THE AUSLANDER BUCHSBAUM FORMULA

HANNO BECKER

Abstract. This is the script for my talk about the Auslander-Buchsbaum formula [AB57, Theorem 3.7] at the Auslander Memorial Workshop, 15th-18th of November 2014 in Bielefeld.

0. Overview

This talk is about the Auslander-Buchsbaum formula:

Theorem (Auslander-Buchsbaum formula). If $R$ is a commutative local Noetherian ring and $M$ a finitely generated $R$-module of finite projective dimension, then

$\text{depth}_R(R) - \text{depth}_R(M) = \text{pdim}_R(M)$.

The plan of the talk is the following:

- In §1 we recall some basic notions from commutative algebra that go into understanding, proving and using the Auslander-Buchsbaum formula.
- In §2 we present the classical proof given in [BH93].
- In §3 we ask how new techniques like derived categories may shed new light on the Auslander-Buchsbaum formula and its proof, and study some of its generalizations.

Conventions

In the following, we denote $R$ a commutative local Noetherian ring with maximal ideal $m$ and residue field $k := R/m$. Further, we denote $R\text{-mod}$ the category of finitely generated left $R$-modules, and $M \in R\text{-mod}$ unless otherwise stated.

1. Basic notions

1.1. Regular sequences. We begin by recalling the notion of a regular sequence.

Definition 1.1. Let $M$ be an $R$-module, $x \in m$ and $\underline{x} = (x_1, \ldots, x_n) \in m^n$.

(i) $x$ is called $M$-regular if $M \xrightarrow{x} M$ is injective.
(ii) A sequence $\underline{x} = (x_1, \ldots, x_n)$ is called $M$-regular if $x_1$ is $M$-regular and $(x_2, \ldots, x_n)$ is $M/x_1M$-regular.

If $M = R$, $x$ resp. $\underline{x}$ are simply called regular.
**Definition 1.2.** Let $M$ be an $R$-module. The *depth* of $M$ (as an $R$-module) is the maximal length of an $M$-regular sequence in $m$. It is denoted $\text{depth}_R(M)$.

**Remark 1.3.** Note that a priori it is not clear that $\text{depth}_R(M) < \infty$ or that any two maximal $M$-regular sequences have the same length. Both however is true and will be established below in Proposition 1.9.

By definition, the non-regular elements of $M$ are those contained in the union

$$\bigcup_{m \in M \setminus \{0\}} \text{Ann}_R(m) = \bigcup_{I \triangleleft R} I = \bigcup_{p \triangleleft R \text{ prime}} p,$$

where for the last equality we used the fact that the ideals maximal among those of the form $\text{Ann}_R(m)$ for $m \in M \setminus \{0\}$ are prime.

**Definition 1.4.** A prime ideal $p \triangleleft R$ is called an *associated prime* of $M$ if there exists an embedding $R/p \rightarrow M$, i.e. if there exists some $m \in M \setminus \{0\}$ such that $p = \text{Ann}_R(m)$. The set of associated primes of $M$ is denoted $\text{Ass}_R(M)$.

**Fact 1.5.** For a short exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ of $R$-modules, we have $\text{Ass}_R(M) \subseteq \text{Ass}_R(M') \cup \text{Ass}_R(M'')$.

**Proof.** If $p \in \text{Ass}_R(M)$ and $i : R/p \rightarrow M$ is an embedding, we distinguish between $\text{im}(i) \cap M' = \{0\}$ and $\text{im}(i) \cap M' \neq \{0\}$. In the first case, the $R/p \rightarrow M \rightarrow M''$ is injective, so $p \in \text{Ass}_R(M'')$. In the second case, there exists some $x \in R \setminus p$ with $i(x) \in M'$, and then $R/p \rightarrow R/p \rightarrow M$ factors through $M'$, so $p \in \text{Ass}_R(M')$. \qed

**Fact 1.6.** For any finitely generated $R$-module $M$, $\text{Ass}_R(M)$ is finite.

**Proof.** Any finitely generated $R$-module admits a finite filtration $0 = M_0 \subset M_1 \subset ... \subset M_{n-1} \subset M_n = M$ such that $M_i/M_{i-1} \cong R/p_i$ for prime ideals $p_i \triangleleft R$, $i = 1, 2, ..., n$. Then $\text{Ass}_R(M) \subseteq \bigcup_i \text{Ass}_R(R/p_i) = \{p_1, ..., p_n\}$. \qed

**Fact 1.7.** Let $M$ be a finitely generated $R$-module and $I \triangleleft R$ an ideal containing no $M$-regular element. Then $I \subset p$ for some associated prime $p \in \text{Ass}_R(M)$ of $M$.

