SELF-INJECTIVE ALGEBRAS: COMPARISON WITH FROBENIUS ALGEBRAS

ROLF FARNSTEINER

Let Λ be a finite dimensional algebra, defined over a field k. The category of finite dimensional left Λ-modules and the set of isoclasses of simple Λ modules will be denoted by $\operatorname{mod}\Lambda$ and $\mathcal{S}(\Lambda)$, respectively. We will occasionally identify $\mathcal{S}(\Lambda)$ with a complete set of its representatives. Given a simple Λ-module S, we consider its projective cover $P(S)$ and its injective envelope $E(S)$. Recall that $\operatorname{Top}(P(S)) \cong S$ and $\operatorname{Soc}(E(S)) \cong S$.

In our lecture [2], we observed that Λ is self-injective if and only if $S \mapsto \operatorname{Soc}(P(S))$ defines a permutation $\nu : \mathcal{S}(\Lambda) \longrightarrow \mathcal{S}(\Lambda)$, the so-called *Nakayama permutation* [2, Theorem]. In early articles, these algebras were referred to as *quasi-Frobenius algebras* (cf. [3]). The purpose of this lecture is to understand when a self-injective algebra is a Frobenius algebra. This class of algebras was introduced by Brauer and Nesbitt in [1], who provided the following chacterization:

Lemma 1. *Let $\pi : \Lambda \longrightarrow k$ be a linear map. Then the following statements are equivalent:*

(1) $\ker \pi$ *does not contain a non-zero left ideal.*

(2) *The Λ-linear map*

$$\Phi_\pi : \Lambda \longrightarrow \Lambda^* \quad ; \quad x \mapsto x.\pi$$

is an isomorphism.

(3) $\ker \pi$ *does not contain a non-zero right ideal.*

Proof. (1) $\Rightarrow$ (2). Consider the left ideal $J := \ker \Phi_\pi$. Since

$$0 = \Phi_\pi(x)(1) = \pi(x) \quad \forall\, x \in J,$$

we obtain the injectivity and hence the bijectivity of Φ_π.

(2) $\Rightarrow$ (3). Let $J \subset \ker \pi$ be a right ideal. Given a linear form $f \in \Lambda^*$, condition (2) provides an element $x \in \Lambda$ such that $f = x.\pi$. For $y \in J$ we thus obtain

$$f(y) = (x.\pi)(y) = \pi(yx) \in \pi(J) = (0).$$

Consequently, $J = (0)$.

(3) $\Rightarrow$ (1). Since (1) $\Rightarrow$ (3) holds for every algebra, application to the opposite algebra Λ^{op} yields the desired conclusion. □

Definition. A k-algebra Λ is a *Frobenius algebra* if there exists a linear form $\pi \in \Lambda^*$ such that $\ker \pi$ contains no non-zero left ideals.

Given a linear form $\pi : \Lambda \longrightarrow k$, we consider the bilinear form

$$(\ ,\ )_\pi : \Lambda \times \Lambda \longrightarrow k \quad ; \quad (a, b) := \pi(ab) \quad \forall\, a, b \in \Lambda.$$

This form is *associative*, that is,

$$(ax, b)_\pi = (a, xb)_\pi \quad \forall\, a, b, x \in \Lambda.$$

Date: October 27, 2005.

Conversely, any associative form $(\ ,\ ) : \Lambda \times \Lambda \longrightarrow k$ is obtained in this fashion: $(\ ,\ ) = (\ ,\ )_\pi$, where $\pi(a) = (a, 1)$.

Corollary 2. *The algebra Λ is a Frobenius algebra if and only if there exists a non-degenerate, associative form on Λ.* $\square$

If $(\ ,\ ) : \Lambda \times \Lambda \longrightarrow k$ is such a form, then there exists an automorphism $\mu : \Lambda \longrightarrow \Lambda$ such that

$$(y, x) = (\mu(x), y) \quad \forall\ x, y \in \Lambda.$$

Another such automorphism μ', induced by a form $\{\ ,\ \}$, is related to μ via an invertible element $u \in \Lambda$, i.e.

$$\mu'(x) = u\mu(x)u^{-1} \quad \forall\ x \in \Lambda.$$

These automorphisms are referred to as *Nakayama automorphisms* of the Frobenius algebra Λ.

Given an automorphism $\alpha : \Lambda \longrightarrow \Lambda$ and a Λ-module M, we denote by $M^{(\alpha)}$ the Λ-module with underlying k-space M and action

$$a{\centerdot}m := \alpha^{-1}(a)m \quad \forall\ a \in \Lambda, m \in M.$$

Since twisting M by an inner automorphism reproduces M, the Nakayama automorphisms induce naturally equivalent automorphisms on $\operatorname{mod}\Lambda$.

