THE THEOREM OF WEDDERBURN-MALCEV: $\mathbf{H^2(A,N)}$ AND EXTENSIONS

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Throughout, A denotes a finite dimensional algebra over a field k. We let $\operatorname{Rad}(A)$ be the Jacobson (nilpotent) radical of A. Wedderburn's classical result [7, Thm.17,Thm.22] tells us that the semi-simple factor algebra $A/\operatorname{Rad}(A)$ is a direct sum of matrix algebras

$$A/\operatorname{Rad}(A) \cong \bigoplus_{i=1}^{m} \operatorname{Mat}_{n_i}(\Delta_i),$$

where each Δ_i is a division algebra over k.

To understand the structure of A, two problems remain, namely,

- the determination of the structure of $\operatorname{Rad}(A)$, and
- the interaction of the constituents $A/\operatorname{Rad}(A)$ and $\operatorname{Rad}(A)$.

The former usually is a hopeless endeavor, the latter was answered by Wedderburn [7, Thm.24,Thm.28] and extended by Malcev [6].

Theorem. *The following statements hold:*

(1) *If the algebra* $A/\operatorname{Rad}(A)$ *is separable, then there exists a subalgebra* $S \subset A$ *such that* $A = S \oplus \operatorname{Rad}(A)$ (Wedderburn).

(2) *Let* $T \subset A$ *be any separable subalgebra of* A*, then there exists* $n \in \operatorname{Rad}(A)$ *such that* $(1+n)T(1+n)^{-1} \subset S$. (Malcev)

Remarks. Part (1) of the foregoing result is often referred to as *Wedderburn's Principal Theorem*. In [7, p.109] the result is stated without assuming separability. In a later paper [8], Wedderburn "amplifies" his earlier result by proving it for commutative algebras in case $A/\operatorname{Rad}(A)$ is separable. Referring to [7] he writes [8, p.854]: "Inseparable extensions were not considered in that paper." In [7] Wedderburn also states that the semi-simple summand is not unique, but determined up to isomorphism.

The entire result is now sometimes called the *Theorem of Wedderburn-Malcev*.

In these lectures we present Hochschild's cohomological proof (cf. [2]) (initially only for fields of $\operatorname{char}(k) = 0$ and extended to arbitrary fields in [3]) of this result, which appears to have been his motivation for the development of associative cohomology [3, 4]. The methods involved can be transferred mutatis mutandis to prove similar results for finite groups (Schur-Zassenhaus Theorem (cf. [1])) and Lie algebras (Theorems of Levi and Malcev-Harish-Chandra (cf. [5])). In fact, in [2] Hochschild transfers the Whitehead Lemmas for Lie algebras to the associative setting. In his introduction to [2] he thanks C. Chevalley (his advisor) for supplying the proofs of Whitehead's results, which were apparently not recorded in the literature.

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We recall the *bar resolution* $\mathcal{A}$ of the (A, A)-bimodule A. We put $\mathcal{A}_n := A^{\otimes(n+2)}$ for $n \geq -1$ and define differentials

$$d_n : \mathcal{A}_n \longrightarrow \mathcal{A}_{n-1} \;\; ; \;\; a_0 \otimes \cdots \otimes a_{n+1} \mapsto \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_{n+1}$$

for $n \geq 0$. Then $\mathcal{A} := (\mathcal{A}_n, d_n)_{n \geq 0}$ is a projective resolution of the module A over the enveloping algebra $A^e := A \otimes_k A^{\mathrm{op}}$. Using this resolution, one obtains the familiar formular for the *Hochschild cohomology groups*

$$H^n(A, M) := \mathrm{Ext}^n_{A^e}(A, M) \qquad \forall\; n \geq 0.$$

with coefficients in the A^e-module M. In this lecture we shall only require the case where $n = 2$.

A k-bilinear map $f : A \times A \longrightarrow M$ is a 2-*cocycle* if

$$a.f(b, c) - f(ab, c) + f(a, bc) - f(a, b).c = 0 \qquad \forall\; a, b, c \in A.$$

If there exists a k-linear map $g : A \longrightarrow A$ with

$$f(a, b) = a.g(b) - g(ab) + g(a).b \qquad \forall\; a, b \in A,$$

then f is a 2-*coboundary.* From the bar resolution we obtain that

$$H^2(A, M) = Z^2(A, M)/B^2(A, M)$$

is the factor space of 2-cocycles by 2-coboundaries.

We shall use the cohomology groups $H^2(A, M)$ to describe extensions

$$(0) \longrightarrow M \longrightarrow E \longrightarrow A \longrightarrow (0)$$

of A by kernels M satisfying $M^2 = (0)$. In this case the ideal structure of $M \subset E$ is determined by its induced structure of an A^e-module.

As a vector space, $E = A \oplus M$, and the multiplication is given via

$$(a, m) \cdot (b, n) := (ab, a.n + m.b + f(a, b)) \qquad \forall\; a, b \in A,\;\; m, n \in M.$$

The associativity of E implies that f is a 2-cocycle. Conversely, every 2-cocycle f defines a square zero extension

$$E_f : \qquad (0) \longrightarrow M \longrightarrow A \ltimes_f M \longrightarrow A \longrightarrow (0)$$

of A by M.