**Proof.** By Fact 1.6 there are only finitely many associated primes, and by assumption $I$ is contained in their union. By prime avoidance, the claim follows. \qed

In the special case $I = m$ we obtain:

**Corollary 1.8.** Let $M$ be an $R$-module. Then $\text{depth}_R(M) = 0$ if and only if $m \in \text{Ass}_R(M)$, i.e. if and only if $\text{Hom}_R(k, M) \neq 0$.

In fact, this Corollary admits the following very useful generalization giving an alternative description of the depth of a module:
Proposition 1.9. For any finitely generated $R$-module $M$, we have
\[
\text{depth}_R(M) = \min\{i \in \mathbb{N}_{\geq 0} \mid \text{Ext}^i_R(k, M) \neq 0\} < \infty.
\]
Moreover, any maximal $M$-regular sequence has length $\text{depth}_R(M)$.

This follows from Corollary 1.8 and the following Lemma:

Lemma 1.10. Let $M, N$ be $R$-modules and $x_1, \ldots, x_n$ an $M$-regular sequence in $M$ contained in $\text{Ann}_R(N)$. Then there is a canonical isomorphism
\[
\text{Ext}^n_R(N, M) \cong \text{Hom}_R(N, M/(x_1, \ldots, x_k)M).
\]

1.2. Geometric considerations. We want to provide some geometric intuition for regularity and associated primes. For this, recall first that any commutative ring $R$ can be viewed as the ring of functions on its spectrum $\text{Spec}(R)$, with ideals (resp. prime ideals) of $R$ corresponding to closed (resp. closed and irreducible) subsets of $\text{Spec}(R)$; in particular, the minimal prime ideals $p_1, \ldots, p_n$ of $R$ correspond to the irreducible components $Z_1, \ldots, Z_n$ of $\text{Spec}(R)$. Now, if $x \in p_1 \cap \ldots \cap p_{n-1} \setminus p_n$, then intuitively $x$ is nonzero on $Z_n$ but vanishes on all the $Z_1, \ldots, Z_{n-1}$, so should be annihilated by any $y \in p_n$. It turns out, however, that $p_1 \cap \ldots \cap p_n$ is not necessarily $\{0\}$, but consists precisely of the nilpotent elements of $R$ – nevertheless, one can construct some element $x \in p_1 \cap \ldots \cap p_{n-1}$ such that $\text{Ann}_R(x) = p_n$, and we have:

Fact 1.11. Any minimal prime ideal is associated.

The presence of nilpotent elements is a bit challenging from the elements-as-functions viewpoint, but one might imagine them as formal Taylor approximations in some “virtual” direction of $\text{Spec}(R)$. The actual subset of $\text{Spec}(R)$ they are “virtually extending” is then given by the zero set of their annihilator, so that we obtain the following geometric interpretation of associated primes:

Intuition. Any associated prime corresponds either to an irreducible component of $\text{Spec}(R)$ or to an irreducible subset of $\text{Spec}(R)$ along which there is some “infinitesimal extension”, which one might think of as a “virtual irreducible component”.

Building on this intuition, the regular elements of $R$ can be thought of as those functions on $\text{Spec}(R)$ that do not vanish on any actual or “virtual” irreducible component.