Let $(\ ,\ ) : \Lambda \times \Lambda \longrightarrow k$ be a non-degenerate associative form with Nakayama automorphism μ; then $\gamma := \mu \otimes \mathrm{id}_\Lambda$ is an automorphism of the enveloping algebra $\Lambda^e := \Lambda \otimes_k \Lambda^{\mathrm{op}}$. The map

$$\Psi : \Lambda^{\gamma^{-1}} \longrightarrow \Lambda^* \ \ ; \ \ \Psi(x)(y) = (x, y)$$

is an isomorphism of Λ^e-modules. Let us look at left linearity. For $r, x, y \in \Lambda$ we have

$$\Psi(r{\centerdot}x)(y) = \Psi(\mu(r)x)(y) = (\mu(r)x, y) = (\mu(r), xy) = (xy, r) = (x, yr) = \Psi(x)(yr) = (r{\centerdot}\Psi(x))(y).$$

As a result, we have

$$\Lambda^* \otimes_\Lambda M \cong M^{(\mu^{-1})} \quad \forall\ M \in \operatorname{mod}\Lambda.$$

Watt's theorem tells us that the functor $M \mapsto \Lambda^* \otimes_\Lambda M$ is naturally isomorphic to the Nakayama functor (see [2] for the definition). Here is a low brow argument involving adjoint isomorphisms: We have the following isomorphisms of Λ-modules:

$$\begin{aligned} \Lambda^* \otimes_\Lambda M &\cong (\Lambda^* \otimes_\Lambda M)^{**} \cong \mathrm{Hom}_k(\Lambda^* \otimes_\Lambda M, k)^* \\ &\cong \mathrm{Hom}_\Lambda(M, \mathrm{Hom}_k(\Lambda^*, k))^* \cong \mathrm{Hom}_\Lambda(M, \Lambda)^* = \mathcal{N}(M). \end{aligned}$$

As an upshot, we obtain natural isomorphisms

$$\mathcal{N}(M) \cong M^{(\mu^{-1})}.$$

By combining this with [2, Lemma 2] we conclude that the Nakayama permutation is given by

$$\nu(S) \cong S^{(\mu)} \quad \forall\ S \in \mathcal{S}(\Lambda).$$

We have thus verified one implication of our main result:

Theorem 3. *Let Λ be a self-injective algebra. Then the following statements are equivalent:*

(1) *Λ is a Frobenius algebra.*
(2) *$dim_k \operatorname{Soc}(P(S)) = \dim_k S \quad \forall\ S \in \mathcal{S}(\Lambda)$.*

Proof. (2) $\Rightarrow$ (1). Since Λ is self-injective, we have a Nakayama permutation $\nu : \mathcal{S}(\Lambda) \longrightarrow \mathcal{S}(\Lambda)$. Since $\mathcal{N}(S) \cong \nu^{-1}(S)$, $\mathcal{N}$ induces an injection $\mathrm{End}_\Lambda(S) \hookrightarrow \mathrm{End}_\Lambda(\nu^{-1}(S))$, so that iteration implies

$$\mathrm{End}_\Lambda(S) \cong \mathrm{End}_\Lambda(\nu(S)) \qquad \forall\ S \in \mathcal{S}(\Lambda).$$

Writing $\Lambda = \bigoplus_{S\in\mathcal{S}(\Lambda)} n_S P(S)$, application of $\mathrm{Hom}_\Lambda(-, S)$ yields

$$(*) \qquad n_S = \frac{\dim_k S}{\dim_k \mathrm{End}_\Lambda(S)} = \frac{\dim_k \nu(S)}{\dim_k \mathrm{End}_\Lambda(\nu(S))} = n_{\nu(S)}.$$

In view of $P(S) \cong E(\nu(S))$ and [2, Lemma 2] we thus have

$$(\Lambda_\Lambda)^* \cong \mathcal{N}(\Lambda) \cong \bigoplus_{S\in\mathcal{S}(\Lambda)} n_S E(S) \cong \bigoplus_{S\in\mathcal{S}(\Lambda)} n_S P(S) \cong \Lambda.$$

If $\Phi : \Lambda \longrightarrow \Lambda^*$ is the corresponding isomorphism of Λ-modules, then $\pi := \Phi(1)$ renders Λ a Frobenius algebra. □

Corollary 4. *Every self-injective, basic algebra Λ is a Frobenius algebra.*

Proof. Returning to the proof of Theorem 3 we recall that our present assumption means $n_S = 1$ for every $S \in \mathcal{S}(\Lambda)$. Equation $(*)$ then implies $\dim_k S = \dim_k \nu(S)$, so that Λ is a Frobenius algebra. □

References

[1] R. Brauer and C. Nesbitt. *On the regular representations of algebras.* Proc. Nat. Acad. Sci. U.S.A. **23** (1937), 236-240

[2] R. Farnsteiner. *Self-injective algebras: The Nakayama permutation.* Lecture Notes, available at `http://www.mathematik.uni-bielefeld.de/~sek/selected.html`

[3] T. Nakayama. *On Frobeniusean algebras I.* Ann. of. Math. **40** (1939), 611-633