Functorial Methods in Representation Theory

Jeremy Russell

The College of New Jersey

Maurice Auslander Memorial Workshop
November 15, 2014
So, how was your day?
The Big Bang Theory

Stephanie  So, how was your day?
Leonard  You know, I’m a physicist, so I thought about stuff.
The Big Bang Theory

Stephanie So, how was your day?

Leonard You know, I’m a physicist, so I thought about stuff.

Stephanie That’s it?
Mindset of the Functorial Approach

The Big Bang Theory

Stephanie  So, how was your day?
Leonard   You know, I’m a physicist, so I thought about stuff.
Stephanie  That’s it?
Leonard   I wrote some of it down.
Thesis of the Talk

The Functorial Approach

Functors are Modules.
Ab - category of abelian groups.
Notation: (Fixed)

- $\text{Ab}$ - category of abelian groups.
- $\Lambda$ - finite dimensional $k$-algebra.
Notation: (Fixed)

\[ \text{Ab} - \text{category of abelian groups.} \]

\[ \Lambda - \text{finite dimensional } k\text{-algebra.} \]
### Notation: (Fixed)

- **Ab** - category of abelian groups.
- **Λ** - finite dimensional $k$-algebra.

### Notation: (For Now)

- **$R$** - ring.
Notation: (Fixed)

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Λ - finite dimensional $k$-algebra.

Notation: (For Now)

$R$ - ring.
Mod($R$) - category of right modules.
### Notation: (Fixed)

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$\text{Ab} - \text{category of abelian groups.}$

$\Lambda - \text{finite dimensional } k\text{-algebra.}$

Notation: (For Now)

$R - \text{ring.}$

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$\text{Mod}(R^{op}) - \text{category of left modules.}$

$\text{mod}(R) - \text{category of finitely presented right modules}$
Notation

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<tr>
<td>$\mathcal{A}$</td>
<td>skeletally small abelian category.</td>
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The Functor Category

Definition

The category $\text{(}A, \text{Ab}\text{)}$ consists of all additive covariant functors $F: A \to \text{Ab}$ together with the natural transformations between them.

Definition

A functor $F: A \to \text{Ab}$ is called representable if $F \cong \text{Hom}_A(X, -)$ for some $X \in A$. 
### Definition

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The Functor Category

Definition

The category $(\mathcal{A}, \text{Ab})$ consists of all additive covariant functors $F: \mathcal{A} \rightarrow \text{Ab}$ together with the natural transformations between them.

Definition

A functor $F: \mathcal{A} \rightarrow \text{Ab}$ is called representable if

$$F \cong \text{Hom}_\mathcal{A}(X, \_ )$$

for some $X \in \mathcal{A}$. 
Notation:

We will abbreviate the representable functor $\text{Hom}_A(X, \_)$ by

\[(X, \_)
\]

and abbreviate $\text{Hom}_{(A, \text{Ab})}(F, \_)$ by

\[\text{Nat}(F, \_)
\]

or

\[(F, \_)
\]

depending on the situation.
Properties of $(\mathcal{A}, \text{Ab})$

The functor category $(\mathcal{A}, \text{Ab})$ has some interesting homological properties which are essentially inherited from the category Ab.
Properties of \((\mathcal{A}, \text{Ab})\)

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- Given \(\alpha, \beta \in \text{Nat}(F, G)\)

\[
(\alpha + \beta)_X := \alpha_X + \beta_X
\]
Properties of \((\mathcal{A}, \text{Ab})\)

The functor category \((\mathcal{A}, \text{Ab})\) has some interesting homological properties which are essentially inherited from the category \(\text{Ab}\).

- Given \(\alpha, \beta \in \text{Nat}(F, G)\)

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- The category \((\mathcal{A}, \text{Ab})\) is abelian.
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- **Given** $\alpha, \beta \in \text{Nat}(\mathcal{F}, \mathcal{G})$

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- The category $\mathcal{A}, \text{Ab}$) is abelian.
- A natural transformation $\alpha: \mathcal{F} \to \mathcal{G}$ is
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  - a monomorphism \iff \(\alpha_X\) is a monomorphism for all \(X \in \mathcal{A}\).
  - an epimorphism \iff \(\alpha_X\) is an epimorphism for all \(X \in \mathcal{A}\).
  - a kernel of \(\beta\) \iff \(\alpha_X\) is a kernel of \(\beta_X\) for all \(X \in \mathcal{A}\).
The functor category \((\mathcal{A}, \text{Ab})\) has some interesting homological properties which are essentially inherited from the category \text{Ab}.