Example 1.12. Consider $R := k[[x, y]]/(xy)$, the functions on the union of the two coordinate axis in the plane. Then $R$ is reduced, and its associated primes are precisely the minimal primes $(x)$ and $(y)$ corresponding to the $y$-axis and $x$-axis, respectively. Now, passing to the quotient $R' := R/(y^2)$, geometrically the $y$-axis has vanished, but an infinitesimal part of it survives as witnessed by the nilpotent function $y \in R'$, based at $\text{Ann}_R(y) = (x)$. In other words, $R'$ is the coordinate ring of the $x$-axis with an additional, infinitesimal $y$-axis attached to it at the origin, and $\text{Ass}(R') = \{(x, y), (y)\}$. 
The following fits well with the above intuition:

**Proposition 1.13.** The following inequalities holds:

\[
\text{depth}_R(R) \leq \min\{\dim R/p \mid p \in \text{Ass}(R)\} \leq \max\{\dim R/p \mid p \in \text{Ass}(R)\} = \dim(R).
\]

**Proof.** This follows from $\text{Ext}^i_R(N \hookrightarrow M) = 0$ for all finitely generated $R$-modules $N \hookrightarrow M$ with $i < \text{depth}_R(M) - \dim(N)$ (Ischebeck’s Theorem, see [Mat89, Theorem 17.1]); note that $\dim(N) = 0$ is essentially the definition of depths, and the general case is obtained by induction, wlog restricting to the case $N = R/p$ by the argument in the proof of Fact 1.6.

**Definition 1.14.** $R$ is called Cohen-Macaulay if $\text{depth}_R(R) = \dim(R)$.

For example, it follows from Proposition 1.13 that any Cohen-Macaulay ring is unmixed in the sense that it has no non-minimal associated primes, and all irreducible components of $\text{Spec}(R)$ have the same dimension.

### 2. Classical Proof

**Theorem 2.1** (Auslander Buchsbaum Formula). Let $R$ be a local commutative Noetherian ring and $M$ a finitely generated $R$-module with $\text{pdim}_R(M) < \infty$. Then

\[
\text{depth}_R(R) - \text{depth}_R(M) = \text{pdim}_R(M).
\]

**Proof.** We present the proof given in [BH93, Theorem 1.3.3], which goes by ascending induction on $\text{depth}_R(R)$.

If $\text{depth}_R(R) = 0$, then Lemma 2.2 tells us that $\text{pdim}_R(M) = 0$, hence $M$ is free. In particular, $\text{depth}_R(M) = \text{depth}_R(R) = 0$, and (AB) holds.

Suppose now that $\text{depth}_R(R) > 0$. If $\text{pdim}_R(M) = 0$, again (AB) is trivial. If not, and if $\text{depth}_R(M) = 0$, then denoting $\Omega M$ a first syzygy of $M$ we have $\text{pdim}_R(\Omega M) = \text{pdim}_R(M) - 1$ while $\text{depth}_R(\Omega M) = \text{depth}_R(M) + 1$ by the Depth lemma 2.3. Hence, the Auslander-Buchsbaum formulas for $M$ and $\Omega M$ are equivalent, and we may consequently assume $\text{depth}_R(M) > 0$. In this case, we have $m \notin \text{Ass}_R(M)$, and since also $m \notin \text{Ass}_R(R)$, prime avoidance implies that there exists $x \in m$ which is both $M$- and $R$-regular. Then

\[
\begin{align*}
\text{depth}_{R/xR}(M/xM) &= \text{depth}_R(M/xM) = \text{depth}_R(M) - 1, \\
\text{depth}_{R/xR}(R/xR) &= \text{depth}_R(R/xR) = \text{depth}_R(R) - 1, \quad \text{and} \\
\text{pdim}_{R/xR}(M/xM) &= \text{pdim}_R(M),
\end{align*}
\]

and the Auslander Buchsbaum formula for $R$ and $M$ follows by induction from the Auslander Buchsbaum formula for $R/xR$ and $M/xM$.

