Torsion pairs induced from Harder-Narasimhan filtration

(based on talks by Markus Reineke at the ICRA 12 conference in Torun, August 2007)

21 November 2007

In this lecture we want to present how to define torsion pairs from the Harder-Narasimhan filtration of a representation of a quiver. Therefore we start with a quick revision of torsion theory.

1 Torsion theory

Let A be a finite-dimensional, basic, connected algebra over a fixed algebraically closed field k. Denote by $\operatorname{mod} A$ the category of all finite-dimensional left A-modules.

A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of a module category is called a *torsion pair* (or *torsion theory*) if the following conditions are satisfied:

(i) $\operatorname{Hom}(M, N) = 0$ for all $M \in \mathcal{T}$, $N \in \mathcal{F}$.

(ii) $\operatorname{Hom}(M, -)|_{\mathcal{F}} = 0$ implies $M \in \mathcal{T}$.

(iii) $\operatorname{Hom}(-, N)|_{\mathcal{T}} = 0$ implies $N \in \mathcal{F}$.

That is, there is no non-zero homomorphism from an object in $\mathcal{T}$ to an object in $\mathcal{F}$ and the two subcategories are maximal with respect to this property. $\mathcal{T}$ is called the *torsion class*, $\mathcal{F}$ the *torsion-free class*.

Each torsion pair induces an idempotent radical, called *torsion radical*, and conversely: $\mathcal{T}$ is a torsion class of some $(\mathcal{T}, \mathcal{F})$ if and only if there exists an idempotent radical t such that $\mathcal{T} = \{M \mid tM = M\}$. So for $M \in \operatorname{Mod}-A$, $tM \in \mathcal{T}$ and $M/tM \in \mathcal{F}$. Also there is always the canonical short exact sequence $0 \to tM \to M \to M/tM \to 0$.

A torsion pair $(\mathcal{T}, \mathcal{F})$ is called *splitting* if each indecomposable module M either lies in $\mathcal{T}$ or in $\mathcal{F}$. Then the canonical sequence above splits. One can also show:

Proposition 1.1. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\operatorname{mod} A$. Then $(\mathcal{T}, \mathcal{F})$ is splitting if and only if $\operatorname{Ext}^1_A(M, N) = 0$ for all $M \in \mathcal{T}, N \in \mathcal{F}$.*

Of course, not every torsion pair is splitting.

2 Harder-Narasimhan filtration

Let Q to be a finite quiver with set of vertices I, and let $\theta : \mathbb{Z}I \to \mathbb{Z}$ be a linear function, called *stability*. We also define dim on $\mathbb{Z}I$ by $\dim d = \sum_{i \in I} d_i$. For a non-zero dimension vector $d \in \mathbb{N}I$, we define its *slope* by $\mu(d) = \frac{\theta(d)}{\dim d} \in \mathbb{Q}$. We define the slope of a non-zero representation X of Q (over some field) as the slope of its dimension vector, thus $\mu(X) = \mu(\dim X) \in \mathbb{Q}$.

We call the representation X *semistable* if $\mu(U) \leq \mu(X)$ for all non-zero subrepresentations U of X, and we call X *stable* if $\mu(U) < \mu(X)$ for all non-zero proper subrepresentations U of X.

Definition 2.1. *A filtration* $0 = X_0 \subset X_1 \subset \ldots \subset X_s = X$ *of a representation* X *is called* Harder-Narasimhan *(abbreviated by* HN*) if the subquotients* X_i/X_{i-1} *are semistable for* $i = 1, \ldots, s$ *and* $\mu(X_1/X_0) > \mu(X_2/X_1) > \ldots > \mu(X_s/X_{s-1})$.

It was shown in a previous lecture that any non-zero representation X possesses a unique Harder-Narasimhan filtration, which was done with the help of the following concept:

Definition 2.2. *A subrepresentation* U *of a representation* X *is called* strongly contradicting semistable *(or just* scss*) if its slope is maximal among the slopes of subrepresentations of* X*, that is,* $\mu(U) = max\{\mu(V) \,|\, V \subset X\}$*, and it is of maximal dimension with this property.*

3 Functorial properties of the HN-filtration

The Harder-Narasimhan filtration can be interpreted functorially. Introduce for a given slope μ and each representation X a family of representations $\{X^{(a)}\}$, for $a \in \mathbb{Q}$ from the Harder-Narasimhan filtration as follows: Define

$$X^{(a)} = X_k \text{ if } \mu(X_k/X_{k-1}) \geq a > \mu(X_{k+1}/X_k),$$

$$X^{(a)} = X, \text{ if } a \leq \mu(X_i/X_{i-1}),\ i = 1, \ldots, s,$$
$$X^{(a)} = 0, \text{ if } a > \mu(X_i/X_{i-1}),\ i = 1, \ldots, s.$$

Recall the following results on maps between semistable representations: Let X, Y be semistable and let $f : X \to Y$ a non-zero homomorphism. Then $\mu(X) \leq \mu(Y)$. Also, each homomorphism $f : X \to Y$ with $\mu(X) > \mu(Y)$ is zero.