- **Given** \(\alpha, \beta \in \text{Nat}(F, G)\)
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  (\alpha + \beta)_X := \alpha_X + \beta_X
  \]

- **The category** \((\mathcal{A}, \text{Ab})\) is abelian.
- **A natural transformation** \(\alpha : F \to G\) is
  \begin{itemize}
  \item a monomorphism \(\iff\) \(\alpha_X\) is a monomorphism for all \(X \in \mathcal{A}\).
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  \item a cokernel of \(\beta\) \(\iff\) \(\alpha_X\) is a cokernel of \(\beta_X\) for all \(X \in \mathcal{A}\).
  \end{itemize}
Exactness in \((\mathcal{A}, \text{Ab})\)

A sequence in \((\mathcal{A}, \text{Ab})\):

\[
F \longrightarrow G \longrightarrow H
\]

is exact if and only if

\[
F(X) \longrightarrow G(X) \longrightarrow H(X)
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is exact for all \(X \in \mathcal{A}\).
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- All universal objects such as the kernel, cokernel, pullback, pushout, etc. are constructed componentwise.
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- All universal objects such as the kernel, cokernel, pullback, pushout, etc. are constructed componentwise.

- For each \(X \in \mathcal{A}\), the evaluation functor
  
  \[
  \text{ev}_X : (\mathcal{A}, \text{Ab}) \rightarrow \text{Ab}
  \]

  is exact.
Yoneda’s Lemma

Lemma (Yoneda)

For any $X \in \mathcal{A}$ and any $F \in (\mathcal{A}, \text{Ab})$,

$$\text{Nat}((X, _), F) \cong F(X)$$

1. The isomorphism is defined by $\alpha \mapsto \alpha_X(1_X)$.
2. The isomorphism is natural in both $X$ and $F$. 

According to Martsinkovsky, Auslander used to say that if you cannot prove something using Yoneda’s lemma, then it isn’t true.
Yoneda’s Lemma

Lemma (Yoneda)
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According to Martsinkovsky, Auslander used to say that if you cannot prove something using Yoneda’s lemma, then it isn’t true.
Consequence of Yoneda’s Lemma

Take an exact sequence in \((\mathcal{A}, \text{Ab})\):

\[ F \longrightarrow G \longrightarrow H \]
Consequence of Yoneda’s Lemma

Take an exact sequence in $(\mathcal{A}, \text{Ab})$:

\[
F \longrightarrow G \longrightarrow H
\]

Apply $\text{Nat}((X, \_), \_)$:

\[
((X, \_), F) \longrightarrow ((X, \_), G) \longrightarrow ((X, \_), H)
\]
Consequence of Yoneda’s Lemma

Take an exact sequence in $(\mathcal{A}, \text{Ab})$:

$$F \rightarrow G \rightarrow H$$

Apply $\text{Nat}((X, _), _) :$

$$((X, _), F) \rightarrow ((X, _), G) \rightarrow ((X, _), H)$$

$$\cong \quad \cong \quad \cong$$

$$F(X) \rightarrow G(X) \rightarrow H(X)$$
Consequence of Yoneda’s Lemma

Take an exact sequence in \((\mathcal{A}, \text{Ab})\):

\[
F \xrightarrow{} G \xrightarrow{} H
\]

Apply \(\text{Nat}((X, _), _)\):

\[
((X, _), F) \xrightarrow{\cong} ((X, _), G) \xrightarrow{\cong} ((X, _), H)
\]

\[
\downarrow \cong \quad \downarrow \cong \quad \downarrow \cong
\]

\[
F(X) \xrightarrow{} G(X) \xrightarrow{} H(X)
\]

The bottom row is exact resulting in exactness of the top row.
Therefore representable functors are projectives in $(\mathcal{A}, \text{Ab})$. 
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**Proposition**

The **Yoneda embedding** \(Y : \mathcal{A} \to (\mathcal{A}, \text{Ab})\) defined by

\[
Y(X) = (X, \_)
\]

is

1. **full**
Consequences of Yoneda’s Lemma

Therefore representable functors are projectives in \((\mathcal{A}, \text{Ab})\).

