# Analytic ideas applied to triangulated categories, 1

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### Overview

- 1 t-structures: examples and formal definition
- 2 Ancient history
- 3 First application: a conjecture of Antieau, Gepner and Heller
- Something about the proof

# Example (the standard t-structure on $\mathbf{D}(A)$ )

Let A be an abelian category. We define two full subcategories of  $\mathbf{D}(A)$ :

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$$\mathbf{D}(A)^{\leq 0} = \{A^* \in \mathbf{D}(A) \mid H^i(A^*) = 0 \text{ for all } i > 0\}$$

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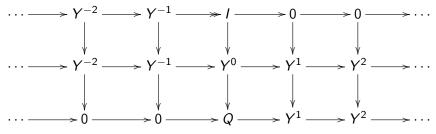
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with  $X \in \mathbf{D}(\mathcal{A})^{\leq 0}[1]$  and with  $Z \in \mathbf{D}(\mathcal{A})^{\geq 0}$ .

Put  $I = \operatorname{Im}(Y^{-1} \to Y^0)$ , and  $Q = Y^0/I$ .

For every  $Y \in \mathbf{D}(A)$  we have produced an exact triangle

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- For every object  $B \in \mathcal{T}$  there exists a triangle  $A \longrightarrow B \longrightarrow C \longrightarrow$  with  $A \in \mathcal{T}^{\leq 0}[1]$  and  $C \in \mathcal{T}^{\geq 0}$ .

Given an object  $B \in \mathcal{T}$ , the third property of a t-structure says that there **exists** an exact triangle

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This triangle is often written

$$B^{\leq -1} \longrightarrow B \longrightarrow B^{\geq 0} \longrightarrow B^{\leq -1}[1]$$

#### **Notation**

For  $n \in \mathbb{Z}$  we adopt the shorthand

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### Definition (Bounded t-Structures)

A t-structure  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  is called bounded if, for every object  $X \in \mathcal{T}$ , there exists an integer n > 0 with

$$X[n] \in \mathcal{T}^{\leq 0}$$

and 
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Resolving F by vector bundles, we may represent it as a complex

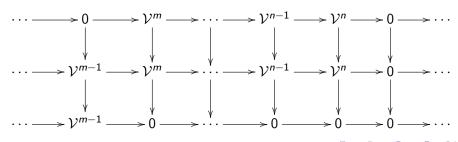
$$\cdots \longrightarrow \mathcal{V}^{m-1} \longrightarrow \mathcal{V}^m \longrightarrow \cdots \longrightarrow \mathcal{V}^{n-1} \longrightarrow \mathcal{V}^n \longrightarrow 0 \longrightarrow \cdots$$

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$$E \longrightarrow F \longrightarrow D[1],$$

with  $E \in \mathbf{D}^{\mathrm{perf}}(X)$  and  $D \in \mathbf{D}^{-}_{\operatorname{coh}}(X)^{\leq m}$ .





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For a proof that works in the relative context, that is given  $F \in \mathbf{D}^-_{\mathsf{coh},Z}(X)$  it produces a triangle

$$D \longrightarrow E \longrightarrow F \longrightarrow D[1],$$

with  $E \in \mathbf{D}_Z^{\mathrm{perf}}(X)$  and  $D \in \mathbf{D}_{\mathbf{coh},Z}^-(X)^{\leq m}$ , see

Tag 36.14 in the Stacks Project.

Let  $\mathcal M$  be a model category with homotopy category  $\mathcal T$ , and assume  $\mathcal T$  has a bounded t-structure. Antieau, Gepner and Heller proved:

- If the abelian category  $\mathcal{T}^{\heartsuit}$  is noetherian, then  $K_n(\mathcal{M}) = 0$  for n < 0.
- **2** Unconditionally we have  $K_{-1}(\mathcal{M}) = 0$ .

Benjamin Antieau, David Gepner, and Jeremiah Heller, *K-theoretic obstructions to bounded t-structures*, Invent. Math. **216** (2019), no. 1, 241–300.

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If  $\mathcal{A}$  is an abelian category, and  $\mathcal{T} = \mathbf{D}^b(\mathcal{A})$  with the usual model structure, the vanishing in negative K-theory is due to Schlichting.

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### Corollary

Let X be a finite-dimensional, noetherian scheme. Assume  $K_{-1}(X)$  is nonzero. Then the category  $\mathbf{D}^{\mathrm{perf}}(X)$  has no bounded t-structure.



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If  $K_n(X)$  is nonzero for  $n \le -2$ , then any bounded t-structure on  $\mathbf{D}^{\mathrm{perf}}(X)$  cannot have a noetherian heart.

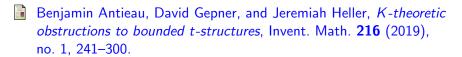


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#### This can be found as Corollary 1.4 in



### Conjecture

Let X be a finite-dimensional, noetherian scheme. The category  $\mathbf{D}^{\mathrm{perf}}(X)$  has a bounded t-structure if and only if X is regular, in which case  $\mathbf{D}^{\mathrm{perf}}(X) = \mathbf{D}^{b}_{\mathrm{coh}}(X)$ .

This can be found as Conjecture 1.5 in



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Let X be a scheme, and let  $Z \subset X$  be a closed subset. Let  $\mathbf{D}_Z^{\mathrm{perf}}(X)$  be the derived category, with objects the perfect complexes on X whose restriction to X-Z is acyclic.



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# For the proof see



Amnon Neeman, *Bounded t-structures on the category of perfect complexes*, https://arxiv.org/abs/2202.08861.

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R is a regular local ring if and only if R/m is of finite projective dimension, if and only if every module is of finite projective dimension.