**Lemma 2.2.** If $\text{depth}_R(R) = 0$ and $\text{pdim}_R(M) < \infty$, then $\text{pdim}_R(M) = 0$. 

\[\square\]
Proof. From our assumption $\text{depth}_R(R) = 0$ we infer $\mathfrak{m} \in \text{Ass}_R(R)$, i.e. we have an embedding $\iota : k \hookrightarrow R$. Then, if $\varphi : F \to G$ is a homomorphism between nonzero free $R$-modules $F$ and $G$, we have a commutative diagram of $R$-modules

\[
\begin{array}{ccc}
F \otimes_R k & \xrightarrow{\text{id}_F \otimes \iota} & F \\
\varphi \otimes \text{id}_k & & \varphi \\
G \otimes_R k & \xrightarrow{\text{id}_G \otimes \iota} & G
\end{array}
\]

If here $\varphi$ is chosen to be minimal in the sense that $\varphi \otimes_R \text{id}_k = 0$ (i.e. if the coefficients of $\varphi$, when written as a matrix, all belong to $\mathfrak{m}$), we infer that $\text{im}(F \otimes_R k \xrightarrow{\text{id}_F \otimes \iota} F \cong F) \subseteq \ker(\varphi)$, hence in particular $\varphi$ is not injective. However, in a bounded and minimal free resolution of a non-projective $R$-module the leftmost non-zero differential would be a minimal and injective homomorphism between free $R$-modules, so we infer that such a resolution cannot exist, as claimed.

Lemma 2.3 (Depth lemma). Let $\text{depth}_R(M) < \text{depth}_R(R)$ and let $\Omega M$ be a syzygy of $M$. Then $\text{depth}_R(\Omega M) = \text{depth}_R(M) + 1$.

Proof. Pick a short exact sequence $0 \to \Omega M \to P \to M \to 0$ with $P$ free. Then the long exact Ext-sequence shows that for all $i \leq \text{depth}_R(R)$ there is a monomorphism $\text{Ext}^{i-1}_R(k, M) \to \text{Ext}_R^i(k, \Omega M)$, which for $i < \text{depth}_R(R)$ is even an isomorphism. The claim follows. \hfill $\square$

3. An application

Definition 3.1. The finitistic global dimension $\text{f.\,gl.\,dim}(R\text{-mod})$ is defined as

$$\text{f.\,gl.\,dim}(R\text{-mod}) := \sup\{\text{gl.\,dim}_R M \mid M \in R\text{-mod}, \text{gl.\,dim}_R M < \infty\}.$$ 

This definition also makes sense if $R$ is not commutative, and an important first question is to ask whether the finitistic global dimension of some ring is finite. While for finite-dimensional algebras over fields this seems open, the Auslander-Buchsbaum formula [AB] settles the question affirmatively:

Corollary 3.2. For a local (commutative) Noetherian ring $R$, we have

$$\text{f.\,gl.\,dim}(R\text{-mod}) \leq \text{depth}_R(R) < \infty.$$ 

In particular, the finitistic global dimension of $R$ is finite.

Proof. If $M$ is a finitely generated $R$-module with $\text{pd}_R M < \infty$, the Auslander-Buchsbaum formula tells us $\text{pd}_R M = \text{depth}_R R - \text{depth}_R M \leq \text{depth}_R R$. \hfill $\square$
4. New techniques & Generalizations

The key arguments for the results described in this section are contained in [FI03], in particular [FI03, Theorem 2.4].

4.1. A proof using derived categories. Using derived categories, we can provide an alternative proof of the Auslander-Buchsbaum formula. We have the following sequence of isomorphisms in $D(R$-Mod):

$$R \text{Hom}_R(k, R) \otimes^L_R (k \otimes^L_R M) \cong R \text{Hom}_R(k, R) \otimes^L_R M$$

$$\Theta \cong R \text{Hom}_R(k, R \otimes^L_R M)$$

$$\cong R \text{Hom}_R(k, M).$$

Here, the first isomorphism is an instance of the projection formula, and for the second isomorphism note that there is always an arrow as indicated, which is an isomorphism for $M = R$ and hence also for any complex quasi-isomorphic to a bounded complex of finitely generated projective $R$-modules.

The Auslander-Buchsbaum formula now follows by looking at the highest degree in which one has cohomology on both sides of (4.1): for the left hand side, it is $\text{depth}_R(R) + \text{pdim}_R(M)$, while for the right hand side it is $\text{depth}_R(M)$.