**Proposition**

The **Yoneda embedding** \(Y: \mathcal{A} \to (\mathcal{A}, \text{Ab})\) defined by

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is

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Consequences of Yoneda’s Lemma

Therefore representable functors are projectives in $(\mathcal{A}, \text{Ab})$.

**Proposition**

The **Yoneda embedding** $Y : \mathcal{A} \rightarrow (\mathcal{A}, \text{Ab})$ defined by

$$Y(X) = (X, _)$$

is

1. full
2. faithful
3. left exact
Start with exact sequence

\[0 \to A \to B \to C \to 0\]
Start with exact sequence

\[ 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \]

Apply the left exact functor \((\_, \, X)\):

\[ 0 \rightarrow (C, X) \rightarrow (B, X) \rightarrow (A, X) \]
Left Exactness of the Yoneda Embedding

Start with exact sequence

$$0 \to A \to B \to C \to 0$$

Apply the left exact functor $(_, X)$:

$$0 \to (C, X) \to (B, X) \to (A, X)$$

Since the exactness of this sequence holds for all $X \in \mathcal{A}$, the sequence

$$0 \to (C, _) \to (B, _) \to (A, _)$$

is exact in $(\mathcal{A}, \text{Ab})$. 
Consequences of Yoneda’s Lemma

**Definition**

Recall that an object \( A \) is called finitely generated if whenever \( A = \bigsqcup_i A_i \), this sum may be taken to be finite.
Consequences of Yoneda’s Lemma

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Recall that an object \( A \) is called finitely generated if whenever \( A = \bigsqcup_i A_i \), this sum may be taken to be finite.

- Representable functors are finitely generated projectives in \((\mathcal{A}, \text{Ab})\).
Consequences of Yoneda’s Lemma

Definition

Recall that an object \( A \) is called finitely generated if whenever \( A = \bigsqcup_I A_i \), this sum may be taken to be finite.

- Representable functors are finitely generated projectives in \((\mathcal{A}, \text{Ab})\).
- Representable functors generate \((\mathcal{A}, \text{Ab})\) in the sense that given \( F: \mathcal{A} \to \text{Ab} \) there exists \( X_i \in \mathcal{A} \) and exact sequence

\[
\bigsqcup_i (X_i, \_ ) \to F \to 0
\]
A functor $F: \mathcal{A} \to \text{Ab}$ is called \textbf{finitely presented} if there exists an exact sequence

$\left( Y, \_ \right) \to \left( X, \_ \right) \to F \to 0$

In other words, $F$ is a cokernel of a representable transformation.
Definition (Auslander)

A functor $F: \mathcal{A} \to \text{Ab}$ is called **finitely presented** if there exists exact sequence

$$(Y, \_ ) \to (X, \_ ) \to F \to 0$$

In other words, $F$ is a cokernel of a representable transformation.

Definition

$\text{fp}(\mathcal{A}, \text{Ab}) = \text{category of finitely presented functors.}$
fp(\mathcal{A}, \text{Ab}) \hookrightarrow (\mathcal{A}, \text{Ab})

- fp(\mathcal{A}, \text{Ab}) is abelian, the inclusion functor is exact, and reflects exact sequences.
Properites of Finitely Presented Functors

\[ \text{fp}(\mathcal{A}, \text{Ab}) \leftrightarrow (\mathcal{A}, \text{Ab}) \]

- \( \text{fp}(\mathcal{A}, \text{Ab}) \) is abelian, the inclusion functor is exact, and reflects exact sequences.
- \( \text{fp}(\mathcal{A}, \text{Ab}) \) has enough projectives and they are precisely the representable functors.
Properites of Finitely Presented Functors

\[ \text{fp} (\mathcal{A}, \text{Ab}) \hookrightarrow (\mathcal{A}, \text{Ab}) \]

