Selected topics in representation theory

– Modules with standard filtration II –

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Let A be an Artin algebra, and denote the category of finitely generated left A-modules by $\operatorname{mod} A$.

1 Reminder

Notation. Let $\Theta = \{\Theta(1), \ldots, \Theta(n)\}$ be a sequence of A-modules with $\operatorname{Ext}^1_A(\Theta(j), \Theta(i)) = 0$ for all $j \geq i$. Denote by $\mathcal{F}(\Theta)$ the full subcategory of $\operatorname{mod} A$ of modules with filtration factors in Θ.

Theorem (Ringel). *The subcategory $\mathcal{F}(\Theta)$ is functorially finite in* $\operatorname{mod} A$*.*

Note that $\mathcal{F}(\Theta)$ is generally *not* closed under direct summands.

Let $\mathcal{X}$ be a full subcategory of $\operatorname{mod} A$, and denote by $\mathcal{Y}$ the full subcategory of $\operatorname{mod} A$ of all modules Y with $\operatorname{Ext}^1_A(X, Y) = 0$ for all $X \in \mathcal{X}$.

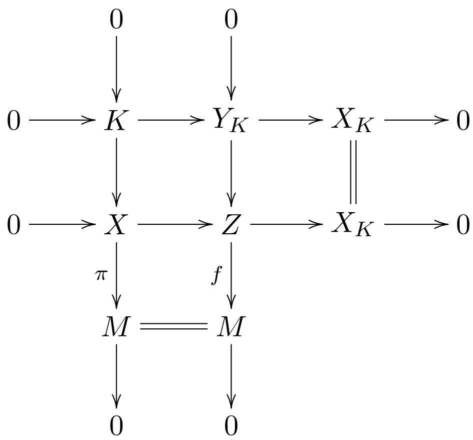
Lemma. *Let $0 \to Y \to X \xrightarrow{f} M \to 0$ with $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ be exact. Then f is a right $\mathcal{X}$-approximation of M.*

Lemma. *Suppose that $\mathcal{X}$ is closed under extensions and for every $N \in \operatorname{mod} A$ there is an exact sequence $0 \to N \to Y_N \to X_N \to 0$ with $Y_N \in \mathcal{Y}$ and $X_N \in \mathcal{X}$. Then every module $M \in \operatorname{mod} A$ has a right $\mathcal{X}$-approximation.*

The main step in the proof and the construction of the right $\mathcal{X}$-approximation can be seen already in the following case:

Let $M \in \operatorname{mod} A$. **There is an epimorphism** $\pi : X \to M$ **with** $X \in \mathcal{X}$**.**

Let $K = \ker \pi$. We get a commutative diagram with exact rows and columns (taking the push out sequences):



Then f is a right $\mathcal{X}$-approximation of M.

Let now $\Theta = \{\Theta(1), \ldots \Theta(n)\}$ be a sequence of A-modules as above, $\mathcal{X} = \mathcal{F}(\Theta)$, and $\mathcal{Y} = \mathcal{Y}(\Theta) = \{Y \in \operatorname{mod} A \mid \operatorname{Ext}^1_A(X, Y) = 0 \ \forall X \in \mathcal{F}(\Theta)\} = \{Y \in \operatorname{mod} A \mid \operatorname{Ext}^1_A(\Theta(i), Y) = 0 \ \forall i = 1, \ldots, n\}$.

Lemma. *Let* $t \in \{1, \ldots, n\}$, *and* $N \in \operatorname{mod} A$ *such that* $\operatorname{Ext}^1_A(\Theta(j), N) = 0$ *for all* $j > t$. *Then there is an exact sequence* $0 \to N \to N' \to Q \to 0$ *with* $Q = \Theta(t)^{r_N}$ *and* $\operatorname{Ext}^1_A(\Theta(j), N') = 0$ *for all* $j \geq t$.

The proof of this Lemma uses universal extensions and a little homological algebra.

The sequence $0 \to N \to N' \to Q \to 0$ is given by the universal extension of N by copies of $\Theta(t)$:

Let $[\varepsilon_1], \ldots, [\varepsilon_{r_N}]$ generate $\operatorname{Ext}^1_A(\Theta(t), N)$ as a left $\operatorname{End}_A(\Theta(t))$-module. Then the universal extension is given by the exact sequence ε such that the s-th inclusion of $\Theta(t)$ into $Q = \Theta(t)^{r_N}$ induces the sequence ε_s for $s = 1, \ldots, r_N$.

