STABLE REPRESENTATION QUIVERS: GROWTH NUMBERS AND ZHANG'S THEOREM

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Let Λ be an artin algebra. As shown in our previous lecture [3], Riedtmann's Theorem sets the stage for the determination of the regular components of the Auslander-Reiten quiver of Λ. More precisely, given such a component Q, abstract considerations, which make no reference to the category $\operatorname{mod}\Lambda$ of finitely generated left Λ-modules, provide a directed tree T_Q and an admissible subgroup $\Pi \subset \operatorname{Aut}(\mathbb{Z}[T_Q])$ such that

$$Q \cong \mathbb{Z}[T_Q]/\Pi.$$

We say that Q is *tree-infinite* if the tree class $\bar{T}_Q$ of Q is an infinite tree.

In this lecture, we determine T_Q and Π of tree-infinite components subject to some hypotheses concerning the growth numbers of the modules belonging to Q. The relevant conditions obtain in classical contexts, such as group algebras of finite groups.

Here is the magic number:

$$\omega := \sqrt[3]{\frac{1}{2} + \sqrt{\frac{23}{108}}} + \sqrt[3]{\frac{1}{2} - \sqrt{\frac{23}{108}}}.$$

Theorem ([6]). *Let Q be a non-periodic, tree-infinite, regular component of the Auslander-Reiten quiver of Λ. If the growth numbers ρ_Q^ℓ and ρ_Q^r of Q are smaller that ω, then*

$$Q \cong \mathbb{Z}[T], \text{ where } T \in \{A_\infty, A_\infty^\infty, B_\infty, C_\infty, D_\infty\}.$$

The result refers to valued quivers, thus the distinction between A_∞, B_∞ and C_∞. To ease the technical aspects, we will ignore valuations, thereby eliminating B_∞ and C_∞. This simplification is legitimate in case Λ is a finite dimensional algebra over an algebraically closed field.

An AR-component Q is called *regular* if it contains neither projective nor injective vertices. Regular components are stable representation quivers with $\tau = \mathrm{DTr}$.

Let Q be a regular AR-component. We say that Q is *periodic* if for each isoclass $[M] \in Q$ there exists a natural number $m \geq 1$ such that

$$\mathrm{DTr}^m(M) \cong M.$$

Thanks to a result due to Happel-Preiser-Ringel [4, Cor.2], periodic components are of the form $Q \cong \mathbb{Z}[A_\infty]/\langle\tau^m\rangle$ for some $m \geq 1$.

We let k be a commutative artinian ring such that Λ is a finitely generated k-module. The length of a k-module module M will be denoted $\ell(M)$. There exists a natural number ℓ_Λ such that

$$\max\{\ell(\mathrm{DTr}(M)), \ell(\mathrm{TrD}(M))\} \leq \ell_\Lambda\, \ell(M)$$

for every $M \in \operatorname{mod}\Lambda$. We may thus make the following definition, which does not depend on the choice of k:

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Definition. Let $M \in \operatorname{mod}\Lambda$. Then
$$\rho^\ell_M := \limsup_{n\to\infty} \sqrt[n]{\ell(\mathrm{DTr}^n(M))} \ \text{ and } \ \rho^r_M := \limsup_{n\to\infty} \sqrt[n]{\ell(\mathrm{TrD}^n(M))}$$
are called the left and right *growth numbers* of M, respectively.

Remark. By definition, we have
$$\ell(\mathrm{DTr}^n(M)) \approx (\rho^\ell_M)^n$$
for infinitely many n, so that our notion of growth refers to exponential growth.

Lemma 1. *If Q is a regular component of the AR-quiver of $\operatorname{mod}\Lambda$, then*
$$\rho^\ell_M := \rho^\ell_N \quad \textit{and} \quad \rho^r_M := \rho^r_N \qquad [M],[N] \in Q. \qquad \square$$

Accordingly, we can define the left and right growth numbers
$$\rho^\ell_Q := \rho^\ell_M \ \text{ and } \ \rho^\ell_Q := \rho^\ell_M \quad [M] \in Q$$
of the component Q.

The proof of our Theorem rests on a comparison between the growth numbers of Q and the spectral radius of the Coxeter transformation of T_Q.