- \text{fp}(\mathcal{A}, \text{Ab}) \) is abelian, the inclusion functor is exact, and reflects exact sequences.
- \text{fp}(\mathcal{A}, \text{Ab}) \) has enough projectives and they are precisely the representable functors.
- All finitely presented functors have projective dimension at most 2:

\[
0 \rightarrow (Z, \_ ) \rightarrow (Y, \_ ) \rightarrow (X, \_ ) \rightarrow F \rightarrow 0
\]
Proposition (Auslander)

$\text{Ext}^n(X, _) \in \text{fp}(\text{Mod}(R^{op}), \text{Ab}).$
Examples of Finitely Presented Functors

Proposition (Auslander)

1. \( \text{Ext}^n(X, \_ ) \in \text{fp}(\text{Mod}(R^{op}), \text{Ab}). \)
2. \( \_ \otimes X \in \text{fp}(\text{Mod}(R^{op}), \text{Ab}) \) if and only if \( X \in \text{mod}(R^{op}). \)
Construction of $w: \text{fp}(\mathcal{A}, \text{Ab}) \to \mathcal{A}$

Auslander constructed a contravariant functor

$$w: \text{fp}(\mathcal{A}, \text{Ab}) \to \mathcal{A}$$
Construction of $w: \text{fp}(\mathcal{A}, \text{Ab}) \to \mathcal{A}$

Auslander constructed a contravariant functor

$$w: \text{fp}(\mathcal{A}, \text{Ab}) \to \mathcal{A}$$

**Step 1:** Start with presentation

$$(Y, _) \to (X, _) \to F \to 0$$
Construction of $w: \text{fp}(\mathcal{A}, \text{Ab}) \to \mathcal{A}$

Auslander constructed a contravariant functor

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**Step 1:** Start with presentation

$$(Y, \_ ) \to (X, \_ ) \to F \to 0$$

**Step 2:** By Yoneda $(Y, \_ ) \to (X, \_ )$ comes from a unique morphism

$$X \to Y$$
Construction of $w : \text{fp}(\mathcal{A}, \text{Ab}) \to \mathcal{A}$

Auslander constructed a contravariant functor

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Step 2: By Yoneda $(Y, _) \to (X, _) \to (X, _) \to F \to 0$ comes from a unique morphism

$$X \to Y$$

Step 3: The exact sequence

$$0 \to w(F) \to X \to Y$$

completely determines $w$. 
Properties

Proposition (Auslander)

1. $w$ does not depend on any choices of presentation.
Properties

Proposition (Auslander)

1. $w$ does not depend on any choices of presentation.
2. $w$ is exact.
Proposition (Auslander)

1. $w$ does not depend on any choices of presentation.
2. $w$ is exact.
3. $w(X, \_ ) = X$. 
Take presentation of $F$:

$$0 \to (Z, \_ ) \to (Y, \_ ) \to (X, \_ ) \to F \to 0$$
What $w$ Measures

Take presentation of $F$:

$$0 \rightarrow (Z, \_ ) \rightarrow (Y, \_ ) \rightarrow (X, \_ ) \rightarrow F \rightarrow 0$$

Apply $w$:

$$0 \rightarrow w(F) \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$
What $w$ Measures

**Proposition (Auslander)**

For $F \in \text{fp}(\mathcal{A}, \text{Ab})$ the following are equivalent:
Proposition (Auslander)

For $F \in \text{fp}(\mathcal{A}, \text{Ab})$ the following are equivalent:

1. $w(F) = 0$
Proposition (Auslander)

For $F \in \text{fp}(\mathcal{A}, \text{Ab})$ the following are equivalent:

1. $w(F) = 0$

2. All presentations of $F$ arise from short exact sequences.
What \( w \) Measures

**Proposition (Auslander)**

For \( F \in \text{fp}(\mathcal{A}, \text{Ab}) \) the following are equivalent:

1. \( w(F) = 0 \)
2. All presentations of \( F \) arise from short exact sequences.
3. There exists short exact sequence in \( \mathcal{A} \)

\[
0 \to X \to Y \to Z \to 0
\]

such that the following is a presentation of \( F \):

\[
0 \to (Z, _) \to (Y, _) \to (X, _) \to F \to 0
\]
Zeroth Derived Functors

Proposition (Auslander)