It suffices to show that the inclusion  $\mathbf{D}_Z^{\mathrm{perf}}(X) \longrightarrow \mathbf{D}_{\mathsf{coh},Z}^b(X)$  is an equivalence.

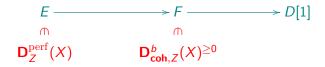
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Let  $\mathcal{T}$  be a triangulated category. Two t-structures  $(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0})$  and  $(\mathcal{T}_2^{\leq 0}, \mathcal{T}_2^{\geq 0})$  are declared equivalent if there exists an integer n>0 with

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We are given a bounded t-structure on  $\mathbf{D}_Z^{\mathrm{perf}}(X)$ , and we would like to compare it to the standard t-structure on  $\mathbf{D}_{\mathrm{coh},Z}^b(X)$ . For technical reasons this is easiest to do in  $\mathbf{D}_{\mathrm{qc},Z}(X)$ .





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# We appeal to Theorem A.1 in



Leovigildo Alonso Tarrío, Ana Jeremías López, and María José Souto Salorio, *Construction of t-structures and equivalences of derived categories*, Trans. Amer. Math. Soc. **355** (2003), no. 6, 2523–2543 (electronic).

Let  $\mathcal T$  be a triangulated category with coproducts, and let  $\mathcal A\subset \mathcal T$  be a set of compact objects satisfying  $\mathcal A[1]\subset \mathcal A$ .

Let  $\operatorname{Coprod}(\mathcal{A})$  be the smallest full subcategory of  $\mathcal{T}$ , containing  $\mathcal{A}$  and closed under coproducts and extensions.

Then  $\left(\operatorname{Coprod}(\mathcal{A}),\operatorname{Coprod}(\mathcal{A})[1]^{\perp}\right)$  is a t-structure on  $\mathcal{T}$ .

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Now suppose we are given a t-structure (A, B) on  $T^c$ ,

## Lemma

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Now we are assuming that we are given a bounded t-structure  $(\mathcal{A},\mathcal{B})$  on the category  $\mathbf{D}_Z^{\mathrm{perf}}(X)$ , which is the category of compact objects in  $\mathbf{D}_{\mathbf{gc},Z}(X)$ .

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Suppose we could prove the inclusions

$$\mathbf{D}_{\mathbf{qc},Z}(X)^{\leq -n} \subset \mathsf{Coprod}(\mathcal{A}) \subset \mathbf{D}_{\mathbf{qc},Z}(X)^{\leq n}$$

for some integer n.

Now we are assuming that we are given a bounded t-structure  $(\mathcal{A},\mathcal{B})$  on the category  $\mathbf{D}_Z^{\mathrm{perf}}(X)$ , which is the category of compact objects in  $\mathbf{D}_{\mathbf{qc},Z}(X)$ .

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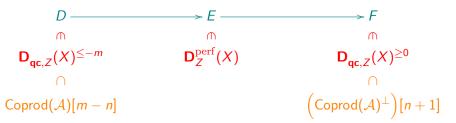
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We will sketch how to do half of this, that is prove the inclusion

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For simplicity we assume that X is projective and that Z = X.

Pick any object  $F \in \mathbf{D}_{\mathbf{qc}}(X)^{\leq 0}$ . Resolving it, we may produce an isomorph

$$\cdots \longrightarrow \mathcal{V}^{m-1} \longrightarrow \mathcal{V}^m \longrightarrow \cdots \longrightarrow \mathcal{V}^{-1} \longrightarrow \mathcal{V}^0 \longrightarrow 0 \longrightarrow \cdots$$

where each  $V^i$  is a coproduct of line bundles  $\mathcal{O}(-\ell)$  for  $\ell > 0$ .

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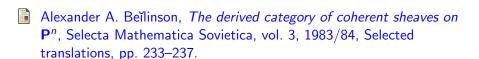
Hence, given any integer N > 0, we can find an integer M > 0 such that

$$\mathcal{O}(-\ell) \in \mathcal{A}[-M]$$
 for all  $0 \le \ell \le N$ .



Alexander A. Beĭlinson, *The derived category of coherent sheaves on*  $\mathbf{P}^n$ , Selecta Mathematica Sovietica, vol. 3, 1983/84, Selected translations, pp. 233–237.





Dmitri O. Orlov, *Smooth and proper noncommutative schemes and gluing of DG categories*, Adv. Math. **302** (2016), 59–105.

## Let R be a commutative ring. On $\mathbb{P}_{R}^{n}$ we have a surjection



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Tensoring together n+1 of these we deduce a quasi-isomorphism of R with the Koszul complex

$$\bigotimes_{i=0}^{n} \left( R[x_i] \xrightarrow{X_i} R[x_i] \right)$$

$$0 \longrightarrow \mathcal{O}(-n) \longrightarrow \oplus \mathcal{O}(-n+1) \longrightarrow \cdots \longrightarrow \oplus \mathcal{O}(-1) \longrightarrow \oplus \mathcal{O} \longrightarrow 0$$

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Tensoring this with itself  $\ell > 0$  times yields a quasi-isomorphism of  $\mathcal{O}(\ell)$  with some complex

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This brutal truncation defines a class in

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Hence the brutal truncation must be quasi-isomorphic to  $\mathcal{O}(\ell) \oplus \mathcal{V}[n]$  for some vector bundle  $\mathcal{V}$ .

Applying the functor  $(-)^{\vee} = \mathcal{RH}om(-,\mathcal{O})$ , we obtain a quasi-isomorphism of  $\mathcal{O}(-\ell) \oplus \mathcal{V}^{\vee}[-n]$  with

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Thus if A[-M] contains

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But then

$$\mathbf{D}_{\mathbf{qc}}(X)^{\leq 0} \subset \mathsf{Coprod}(\mathcal{A}[-M])$$
.

## Thank you!