QUASI-HEREDITARY ALGEBRAS: HOMOLOGICAL PROPERTIES

ROLF FARNSTEINER

Throughout, we let A be a finite-dimensional algebra over an algebraically closed field k[1]. We fix a finite, partially ordered set $(I, \leq)$ with $(S(i))_{i\in I}$ being a complete set of representatives for the isomorphism classes of the simple A-modules. For each $i \in I$, let $P(i)$ and $I(i)$ be the projective cover and the injective hull of $S(i)$, respectively. We will be working in the category $\operatorname{mod} A$ of finite-dimensional A-modules. The Jordan-Hölder multiplicity of $S(i)$ in M will be denoted $[M : S(i)]$. As in [2], we consider the *standard modules* $(\Delta(i))_{i\in I}$, satisfying

(a) $\operatorname{Top}(\Delta(i)) \cong S(i)$ and $[\Delta(i) : S(i)] = 1$, as well as

(b) $[\Delta(i) : S(j)] = 0$ for $j \not\leq i$.

The full subcategory of $\operatorname{mod} A$ consisting of the Δ-*good modules* will be denoted $\mathcal{F}(\Delta)$. Thus, each object $M \in \mathcal{F}(\Delta)$ affords a filtration, whose factors are standard modules. We let $(M : \Delta(i))$ be the multiplicity of $\Delta(i)$ in M. As usual, Ω_A denotes the Heller operator of $\operatorname{mod} A$.

Definition. The algebra A is *quasi-hereditary* if

(a) each $P(i)$ is Δ-good, and

(b) $(P(i) : \Delta(i)) = 1$ and $(P(i) : \Delta(j)) = 0$ for $i \not\leq j$.

If, in addition, there exists a duality $D : \operatorname{mod} A \longrightarrow \operatorname{mod} A$ with $D(S(i)) \cong S(i)$, then A is called a *BGG-algebra*.

Recall that $\Omega_A(\Delta(i))$ is Δ-good with filtration factors belonging to $\{\Delta(\ell) \ ; \ \ell > i\}$, see [2, Lemma 1].

Proposition 1. *Let A be quasi-hereditary. Then A has finite global dimension.*

Proof. (1) Let $\operatorname{mod}_{\text{fin}} A$ be the full subcategory of $\operatorname{mod} A$ consisting of the modules of finite projective dimension. Given an exact sequence

$$(0) \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow (0),$$

one of its terms belongs to $\operatorname{mod}_{\text{fin}} A$ whenever the other two terms do. Moreover, $M \in \operatorname{mod}_{\text{fin}} A$ if and only if $\Omega^n_A(M) \in \operatorname{mod}_{\text{fin}} A$ for some $n \geq 0$.

Suppose there exists a standard module $\Delta(i)$ of infinite projective dimension, and let $i_0 \in I$ be maximal subject to this property. Since $\Omega_A(\Delta(i))$ is Δ-good with filtration factors of the form $(\Delta(\ell))_{\ell > i_0}$, the choice of i_0 implies that $\Omega_A(\Delta(i_0))$ has finite projective dimension. Hence $\Delta(i_0) \in \operatorname{mod}_{\text{fin}} A$, a contradiction. We conclude that all standard modules belong to $\operatorname{mod}_{\text{fin}} A$.

To show that $S(i) \in \operatorname{mod}_{\text{fin}} A$ for all $i \in I$, we assume that there exists a minimal element $i_1 \in I$ such that $S(i_1)$ has infinite projective dimension. Since all composition factors of $\operatorname{Rad}(\Delta(i_1))$ belong to $\{S(\ell) \ ; \ \ell < i_1\}$, we obtain $\operatorname{Rad}(\Delta(i_1)), \Delta(i_1) \in \operatorname{mod}_{\text{fin}} A$. Consequently, $S(i_1) \in \operatorname{mod}_{\text{fin}} A$, a contradiction. As a result, all simple modules have finite projective dimension, so that A has finite global dimension. □

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[1]The reader may consult [1] for algebras over arbitrary fields.

The definition of standard modules provides an upper triangular matrix $(a_{ij})_{i,j\in I} \in \mathrm{Mat}_n(\mathbb{Z})$ with $a_{ii} = 1$ such that the classes $[\Delta(j)]$ and $[S(i)]$ in the Grothendieck group $K_0(A)$ are related via

$$[\Delta(j)] = \sum_{i\in I} a_{ij}[S(i)] \qquad \forall\ j \in I.$$

As a result, $\{[\Delta(i)]\ ;\ i \in I\}$ is a basis of $K_0(A)$. Given $i \in I$, we consider the unique $\mathbb{Z}$-linear map $f_i : K_0(A) \longrightarrow \mathbb{Z}$ with $f_i([\Delta(j)]) = \delta_{ij}$ for $j \in I$. We may thus define

$$(M\!:\!\Delta(i)) := f_i([M])$$

for an arbitray $M \in \mathrm{mod}\, A$. For $M \in \mathcal{F}(\Delta)$, this number coincides with the filtration multiplicity defined before.