For any finitely presented functor $F$ there is an exact sequence:

$$0 \to F_0 \to F \to (w(F), \_ ) \to F_1 \to 0$$

If $A$ has enough injectives, then $w(F)$ is zero.
Proposition (Auslander)

For any finitely presented functor $F$ there is an exact sequence:

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- If $\mathcal{A}$ has enough injectives, then $(w(F), _) = R^0 F$. 
Proposition (Auslander)

For any finitely presented functor $F$ there is an exact sequence:

$$0 \rightarrow F_0 \rightarrow F \rightarrow (w(F), \_ ) \rightarrow F_1 \rightarrow 0$$

- If $\mathcal{A}$ has enough injectives, then $(w(F), \_ ) = R^0 F$.
- In this case $F$ vanishes on injectives if and only if $w(F') = 0$. 
For a finite dimensional $k$-algebra $\Lambda$, the category $\text{mod}(\Lambda^{op})$ is abelian and has enough injectives.
Finite Dimensional $k$-Algebras

- For a finite dimensional $k$-algebra $\Lambda$, the category $\text{mod}(\Lambda^{op})$ is abelian and has enough injectives.
- Every finitely presented left $\Lambda$-module $M$ is a finite direct sum of indecomposables

\[ M = \bigoplus_{i=1}^{n} X_i \]
Projective Covers

**Definition**

Recall that an epimorphism \( f : P \to X \) from a projective \( P \) to object \( X \) is called a projective cover if

\[
fh = f
\]

implies that \( h \) is an isomorphism.
### Minimal Resolutions

**Definition**

A projective resolution

\[ \cdots P_k \rightarrow P_{k-1} \rightarrow \cdots P_1 \rightarrow P_0 \rightarrow X \rightarrow 0 \]

is a minimal projective resolution if each

\[ P_n \rightarrow \Omega^n X \]

is a projective cover.
Proposition (Auslander)

All finitely presented functors $F : \text{mod}(\Lambda^{op}) \rightarrow \text{Ab}$ have minimal projective resolutions.
fp(mod(Λ^{op}) Has Minimal Projective Resolutions

Proposition (Auslander)

All finitely presented functors $F : \text{mod}(Λ^{op}) \to \text{Ab}$ have minimal projective resolutions.

- Given $(X, \_ ) \to F \to 0$, one can take $X$ to have smallest dimension. This will be a projective cover.
Simple Functors

**Definition**

A functor $S : \mathcal{A} \to \text{Ab}$ is called simple if $S \neq 0$ and any non-zero morphism $F \to S$ is an epimorphism.
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Proposition (Auslander)

For a simple functor $S: \text{mod}(\Lambda^{op}) \to \text{Ab}$ there exists a unique indecomposable $N$ such that

1. $S(N) \neq 0$
Simple Functors

Definition
A functor $S: \mathcal{A} \to \text{Ab}$ is called simple if $S \neq 0$ and any non-zero morphism $F \to S$ is an epimorphism.

Proposition (Auslander)
For a simple functor $S: \text{mod}(\Lambda^{op}) \to \text{Ab}$ there exists a unique indecomposable $N$ such that

1. $S(N) \neq 0$
2. There is a projective cover $(N, \_ \_ \_ ) \to S \to 0.$
Step 1: Find exact sequence \((N, \_ ) \rightarrow S \rightarrow 0.\)
Sketch of Proof

Step 1: Find exact sequence \((N, \_ ) \rightarrow S \rightarrow 0\).

- Since \(S \neq 0\), there exists \(N \in \text{mod}(\Lambda^{op})\) such that \(S(N) \neq 0\).
Sketch of Proof

Step 1: Find exact sequence \((N, \_ ) \to S \to 0\).

- Since \(S \neq 0\), there exists \(N \in \text{mod}(\Lambda^{op})\) such that \(S(N) \neq 0\).
- Since \(S(N) \neq 0\), there exists non-zero \(x \in S(N)\).
Sketch of Proof

Step 1: Find exact sequence \((N, \_ ) \rightarrow S \rightarrow 0\).