Lemma 1. (1) *If f is a coboundary, then the extension E_f of algebras splits.*

(2) *Let A be an algebra such that $H^2(A/\mathrm{Rad}(A), M) = (0)$ for every $A/\mathrm{Rad}(A)$-bimodule M. Then there exists a subalgebra $S \subset A$ with $A = S \oplus \mathrm{Rad}(A)$.*

Proof. (1) Suppose that f is a coboundary. Then there exists a k-linear map $g : A \longrightarrow A$ such that

$$f(a, b) = a.g(b) - g(ab) + g(a).b.$$

Consider the map

$$\varphi : A \longrightarrow A \ltimes_f M \;\; ; \;\; a \mapsto (a, -g(a)).$$

Then φ is a k-linear splitting of the sequence E_f such that

$$\varphi(ab) = (ab, -g(ab)) = (ab, -a.g(b) + -g(a).b + f(a, b)) = (a, -g(a)) \cdot (b, -g(b)).$$

(2) Since $\mathrm{Rad}(A)$ is nilpotent, there exists a minimal $n \in \mathbb{N}$ with $\mathrm{Rad}(A)^n = (0)$. We proceed by induction on n, the case $n = 1$ being trivial.

Let $n \geq 2$, and consider the algebra $A' := A/\mathrm{Rad}(A)^{n-1}$. Then $\mathrm{Rad}(A')^{n-1} = (0)$ and $A'/\mathrm{Rad}(A') \cong A/\mathrm{Rad}(A)$, so that the inductive hypothesis provides a subalgebra $S' \subset A'$ with

$$A' = S' \oplus \mathrm{Rad}(A').$$

Let $\pi : A \longrightarrow A'$ be the canonical projection. Then $M := \ker \pi$ is an ideal of $S'' := \pi^{-1}(S')$ such that $M^2 = (0)$. Thus, M is an S'^e-module and $S' \cong A'/\operatorname{Rad}(A') \cong A/\operatorname{Rad}(A)$. Consequently,

$$H^2(S', M) \cong H^2(A/\operatorname{Rad}(A), M) = (0),$$

and part (1) provides a subalgebra $S \subset S''$ with $S'' = S \oplus M$. This readily implies $A = S \oplus \operatorname{Rad}(A)$. □

Remarks. The second part of the Lemma can be found in [2, p.688] for $\operatorname{char}(k) = 0$ and in general in [3]. On page 687 of [2] Hochschild notes: "J.H.C. Whitehead has communicated a proof of this result to N. Jacobson, which utilizes the same ideas as the proof given here."

It remains to see when the vanishing condition of Lemma 1(2) obtains. Algebras R with $H^2(R, M) = (0)$ for every R^e-module M can be characterized by a lifting property for homomorphisms which, in the context of finitely generated commutative algebras, amounts to smoothness.

Definition. The k-algebra R is *smooth* if for every algebra S, every ideal $I \lhd S$ with $I^2 = (0)$, and every homomorhism $\varphi : R \longrightarrow S/I$ of k-algebras, there exists a homomorphism $\psi : R \longrightarrow S$ such that $\varphi = \pi \circ \psi$. (Here $\pi : S \longrightarrow S/I$ denotes the canonical projection.)

The vanishing condition certainly holds if the enveloping algebra of A is semi-simple.

Definition. The k-algebra A is *separable* if for every field extension $K : k$, the algebra $A \otimes_k K$ is semi-simple.

Lemma 2. *If A is separable, then A^e is semi-simple.*

Proof. Let K be an algebraic closure of k. By Wedderburn's Theorem we have

$$A^{\mathrm{op}} \otimes_k K \cong \bigoplus_{i=1}^{m} \operatorname{Mat}_{n_i}(K),$$

whence

$$A^e \otimes_k K \cong \bigoplus_{i=1}^{m} A \otimes_k \operatorname{Mat}_{n_i}(K) \cong \bigoplus_{i=1}^{m} \operatorname{Mat}_{n_i}(A \otimes_k K).$$

As $A \otimes_k K$ is semisimple, each matrix ring $\operatorname{Mat}_{n_i}(A \otimes_k K)$ also enjoys this property. Consequently, $A^e \otimes_k K$ is semisimple, thereby implying the desired result. □

Examples. (1) Suppose that $\operatorname{char}(k) = p > 0$ and let $E = k(\alpha)$ be a purely inseparable extension of k of exponent one. Then $a := \alpha^p \in k$ while $\alpha \neq k$. Hence $n := \alpha \otimes 1 - 1 \otimes \alpha$ is a nonzero element of $E \otimes_k E$, while

$$n^p = \alpha^p \otimes 1 - 1 \otimes \alpha^p = a(1 \otimes 1 - 1 \otimes 1) = 0.$$

As a result, the k-algebra E is not separable.

(2) If $E = k(\alpha)$ is an extension field of k with minimal polynomial $f_\alpha \in k[X]$, then E is a separable k-algebra if and only if f_α is separable, that is f_α has no multiple roots in its splitting field. This follows directly from the Chinese Remainder Theorem.

(3) Let G be a finite group. According to Maschke's Theorem, the group algebra kG is semi-simple if and only if $\operatorname{char}(k)$ does not divide the order of G. Since the latter property is invariant under base field extensions, every semi-simple group algebra is separable.

References

[1] L. Evens. *The Cohomology of Groups*. Oxford Mathematical Monographs. Oxford University Press, 1991

[2] G. Hochschild. *Semi-simple algebras and generalized derivations*. Amer. J. Math. **64** (1941), 677-694

[3] ———. *On the cohomology groups of an associative algebra*. Ann. of Math. **46** (1945), 58-67

[4] ———. *On the cohomology theory for associative algebras*. Ann. of Math. **47** (1946), 568-579

[5] N. Jacobson. *Lie Algebras*. Dover Publications, 1979

[6] A. Malcev. *On the representation of an algebra as a direct sum of its radical and a semi-simple subalgebra*. Dokl. Akad. Nauk **36** (1942), 42-46

[7] J.H.M. Wedderburn. *On hypercomplex numbers*. Proc. London Math. Soc. **6** (1908), 77-118

[8] ———. *Notes on algebras*. Ann. of Math. **38** (1937), 854-856