonomy group G. By the first part of Bieberbach's theorem, Γ satisfies an exact sequence

$$0 \to L \to \Gamma \to G \to 1.$$

Obviously, G acts faithfully on L, so we may assume that G is a subgroup of $\mathrm{Aut}(L)$.[1] Conversely, any extension as above, where L is a free Abelian group of rank n, and G is a finite subgroup of $\mathrm{Aut}(L)$, gives rise to an n-dimensional crystallographic group. The extensions of L by G are parametrized by the elements of $H^2(G, L) = \mathrm{Ext}^2_{\mathbb{Z}G}(\mathbb{Z}, L)$, and the extension corresponding to $\eta \in H^2(G, L)$ is torsionfree if and only if $\mathrm{res}^G_U \eta \neq 0$ for all $1 \neq U \leq G$.

As a consequence of the second part of Bieberbach's theorem, the group of affinites of a flat manifold is closely related to the automorphism group of the corresponding Bieberbach group. First of all, $\mathrm{Aut}(\Gamma) \cong N_{\mathcal{A}_n}(\Gamma)/C_{\mathcal{A}_n}(\Gamma)$. An easy computation gives:

Proposition 6
$Z(\Gamma) = L^G := \{v \in L \mid gv = v \ \forall g \in G\}$ *and* $C_{\mathcal{A}_n}(\Gamma) = (\mathbb{R}^n)^G = \langle L^G \rangle_{\mathbb{R}}$. □

[1] This sequence allows one to prove the third part of Bieberbach's theorem. By the second part, one only needs to consider crystallographic groups up to isomorphism. By a theorem of Jordan, there are only finitely many conjugacy classes of finite subgroups of $GL_n(\mathbb{Z})$. Also, for each finite subgroup G there are only finitely many extensions.

Thus $N_{\mathcal{A}_n}(\Gamma)/\Gamma$ has an Abelian normal subgroup, $C_{\mathcal{A}_n}(\Gamma)\Gamma/\Gamma \cong (\mathbb{R}/\mathbb{Z})^{\mathrm{rank}(Z(\Gamma))}$, and the quotient by this subgroup is isomorphic to $\mathrm{Out}(\Gamma)$, the group of outer automorphism of Γ.

If Γ is a Bieberbach group, $\mathrm{Aff}(\mathbb{R}^n/\Gamma) \cong N_{\mathcal{A}_n}(\Gamma)/\Gamma$. The subgroup $C_{\mathcal{A}_n}(\Gamma)\Gamma/\Gamma$ is the identity component of the group of affinities. It is a torus whose dimension is equal to the rank of $Z(\Gamma)$ by the above discussion. It is known from algebraic topology that $H_1(\mathbb{R}^n/\Gamma)$ is isomorphic to $\Gamma/[\Gamma,\Gamma]$.[2] It is not to hard to show that the torsionfree part of $\Gamma/[\Gamma,\Gamma]$ is isomorphic to $Z(\Gamma)$. Thus, the rank of $Z(\Gamma)$, which is the dimension of $\mathrm{Aff}_0(\mathbb{R}^n/\Gamma)$, happens to be the first Betti number of $\mathbb{R}^n/\Gamma$.

To understand the group of affinities of a flat manifold, it remains to understand the outer automorphism group of its fundamental group. We will now look at $\mathrm{Out}(\Gamma)$ for a crystallographic group Γ, since it makes no difference for the description of the outer automorphism group whether Γ is torsionfree or not. Pick $f \in \mathrm{Aut}(\Gamma)$. By Bieberbach's second theorem, there exists $\alpha = (x, u) \in \mathcal{A}_n$ such that f is just conjugation by α. Thus

$$f(g,l) = (x,u)(g,l)(x^{-1}, -x^{-1}u) = (xgx^{-1}, u + xl - xgx^{-1}u)$$

for any $(g,l) \in \Gamma$. This implies $x \in N := N_{\mathrm{Aut}(L)}(G)$. But N acts on $H^2(G, L)$ in a natural way, and one sees easily that x must fix the element $\eta \in H^2(G, L)$ which corresponds to Γ. Altogether, one gets a homomorphism $r : \mathrm{Aut}(\Gamma) \to N_\eta$, where N_η denotes the stabilizer of η in N. It is rather straightforward to check the following properties of r.

Proposition 7

(i) *r is onto.*

(ii) *The kernel of r consists of those automorphisms of Γ that induce automorphisms of the extension*

$$0 \to L \to \Gamma \to G \to 1.$$

(iii) *The kernel of r, intersected with the group of inner automorphisms of Γ, is isomorphic to $L/L^G = L/Z(\Gamma)$.*

(iv) $\ker(r)/\ker(r) \cap \mathrm{Inn}(\Gamma) \cong H^1(G, L)$ □

Here, $\mathrm{Inn}(\Gamma) \cong \Gamma/Z(\Gamma)$ denotes the group of inner automorphisms of Γ.

Theorem 8

The outer automorphism group of Γ satisfies the following exact sequence:

$$0 \to H^1(G, L) \to \mathrm{Out}(\Gamma) \to N_\eta/G \to 1$$

[2] It is easy to compute the commutator subgroup for the fundamental group of the Klein bottle. It is $\mathbb{Z}(0,2)^t$. Hence the first homology group of the Klein bottle is $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$.