- Since \(S \neq 0\), there exists \(N \in \text{mod}(\Lambda^\text{op})\) such that \(S(N) \neq 0\).
- Since \(S(N) \neq 0\), there exists non-zero \(x \in S(N)\).
- \(x\) determines \(\varepsilon_x: (N, \_ ) \rightarrow S\) where \(\varepsilon_x(1_N) = x\).
Sketch of Proof

Step 1: Find exact sequence \((N, \_ ) \to S \to 0\).

- Since \(S \neq 0\), there exists \(N \in \text{mod}(\Lambda^{op})\) such that \(S(N) \neq 0\).
- Since \(S(N) \neq 0\), there exists non-zero \(x \in S(N)\).
- \(x\) determines \(\varepsilon_x: (N, \_ ) \to S\) where \(\varepsilon_x(1_N) = x\).
- \(x \neq 0\) implies \(\varepsilon_x \neq 0\).
Step 1: Find exact sequence $(N, _) \to S \to 0$.

- Since $S \neq 0$, there exists $N \in \text{mod}(\Lambda^{op})$ such that $S(N) \neq 0$.
- Since $S(N) \neq 0$, there exists non-zero $x \in S(N)$.
- $x$ determines $\varepsilon_x: (N, _) \to S$ where $\varepsilon_x(1_N) = x$.
- $x \neq 0$ implies $\varepsilon_x \neq 0$
- Because $S$ is simple, $\varepsilon_x$ is an epimorphism.
Sketch of Proof

Step 1: Find exact sequence \((N, \_ ) \to S \to 0\).

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Step 2: Choose \(N\) from above to have smallest dimension.
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- \( x \neq 0 \) implies \( \varepsilon_x \neq 0 \)
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- \( N \) will be indecomposable.
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Step 2: Choose \(N\) from above to have smallest dimension.
- \(N\) will be indecomposable.
- Otherwise \(N \cong A \biguplus B\).
- \(S(N) = S(A) \biguplus S(B) \neq 0\).
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- \(N\) will be indecomposable.
- Otherwise \(N \cong A \bigsqcup B\).
- \(S(N) = S(A) \bigsqcup S(B) \neq 0\).
- \(A\) and \(B\) have smaller dimension.
- Either \(S(A) \neq 0\) or \(S(B) \neq 0\).
Step 3: Show that $\varepsilon_x : (N, \_ ) \to S \to 0$ is a projective cover.
Sketch of Proof

Step 3: Show that \( \varepsilon_x: (N, \_ ) \to S \to 0 \) is a projective cover.

- Suppose that we have the following commutative diagram:

\[
\begin{array}{ccc}
(N, \_ ) & \xrightarrow{\varepsilon_x} & F \\
| & | & | \\
(f, \_ ) & \downarrow & 1 \\
| & | & | \\
(N, \_ ) & \xrightarrow{\varepsilon_x} & F
\end{array}
\]
Sketch of Proof

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- This gives $\varepsilon_x = \varepsilon_x(f^n, \_ )$ for all $n \geq 1$
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$\therefore f$ must be an isomorphism.
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\downarrow{(f, \_ )} & & \downarrow{1} \\
(N, \_ ) & \xrightarrow{\varepsilon_x} & F
\end{array}
$$

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  $\therefore (N, \_ ) \to S$ is a projective cover.

- Uniqueness of $N$ follows from uniqueness of projective covers.
$S_N$ denotes simple functor $S$ such that $S(N) \neq 0$ for indecomposable $N$. 
$S_N$ denotes simple functor $S$ such that $S(N) \neq 0$ for indecomposable $N$.

**Theorem (Auslander)**

The simple functors are finitely presented in $(\text{mod}(\Lambda^{op}), \text{Ab})$. 
Recall that we are looking at the category

$$\text{fp}(\text{mod}(\Lambda^{op}), \text{Ab})$$
Recall that we are looking at the category

\[ \text{fp}(\text{mod}(\Lambda^{op}), \text{Ab}) \]

**Proposition**

Suppose that \( N \) is a non-injective indecomposable. Then

\[ w(S_N) = 0. \]
Proof

Since $N$ is not injective, $S_N(I) = 0$ for any indecomposable injective $I$. 