Lemma 3.1. *Any morphism* $f : X \to Y$ *respects the HN-filtration, in the sense that* $f(X^{(a)}) \subset Y^{(a)}$ *for all* $a \in \mathbb{Q}$.

Proof. First, we will prove the following property by induction on k:

If $f(X_k) \subset Y_l \backslash Y_{l-1}$, then $\mu(Y_l/Y_{l-1}) \geq \mu(X_k/X_{k-1})$.

The claim in the lemma follows from this: given $a \in \mathbb{Q}$, we have $X^{(a)} = X_k$ for the index k satisfying $\mu(X_k/X_{k-1}) \geq a > \mu(X_{k+1}/X_k)$ (by definition). Choosing

l minimal such that $f(X_k) \subset Y_l$, we then have $\mu(Y_l/Y_{l-1}) \geq \mu(X_k/X_{k-1}) \geq a$, and thus $Y_l \subset Y^{(a)}$ by definition.

In case $k = 0$ there is nothing to show. For $k = 1$, suppose $f(X_1) \subset Y_l \backslash Y_{l-1}$. Then f induces a non-zero map between the semistable representations X_1 and Y_l/Y_{l-1}, showing $\mu(X_1) \leq \mu(Y_l/Y_{l-1})$ as claimed. For general k, suppose that $f(X_k) \subset Y_l \backslash Y_{l-1}$, and consider the short exact sequences

$$0 \to X_{k-1} \xrightarrow{\alpha} X_k \to X_k/X_{k-1} \to 0$$

$$0 \to Y_{l-1} \to Y_l \xrightarrow{\beta} Y_l/Y_{l-1} \to 0$$

together with the map $f : X_k \to Y_l$.

If the composition $\beta f \alpha$ equals 0, the map f induces a non-zero map $X_k/X_{k-1} \to Y_l/Y_{l-1}$ between semistable representations, and thus $\mu(X_k/X_{k-1}) \leq \mu(Y_l/Y_{l-1})$ as desired.

If $\beta f \alpha$ is non-zero, we have $f(X_{k-1}) \subset Y_l \backslash Y_{l-1}$, and we can conclude by induction that $\mu(Y_l/Y_{l-1}) \geq \mu(X_{k-1}/X_{k-2}) > \mu(X_k/X_{k-1})$, which gives what we wanted. □

4 Torsion pairs from HN-filtration

Let us call the slopes $\mu(X_1/X_0), \ldots, \mu(X_s/X_{s-1})$ in the unique Harder-Narasimhan filtration of X the *weights* of X.

Definition 4.1. *Given $a \in \mathbb{Q}$, define $\mathcal{T}_a$ as the class of all representations X all of whose weights are $\geq a$, and define $\mathcal{F}_a$ as the class of all representations X all of whose weights are $< a$.*

Lemma 4.1. *For each $a \in \mathbb{Q}$, the pair $(\mathcal{T}_a, \mathcal{F}_a)$ defines a torsion pair in* $\operatorname{mod} kQ$. *For $a < b$, we have $\mathcal{T}_a \supseteq \mathcal{T}_b$ and $\mathcal{F}_a \subseteq \mathcal{F}_b$.*

Proof. Assume $X \in \mathcal{T}_a$ and $Y \in \mathcal{F}_a$. In the $\mathbb{Q}$-indexed Harder-Narisimhan filtration, we thus have $X^{(b)} = X$ for all $a \leq b$, and $Y^{(b)} = 0$ for all $a < b$. But any morphism $f : X \to Y$ is already zero, since the slope of X is greater than the slope of Y, proving $\operatorname{Hom}(\mathcal{T}_a, \mathcal{F}_a) = 0$.

Now assume $\operatorname{Hom}(X, \mathcal{F}_a) = 0$ for some representation X. Suppose X has a weight strictly less than a, then certainly the slope of the (semistable) top factor in the Harder-Narasimhan filtration, X/X_{s-1} is strictly less than a, too, thus it belongs to $\mathcal{F}_a$. But the projection map $X \to X/X_{s-1}$ is non-zero, a contradiction. Thus, X belongs to $\mathcal{T}_a$.

Finally, assume $\operatorname{Hom}(\mathcal{T}_a, Y) = 0$ for some representation Y. If Y has a weight $\geq a$, then certainly the slope of its (semistable) scss subrepresentation Y_1 is $\geq a$. Thus Y_1 belongs to $\mathcal{T}_a$. But the inclusion $Y_1 \to Y$ is non-zero, a contradiction. Thus, Y belongs to $\mathcal{F}_a$.

The inclusion properties of the various torsion and free classes follows from the definitions. □

References

[HN] Harder, Narasimhan: *On the cohomology groups of moduli spaces of vector bundles on curves*, Math. Ann. 212, 1975.

[H1] L. Hille, J.A. de la Peña: *Stable representations of quivers*, Journal of Pure and Applied Algebra, 172, 2002.

[R1] M. Reineke: *The Harder-Narasimhan system in quantum groups and cohomology of quiver moduli*, Inventiones math. 152, 2003.

[R2] M. Reineke: *Algebraic aspects of stability*, talks by Markus Reineke at the ICRA 12 conference in Torun, Poland, 2007.