Proof: The properties of r mentioned above yield the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L/L^G & \longrightarrow & \mathrm{Inn}(\Gamma) & \xrightarrow{r} & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \ker(r) & \longrightarrow & \mathrm{Aut}(\Gamma) & \xrightarrow{r} & N_\eta & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H^1(G, L) & & \mathrm{Out}(\Gamma) & & N_\eta/G & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 1 & & 1 & &
\end{array}
$$

An easy diagram chase shows that one can complete the bottom row. □

The cohomology groups $H^i(G, L)$ are finite for $i > 0$, so one gets:

Corollary 9
$\mathrm{Out}(\Gamma)$ *is finite if and only if* N *is finite.* □

Since the quotient $N/C_{\mathrm{Aut}(L)}(G) \lesssim \mathrm{Aut}(G)$ is finite, N is finite if and only if $C_{\mathrm{Aut}(L)}(G)$ is finite. This is the unit group of $\mathrm{End}_{\mathbb{Z}G}(L)$, which is a $\mathbb{Z}$-order in $\mathrm{End}\mathbb{Q}G(L)$. Analyzing unit groups of orders, one can show:

Theorem 10 (Brown-Neubüser-Zassenhaus, 1973)
$\mathrm{Out}(\Gamma)$ *is finite if and only if* $\mathbb{Q} \otimes_{\mathbb{Z}} L$ *is multiplicity free (as a* $\mathbb{Q}G$*-module), and for each simple submodule* V *of* $\mathbb{Q} \otimes_{\mathbb{Z}} L$*, the* $\mathbb{R}G$*-module* $\mathbb{R} \otimes_{\mathbb{Q}} V$ *is simple.* □

Let M be a $\mathbb{Z}G$ lattice, i.e. a $\mathbb{Z}G$-module which is free of finite rank as a $\mathbb{Z}$-module. Suppose that $\mathbb{R} \otimes_{\mathbb{Z}} M$ is simple. Then two cases occur: either $\mathbb{C} \otimes_{\mathbb{Z}} M$ is simple, or $\mathbb{C} \otimes_{\mathbb{Z}} M = V \oplus \overline{V}$, where the character of G afforded by $\overline{V}$ is the complex conjugate of the character afforded by V. In the first case, $\mathrm{End}_{\mathbb{Z}G}(M) \cong \mathbb{Z}$ is as small as it can possibly be. In the second case, $\mathrm{End}_{\mathbb{Q}G}(\mathbb{Q} \otimes_{\mathbb{Z}} M) \cong \mathbb{Q}(\chi_V)$ is an imaginary quadratic number field (χ_V denotes the character afforded by V). The ring of algebraic integers in $\mathbb{Q}(\chi_V)$ is the unique maximal order, and its unit group is $\{\pm 1\}$, unless $\mathbb{Q}(\chi_V) = \mathbb{Q}(i)$ or $\mathbb{Q}(\chi_V) = \mathbb{Q}(\frac{1+i\sqrt{3}}{2})$.

Example: Let $G := M_{11}$ be the Mathieu group on eleven letters. This has no outer automorphisms, so it suffices to consider the centralizer. Using trial and error – and a computer –, I have constructed four $\mathbb{Z}G$-lattices $L_1, \ldots, L_4$ as follows:

(i) The character afforded by L_1 is the sum of the two complex conjugate characters of G of degree ten, the character field is $\mathbb{Q}(i\sqrt{2})$, and $H^2(G, L_1) \cong C_6$.

(ii) The character afforded by L_2 is the sum of the two characters of G of degree 16, the character field is $\mathbb{Q}(i\sqrt{11})$, and $H^2(G, L_2) \cong C_5$.

(iii) The character afforded by L_3 is the irreducible character of G of degree 44 and $H^2(G, L_3)$ is cyclic of order six.

(iv) The character afforded by L_4 is the irreducible character of G of degree 45 and $H^2(G, L_4)$ is cyclic of order eleven.

Let $L := L_1 \oplus \cdots \oplus L_4$. Since the trivial character is no constituent of the character afforded by L, the space L^G of fixed points is zero. This implies, using the long exact sequence of cohomology, that $H^1(G, L)$ is isomorphic to $H^0(G, \mathbb{Q} \otimes_{\mathbb{Z}} L/L) = (\mathbb{Q} \otimes_{\mathbb{Z}} L/L)^G$. A prime p divides the order of this group of fixed points if and only if $(L/pL)^G$ is non-zero. One can easily check – again using a computer –, that $(L_i/pL_i)^G = 0$ for all i and all primes dividing $|G| = 7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$. Thus, $H^1(G, L) = 0$. By the above discussion, $C := C_{\mathrm{Aut}(L)}(G) \cong \{\pm 1\}^4$. Let $\eta := \eta_1 + \cdots + \eta_4$, where each η_i is a generator of $H^2(G, L_i)$. None of the η_i has order two, so $C_\eta := \mathrm{Stab}_C(\eta) = 1$. By 8, the outer automorphism group of Γ is trivial, where Γ is the extension corresponding to η. Also, $Z(\Gamma) = L^G = 0$, so Γ gives a counterexample to Malfait's conjecture, provided it is torsionfree. However, using once more a computer rather then the brain, one checks that $\mathrm{res}^G_U \eta \neq 0$ for each $1 \neq U \leq G$. Note that it suffices to this for representatives of the subgroups of prime order, since restriction is transitive (and "invariant" an conjugacy classes of subgroups). Also, there is nothing to be done for subgroups of order five or eleven, since those are Sylow subgroups (for $P \leq G$ a Sylow p-subgroup, $\mathrm{res}^G_P : H^i(G, L)_p \to H^i(P, L)$ is injective for $i > 0$).