Theorem 2. *Let A be a quasi-hereditary algebra.*

(1) *We have*

$$(M\!:\!\Delta(i)) = \sum_{\ell\geq 0}(-1)^\ell \dim_k \mathrm{Ext}^\ell_A(M, \nabla(i))$$

for every $M \in \mathrm{mod}\, A$.

(2) *If A is a BGG-algebra, then*

$$(M\!:\!\Delta(i)) = \sum_{\ell\geq 0}(-1)^\ell \dim_k \mathrm{Ext}^\ell_A(\Delta(i), M)$$

for every $M \in \mathrm{mod}\, A$.

Proof. (1) By the Euler-Poincaré principle, the map

$$N \mapsto \sum_{\ell\geq 0}(-1)^\ell \dim_k \mathrm{Ext}^\ell_A(N, \nabla(i))$$

defines a $\mathbb{Z}$-linear map $g_i : K_0(A) \longrightarrow \mathbb{Z}$. Owing to [2, Lemma 2], we obtain

$$g_i([P(j)]) = \dim_k \mathrm{Hom}_A(P(j), \nabla(i)) = (P(j)\!:\!\Delta(i)) = f_i([P(j)]).$$

In view of Proposition 1, the set $\{[P(j)]\ ;\ j \in I\}$ is a basis of $K_0(A)$, so that $f_i = g_i$. This implies our assertion.

(2) As in (1), the map

$$N \mapsto \sum_{\ell\geq 0}(-1)^\ell \dim_k \mathrm{Ext}^\ell_A(\Delta(i), N)$$

defines a $\mathbb{Z}$-linear map $h_i : K_0(A) \longrightarrow \mathbb{Z}$. Since A is a BGG-algebra, there exists a duality $D : \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, A$ with $D(S(i)) \cong S(i)$. Consequently, D induces the identity map on $K_0(A)$. Oberserving $D(\Delta(i)) \cong \nabla(i)$ along with D being a duality, we obtain

$$\begin{aligned} h_i([M]) &= h_i([D(M)]) = \sum_{\ell\geq 0}(-1)^\ell \dim_k \mathrm{Ext}^1_A(\Delta(i), D(M)) = \sum_{\ell\geq 0}(-1)^\ell \dim_k \mathrm{Ext}^1_A(M, D(\Delta(i))) \\ &= g_i([M]) = f_i([M]) \end{aligned}$$

for every $M \in \mathrm{mod}\, A$. □

Since $M \mapsto \dim_k M$ also defines a homomorphism $K_0(A) \longrightarrow \mathbb{Z}$, we have

$$\dim_k M = \sum_{i\in I}(M\!:\!\Delta(i)) \dim_k \Delta(i)$$

for every $M \in \operatorname{mod} A$. In the motivating examples, one usually knows the dimensions or the characters of the standard modules $\Delta(i)$. Accordingly, the knowledge of the coefficients $(S(j):\Delta(i))$ provides the dimensions or the characters of the simple modules.

Given $i \leq j \in I$, we define the *distance* between i and j via

$$d(i,j) := \max\{n \in \mathbb{N}_0 \ ; \ \exists\, i = i_0 < i_1 < \cdots < i_n = j\}.$$

Lemma 3. *Let A be quasi-hereditary. Then the following statements hold:*

(1) $\operatorname{Hom}_A(\Delta(i), \Delta(j)) = (0)$ *for* $i \not\leq j$.
(2) *Let* $\ell > 0$ *and* $i \not< j$. *Then* $\operatorname{Ext}^{\ell}_A(\Delta(i), S(j)) = (0) = \operatorname{Ext}^{\ell}_A(\Delta(i), \Delta(j))$.
(3) *If* $i \leq j$ *and* $\ell > d(i,j)$, *then* $\operatorname{Ext}^{\ell}_A(\Delta(i), S(j)) = (0) = \operatorname{Ext}^{\ell}_A(\Delta(i), \Delta(j))$.