4.2. Generalizations. It is instructive to study the proof (4.1) further: The main point in it is the isomorphism $\Theta$, which is essentially the reflexivity of $M$ with respect to $(-)^\vee := R \text{Hom}_R(-, R)$ and the fact that $R \text{Hom}_R(M, N) \cong M^\vee \otimes^L_R N$ for any perfect $M$. Namely, we can rewrite $\Theta$ as a sequence of isomorphisms

$$R \text{Hom}_R(k, R) \otimes^L_R M \cong R \text{Hom}_R(k, R) \otimes^L_R M^{\vee \vee}$$

$$\cong R \text{Hom}_R(M^{\vee}, R \text{Hom}_R(k, R))$$

$$\cong R \text{Hom}_R(M^{\vee} \otimes^L_R k, R)$$

$$\cong R \text{Hom}_R(k, R \text{Hom}_R(M^{\vee}, R))$$

$$\cong R \text{Hom}_R(k, M^{\vee \vee})$$

$$\cong R \text{Hom}_R(k, M);$$

here, it is only the second isomorphism where the perfectness of $M$ actually plays a role, and only the first and last isomorphism where the reflexivity of $M$ is important. Moreover, we see that we use nothing particular about $R$ in the above sequence of isomorphisms, only the assumption that $M$ is reflexive with respect to $R \text{Hom}_R(-, R)$. Hence, we may summarize:

Proposition 4.1. Let $M, \omega$ be complexes of $R$-modules such that $M$ is reflexive w.r.t. $D_\omega := R \text{Hom}_R(-, \omega)$. Then there are canonical isomorphisms in $D(R$-Mod):

$$R \text{Hom}_R(k, M) \cong R \text{Hom}_R(D_\omega M, D_\omega k) \cong R \text{Hom}_k(k \otimes^L_R D_\omega M, D_\omega k).$$
If in the situation of Proposition 4.1 the complex \( \mathbb{D}_\omega M \) is bounded with finitely generated cohomology, then looking at the lowest degree of cohomology in both sides of (4.2), we obtain the equality

\[
\text{depth}_R(M) = \inf R\text{Hom}_R(k, M) = \inf \mathbb{D}_\omega k - \sup \mathbb{D}_\omega M.
\]

**Example 4.2.** We recover the classical Auslander-Buchsbaum formula if \( \omega := R \) and if \( M \) is a finitely generated \( R \)-module of finite projective dimension, since \( \mathbb{D}_\omega k = R\text{Hom}_k(k, R) \) computes depth\(_R\)(\( R \)) and \( \mathbb{D}_\omega M = R\text{Hom}_R(M, R) \) computes \( \text{pdim}_R(M) \) in this case.

**Example 4.3.** More generally, we say that \( M \) is of finite Gorenstein-projective dimension if it is reflexive with respect to \( R\text{Hom}_R(-, R) \) and if \( R\text{Hom}_R(M, R) \) is cohomologically bounded. In this case, the largest degree of cohomology of \( R\text{Hom}_R(M, R) \) is called the Gorenstein-projective dimension \( \text{gp-dim}_R(M) \) of \( M \); see [Chr00], in particular [Chr00, Theorem 2.2.3]. Hence, (4.3) generalizes the Auslander-Buchsbaum formula to the Auslander-Bridger formula [AB69, Theorem 4.13]: For any finitely generated \( R \)-module \( M \) of finite Gorenstein-projective dimension, we have

\[
\text{gp-dim}_R(M) = \text{depth}_R(R) - \text{depth}_R(M).
\]

The Gorenstein-projective dimension is finite for all finitely generated \( R \)-modules if and only if \( R \) is Gorenstein, i.e. of finite injective dimension over itself.

Finally, note there can be no concept of dimension which makes the analogue of the Auslander-Buchsbaum and Auslander-Bridger formulas valid without any assumptions on the module \( M \), as in general depth\(_R\)(\( M \)) \( \not\leq \) depth\(_R\)(\( R \)) : for example, taking \( R := k[\|x, y\|/(xy, y^2)] \), we have depth\(_R\)\( R = 0 \) but depth\(_R\)\( R/(y) = 1 \).

**References**


Mathematisches Institut Universität Bonn, Endenicher Allee 60, 53115 Bonn,
E-mail address: habecker@math.uni-bonn.de