Let H be a hereditary algebra with Grothendieck group $K_0(H) \cong \mathbb{Z}^m$. The *Coxeter transformation* $\Phi : \mathbb{Z}^m \longrightarrow \mathbb{Z}^m$ is defined via
$$\Phi([P(S)]) = -[I(S)],$$
where $P(S)$ and $I(S)$ are the projective cover and the injective hull of the simple H-module S, respectively. If M is an indecomposable, non-projective H-module, then, letting $\underline{\dim}\, M \in \mathbb{Z}^m$ be the dimension vector of M, we have
$$\underline{\dim}\,\mathrm{DTr}(M) = \Phi(\underline{\dim} M).$$
Viewing Φ as a linear map of $\mathbb{C}^m$, we let
$$\rho_H := \max\{|\lambda| \ ; \ \lambda \in \mathrm{Spec}(\Phi)\}$$
be the maximal modulus of all eigenvalues of Φ. By the Perron-Frobenius Theorem ρ_Λ is a simple eigenvalue of Φ and
$$|\lambda| < \rho_\Lambda \quad \forall \ \lambda \neq \rho_\Lambda \in \mathrm{Spec}(\Phi).$$
To get a feeling for the connection between the spectral radius and growth numbers, let us consider the following result:

Proposition 2 ([5])**.** *Let H be a wild connected hereditary algebra. Then there exists $x^+ \in \mathbb{R}^m$ such that for every regular H-module M there is $\alpha_M > 0$ with*
$$\lim_{n\to\infty} \rho_H^{-n}\, \underline{\dim}\,\mathrm{DTr}^n(M) = \alpha_M x^+.$$

Upon application of the continuous function $x \mapsto \sum_{i=1}^m x_i$, we obtain
$$\lim_{n\to\infty} \sqrt[n]{\dim \mathrm{DTr}^n(M)} = \rho_H.$$

Our goal is to establish a similar result for arbitrary artin algebras. Given a finite quiver Γ without oriented cycles, we put $\rho_\Gamma := \rho_{\mathbb{C}[\Gamma]}$.

Lemma 3. *If T and T' are finite directed trees with the same underlying graph, then $\rho_T = \rho_{T'}$.* □

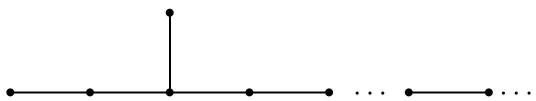
Definition. Let T be a tree. The *spectral radius* $\varrho(T)$ is defined via

$$\varrho(T) = \sup_{T' \subset T \text{ finite}} \rho(T'),$$

where $\rho(T') := \rho_{T'}$.

Proposition 4. *There exist infinite trees $T^1, \ldots, T^5$ with the following properties:*
(1) $\rho(T^1) = \min\{\rho(T^i) \ ; \ 1 \leq i \leq 5\} = \omega$.
(2) *If T be an infinite tree not belonging to $\{A_\infty, A_\infty^\infty, B_\infty, C_\infty, D_\infty\}$, then $T^i \subset T$ for some $i \in \{1, \ldots, 5\}$ and $\rho(T) \geq \omega$.* □

The graph $\mathbf{T^1}$



Proof of the Theorem. Let Q be an AR-component as given in the Theorem. By Riedtmann's Theorem, there exists a directed tree T_Q and an admissible subgroup $\Pi \subset \mathrm{Aut}(\mathbb{Z}[T_Q])$ such that

$$\mathbb{Z}[T_Q]/\Pi \cong Q.$$

By assumption, the tree class $\bar{T}_Q$ is an infinite tree.

If $\bar{T}_Q \notin \{A_\infty, A_\infty^\infty, B_\infty, C_\infty, D_\infty\}$, then $T^i \subset \bar{T}_Q$ for some $i \in \{1, \ldots, 5\}$. Let us assume that $T^1 \subset \bar{T}_Q$ and observe that T^1 can be approximated by wild trees T_j. We can apply Proposition 2 to each T_j and combine it with a theorem by Bautista [1] to arrive at the estimate

$$\min\{\rho_Q^\ell, \rho_Q^r\} \geq \rho(\bar{T}_Q).$$

Our current assumption in conjunction with Proposition 4 now gives a contradiction, so that $\bar{T}_Q \in \{A_\infty, A_\infty^\infty, B_\infty, C_\infty, D_\infty\}$. To complete the proof, we require the following facts concerning automorphisms of $\mathbb{Z}[T_Q]$:

- If $\bar{T}_Q = A_\infty, B_\infty, C_\infty$, then $\mathrm{Aut}(\mathbb{Z}[T_Q]) = \langle \tau \rangle$.
- If $\{1\} \neq \Pi \subset \mathrm{Aut}(\mathbb{Z}[D_\infty])$ is admissible, then there exists $n \in \mathbb{N}$ with $\tau^n \in \Pi$.
- If $\bar{T}_Q \cong A_\infty^\infty$ and Q is regular, then $\Pi = \{1\}$, cf. [2].

Since the component Q is not periodic, the group Π does not contain a positive power of τ and is therefore trivial. □

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