Proof. (1) If $i \not\leq j$, then left-exactness of $\operatorname{Hom}_A(-, \Delta(j))$ implies

$$0 = [\Delta(j):S(i)] = \dim_k \operatorname{Hom}_A(P(i), \Delta(j)) \geq \dim_k \operatorname{Hom}_A(\Delta(i), \Delta(j)).$$

(2) Suppose that $i \not\leq j$. General theory yields

$$\operatorname{Ext}^{\ell}_A(\Delta(i), S(j)) \cong \operatorname{Ext}^{\ell-1}_A(\Omega_A(\Delta(i)), S(j)) \qquad \forall\ \ell \geq 1.$$

Since $\Omega_A(\Delta(i))$ is Δ-good with filtration factors belonging to $\{\Delta(n) \ ; \ n > i\}$, the vanishing of $\operatorname{Ext}^{\ell}_A(\Delta(i), S(j))$ follows by induction on ℓ, with the case $\ell = 1$ being a consequence of (1).

As $\Delta(j)$ has composition factors belonging to $\{S(m) \ ; \ m \leq j\}$, we also obtain $\operatorname{Ext}^{\ell}_A(\Delta(i), \Delta(j)) = (0)$.

(3) We proceed by induction on $d(i,j)$. If $d(i,j) = 0$, then $i = j$ and the assertion follows from (2). Suppose that $d(i,j) > 0$ and $\ell > d(i,j)$. Given $M \in \{\Delta(j), S(j)\}$, we consider $q > i$. If $q \leq j$, then $d(q,j) < d(i,j)$, and the inductive hypothesis yields $\operatorname{Ext}^{\ell-1}_A(\Delta(q), M) = (0)$. Alternatively, $q \not\leq j$ and (2) gives the same result. Thanks to [2, Lemma 1], the module $\Omega_A(\Delta(i))$ is Δ-good with filtration factors belonging to $\{\Delta(q) \ ; \ q > i\}$. Our foregoing observations imply

$$\operatorname{Ext}^{\ell}_A(\Delta(i), M) \cong \operatorname{Ext}^{\ell-1}_A(\Omega_A(\Delta(i)), M) = (0),$$

as desired. □

We can express the above results in terms of certain polynomials.

Definition. Let A be a BGG-algebra. Given $i \leq j \in I$, we consider the polynomial

$$P_{i,j} := \sum_{n \geq 0} (-1)^{d(i,j)-n} \dim_k \operatorname{Ext}^n_A(\Delta(i), S(j)) X^{\frac{d(i,j)-n}{2}} \in \mathbb{Z}[X^{\frac{1}{2}}].$$

Corollary 4. *Let A be a BGG-algebra. Then the following statements hold:*

(1) $P_{i,i} = 1$.
(2) *If* $i < j$, *then* $\deg(P_{i,j}) \leq \frac{d(i,j)-1}{2}$.
(3) *If* $i \leq j$, *then* $P_{i,j}(1) = (-1)^{d(i,j)}(S(j):\Delta(i))$.

Proof. (1) In view of Lemma 3(2), we only have to compute $\dim_k \operatorname{Hom}_A(\Delta(i), \Delta(i))$. Since $\operatorname{Rad}(\Delta(i))$ has composition factors $S(j)$ with $j < i$, we have $\operatorname{Hom}_A(\Delta(i), \operatorname{Rad}(\Delta(i))) = (0)$, whence

$$1 \leq \dim_k \operatorname{Hom}_A(\Delta(i), \Delta(i)) \leq \dim_k \operatorname{Hom}_A(\Delta(i), S(i)) = 1.$$

(2) Suppose that $i < j$. Since $\mathrm{Hom}_A(\Delta(i), S(j)) = (0)$, part (3) of Lemma 3 gives

$$P_{i,j} = \sum_{n=1}^{d(i,j)} (-1)^{d(i,j)-n} \dim_k \mathrm{Ext}_A^n(\Delta(i), S(j)) X^{\frac{d(i,j)-n}{2}},$$

so that $\deg(P_{i,j}) \leq \frac{d(i,j)-1}{2}$.

(3) This follows from Theorem 2. □

Remarks. (1) For blocks of the category $\mathcal{O}$, Kazhdan and Lusztig [5] defined polynomials $\tilde{P}_{i,j}$ algorithmically and conjectured

$$\tilde{P}_{i,j}(1) = (-1)^{d(i,j)}(S(j):\Delta(i)).$$

Vogan showed that this conjecture is equivalent to $\tilde{P}_{i,j} = P_{i,j}$.

(2) In the category of G_rT-modules, values closely related to $P_{i,j}(1)$ occur as coefficients in a formula that expresses the character of a simple module in terms of the (well-known) characters of the baby Verma modules, see [4, (II.9.9)].

References

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