Lemma. *Let* $t \in \{1, \ldots, n\}$, *and* $N \in \operatorname{mod} A$ *such that* $\operatorname{Ext}^1_A(\Theta(j), N) = 0$ *for all* $j > t$. *Then there exists an exact sequence* $0 \to N \to Y \to X \to 0$ *with* $X \in \mathcal{F}(\{\Theta(1), \ldots, \Theta(t)\})$ *and* $Y \in \mathcal{Y}(\Theta)$. [1]

For the proof of this Lemma, use the previous Lemma and reverse induction:

$$N = N_{t+1} \xrightarrow{\mu_t} N_t \xrightarrow{\mu_{t-1}} \cdots \xrightarrow{\mu_1} N_1 = Y,$$

w. l. o. g. all μ_i are inclusions for $i = 1, \ldots, t$, and set $X = \operatorname{cok} \mu_t \circ \cdots \circ \mu_1$.

Now the theorem follows if we just take the dual constructions to get the left $\mathcal{F}(\Theta)$-approximations.

Here is an example of a subcategory filtered by some finite set of modules Θ which is *not* functorially finite in $\operatorname{mod} A$.

Example. *Let* $A = kQ$ *with* $Q : \circ \rightrightarrows \circ$.

Any indecomposable module M *of length* 2 *has self extensions. If* $\Theta = \{M\}$, *then* $\mathcal{F}(\Theta)$ *is neither covariantly (left approximations) nor contravariantly (right approximations) finite. (Every indecomposable* M_n *of dimension* $2n$ *from the same tube as* M *is contained in* $\mathcal{F}(\Theta)$. *Take the preprojective (resp. the preinjective) modules. They do not have left (resp. right)* $\mathcal{F}(\Theta)$*-approximations.)*

2 Δ- and ∇-filtered modules

Let $\mathcal{S} = \{S_1, \ldots, S_n\}$ denote the set of isomorphism classes of simple A-modules, $\mathcal{P} = \{P_1, \ldots, P_n\}$ the set of isomorphism classes of the corresponding projective A-modules, and $\mathcal{I} = \{I_1, \ldots, I_n\}$ the set of isomorphism classes of the corresponding injective A-modules.

Let U_i be the sum of all images of maps $P_j \to P_i$ with $j > i$, and $\Delta_i = P_i/U_i$.

Let ∇_i be the intersection of all kernels of maps $I_i \to I_j$ with $j > i$.

We have

$$\operatorname{Ext}^1_A(\Delta_j, \Delta_i) = 0 \;\forall j \geq i$$

and

$$\operatorname{Ext}^1_A(\nabla_j, \nabla_j) = 0 \;\forall j \leq i.$$

Thus, we can apply the Theorem to both $\Delta = \{\Delta_1, \ldots, \Delta_n\}$ and $\nabla = \{\nabla_n, \ldots, \nabla_1\}$.

Here is a different description of $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$.

[1] Note the the special case $t = n$ gives us the "required" sequences, therefore the right $\mathcal{F}(\Theta)$-approximations for any module $M \in \operatorname{mod} A$.

Let J_i be the sum of the images of all maps $P_j \to_A A$ with $j \geq i$. Then we get a decreasing sequence of ideals

$$A = J_1 \supsetneq \ldots \supsetneq J_n \supsetneq J_{n+1} = 0.$$

We have

$$M \in \mathcal{F}(\Delta) \Leftrightarrow J_i M / J_{i+1} M \text{ is projective as an } A/J_{i+1}\text{-module } \forall i = 1, \ldots, n,$$

and

$$M \in \mathcal{F}(\nabla) \Leftrightarrow J_i M / J_{i+1} M \text{ is injective as an } A/J_{i+1}\text{-module } \forall i = 1, \ldots, n.$$

Hence, both $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ are closed under direct summands.

Using the main theorem by Auslander and Smalø from [1], we get

Theorem. *The subcategories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ have (relative) almost split sequences.*

References

[1] M. Auslander, S. O. Smalø: *Almost split sequences in subcategories.* J. Algebra **69** (1981), no. 2, 426–454.

[2] C. M. Ringel: *The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences.* Math. Z. **208** (1991), no. 2, 209–223.