Algebraic aspects of stability

(based on talks by Markus Reineke at the ICRA 12 conference in Torun, August 2007)

31st October 2007

1 Stable and semistable representations

Let Q be a finite quiver with set of vertices I, and let $\theta : \mathbb{Z}I \to \mathbb{Z}$ be a linear function, called *stability*. We also define dim on $\mathbb{Z}I$ by $\dim d = \sum_{i \in I} d_i$.

Definition 1.1. .

1. *For a non-zero dimension vector $d \in \mathbb{N}I$, we define its* slope *by*

$$\mu(d) = \frac{\theta(d)}{\dim d} \in \mathbb{Q}.$$

We define the slope of a non-zero representation X of Q over some field k as the slope of its dimension vector, thus $\mu(X) = \mu(\dim X) \in \mathbb{Q}$.

2. *We call the representation X* semistable *if $\mu(U) \leq \mu(X)$ for all non-zero subrepresentations U of X, and we call X* stable *if $\mu(U) < \mu(X)$ for all non-zero proper subrepresentations U of X.*

Lemma 1.1. *Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence of non-zero representations of Q. Then the following holds:*

(1) $\mu(X) \leq \mu(Y)$ if and only if $\mu(X) \leq \mu(Z)$ if and only if $\mu(Y) \leq \mu(Z)$.

(2) $\mu(X) < \mu(Y)$ if and only if $\mu(X) < \mu(Z)$ if and only if $\mu(Y) < \mu(Z)$.

(3) $min\{\mu(X), \mu(Z)\} \leq \mu(Y) \leq max\{\mu(X), \mu(Z)\}$.

Proof. Let d and e be the dimension vectors of X and Z, respectively. Then the dimension vector of Y equals $d + e$, and thus the slope of Y equals

$$\mu(Y) = \frac{\theta(d) + \theta(e)}{\dim d + \dim e}.$$

It is now easy verify that

$$\frac{\theta(d)}{\dim d} \leq \frac{\theta(d)+\theta(e)}{\dim d + \dim e} \Leftrightarrow \frac{\theta(d)}{\dim d} \leq \frac{\theta(e)}{+\dim e} \Leftrightarrow \frac{\theta(d)+\theta(e)}{\dim d + \dim e} \leq \frac{\theta(e)}{\dim e},$$

and

$$\frac{\theta(d)}{\dim d} < \frac{\theta(d)+\theta(e)}{\dim d + \dim e} \Leftrightarrow \frac{\theta(d)}{\dim d} < \frac{\theta(e)}{+\dim e} \Leftrightarrow \frac{\theta(d)+\theta(e)}{\dim d + \dim e} < \frac{\theta(e)}{\dim e}$$

hold. The third part then follows immediately. □

Remark. This lemma shows that semistability of a representation X can also be characterised by the condition $\mu(X) \leq \mu(W)$ for any non-zero factor representation W of X.

Denote by $\mathrm{mod}^{\mu}\, kQ$ the full subcategory of $\mathrm{mod}\, kQ$ consisting of semistable representations of slope $\mu \in \mathbb{Q}$. Then we have the following important theorem:

Theorem 1.2. .

(1) Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence of non-zero representations of Q of the same slope μ. Then Y is semistable if and only if X and Z are semistable.

(2) $\mathrm{mod}^{\mu}\, kQ$ is an abelian subcategory of $\mathrm{mod}\, kQ$.

(3) If $\mu > \nu$, then $\mathrm{Hom}(\mathrm{mod}^{\mu}\, kQ, \mathrm{mod}^{\nu}\, kQ) = 0$.

(4) The stable representations of slope μ are precisely the simple objects in the abelian category $\mathrm{mod}^{\mu}\, kQ$. In particular, they are indecomposable, their endomorphism ring is a skew field (or k in case k is algebraically closed), and there are no non-zero morphisms between non-isomorphic stable representations of the same slope.

Proof. Suppose that X and Z are semistable, and let U be a subrepresentation of Y. This yields an induced exact sequence

$$0 \to U \cap X \to U \to (U+X)/X \to 0$$

of subrepresentations of X, Y and Z, respectively. By semistability of X and Z, we have $\mu(U \cap X) \leq \mu(X) = \mu$ and $\mu(U+X)/X) \leq \mu(Z) = \mu$. Applying the third part of the previous lemma, we get $\mu(U) \leq max\{\mu(U \cap X), \mu((U+X)/X))\} \leq \mu = \mu(Y)$, proving semistability of Y.

Conversely, suppose that Y is semistable. A subrepresentation U of X can then be viewed as a subrepresentation of Y, and thus $\mu(U) \leq \mu(Y) = \mu = \mu(X)$, proving semistability of X. A subrepresentation U of Z induces an exact sequence

$$0 \to X \to V \to U \to 0$$

by pullback, and thus $\mu(V) \leq \mu(Y) = \mu = \mu(X)$. Applying the first part of the previous lemma, we get $\mu(U) \leq \mu(V) \leq \mu = \mu(Z)$, proving semistability of Z. This proves the first part. It also proves that the subcategory $\operatorname{mod}^{\mu} kQ$ is closed under extensions.

Given a morphism $f : X \to Y$ in $\operatorname{mod}^{\mu} kQ$, we have $\mu = \mu(X) \leq \mu(Im(f)) \leq \mu(Y) = \mu$ by semistabililty of X and Y, and thus $\mu(Im(f)) = \mu$. Thus, $\operatorname{Ker}(f)$, $Im(f)$ and $Coker(f)$ all have the same slope μ, and they are all semistable by the first part. This proves that the category $\operatorname{mod}^{\mu} kQ$ is abelian.

The same argument proves the third part: If $f : X \to Y$ is a non-zero morphism, then $\mu(X) \leq \mu(Im(f)) \leq \mu(Y)$.

By the definition of stability, a representation is stable of slope μ if and only if it has no non-zero proper subrepresentation in $\operatorname{mod}^{\mu} kQ$, proving that the stables of slope μ are the simples in $\operatorname{mod}^{\mu} kQ$. The remaining statements of the fourth part follow from Schur's Lemma. □

2 Strongly contradicting semistability

Definition 2.1. *A subrepresentation U of a representation X is called* strongly contradicting semistable *(or just* scss*) if its slope is maximal among the slopes of subrepresentations of X, that is, $\mu(U) = max\{\mu(V) \,|\, V \subset X\}$, and it is of maximal dimension with this property.*

Such a subrepresentation clearly exists, since there are only finitely many dimensions and slopes of subrepresentations. By its defining property, it is clearly semistable.

Proposition 2.1. *Any representation X admits a unique scss subrepresentation.*

Proof. Suppose U and V are scss subrepresentations of X, neccessarily of the same slope μ. The exact sequence $0 \to U \cap V \to U \oplus V \to U + V \to 0$ yields $\mu(U \cap V) \leq \mu = \mu(U \oplus V)$, thus $\mu \leq \mu(U + V)$ by the first lemma 1.1. By maximality of the slope μ among subrepresentations of X, we have $\mu(U+V) = \mu$. By maximality of the dimension of U and V, we have $\dim(U + V) \leq \dim U$ and $\dim(U + V) \leq \dim V$. So $U = V$. □

Remark. The uniqueness of the scss of a representation X has some interesting applications: for example, the scss has to be fixed under arbitrary automorphisms ρ of X, since applying ρ to a subrepresentations does not change its dimension vector, and thus also its slope and dimension.

References

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