∴ $S_N(J) = 0$ for any finite sum of indecomposable injectives $J = \sum I_k$. All injectives in $\text{mod}(\Lambda^{\text{op}})$ are of this form.

∴ $S_N$ vanishes on injectives. Because $\text{mod}(\Lambda^{\text{op}})$ has enough injectives, this is equivalent to

$w(S_N) = 0$

Hence there exists an exact sequence in $\text{mod}(\Lambda^{\text{op}})$:

$$0 \to N \to Y \to Z \to 0$$

such that

$$0 \to (Z, S_N) \to (Y, S_N) \to (N, S_N) \to 0$$
Proof

Since $N$ is not injective, $S_N(I) = 0$ for any indecomposable injective $I$.

∴ $S_N(J) = 0$ for any finite sum of indecomposable injectives $J = \bigsqcup_k I_k$. 
Proof

Since \( N \) is not injective, \( S_N(I) = 0 \) for any indecomposable injective \( I \).

\[
\therefore S_N(J) = 0 \quad \text{for any finite sum of indecomposable injectives} \quad J = \bigsqcup_k I_k.
\]

All injectives in \( \text{mod}(\Lambda^{op}) \) are of this form.
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Since $N$ is not injective, $S_N(I) = 0$ for any indecomposable injective $I$.

∴ $S_N(J) = 0$ for any finite sum of indecomposable injectives $J = \bigsqcup_k I_k$.

All injectives in mod($\Lambda^{op}$) are of this form.

∴ $S_N$ vanishes on injectives.
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Since \( N \) is not injective, \( S_N(I) = 0 \) for any indecomposable injective \( I \).

\[ \therefore S_N(J) = 0 \text{ for any finite sum of indecomposable injectives } J = \bigsqcup_k I_k. \]

All injectives in \( \text{mod}(\Lambda^{op}) \) are of this form.

\[ \therefore S_N \text{ vanishes on injectives. } \]

Because \( \text{mod}(\Lambda^{op}) \) has enough injectives, this is equivalent to

\[ w(S_N) = 0 \]
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Because mod($\Lambda^{op}$) has enough injectives, this is equivalent to $w(S_N) = 0$.

Hence there exists an exact sequence in mod($\Lambda^{op}$):

$$0 \to N \to Y \to Z \to 0$$

such that

$$0 \to (Z, \_ ) \to (Y, \_ ) \to (N, \_ ) \to S_N \to 0$$
What Are Almost Split Sequences?

For $N$ indecomposable non-injective:
What Are Almost Split Sequences?

For $N$ indecomposable non-injective:

**Step 1:** Take minimal projective presentation:

$$0 \to (Z, \_ ) \to (Y, \_ ) \to (N, \_ ) \to S_N \to 0$$
For $N$ indecomposable non-injective:

**Step 1:** Take minimal projective presentation:

$$0 \to (Z, _) \to (Y, _) \to (N, _) \to S_N \to 0$$

**Step 2:** Apply $w$ to get the short exact sequence

$$0 \to N \to Y \to Z \to 0$$
What Are Almost Split Sequences?

Start with any morphism \( u: N \to K \)

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & 0 \\
\downarrow{u} & & \downarrow & & \downarrow & & \downarrow & & \\
 & K & & & & & & & \\
\end{array}
\]
What Are Almost Split Sequences?

Start with any morphism \( u : N \rightarrow K \)

\[
\begin{array}{cccccc}
0 & \rightarrow & N & \xrightarrow{f} & Y & \rightarrow & Z & \rightarrow & 0 \\
\downarrow{u} & & \downarrow & & \downarrow & & \downarrow{1} \\
0 & \rightarrow & K & \rightarrow & E & \rightarrow & Z & \rightarrow & 0 \\
\end{array}
\]

push out diagram
Two Possibilities for $u$

$$
\begin{array}{cccccc}
0 & \rightarrow & (Z, \_ \_ ) & \rightarrow & (E, \_ \_ ) & \rightarrow & (K, \_ \_ ) & \rightarrow & F & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & (u, \_ \_ ) & & \\
1 & & \rightarrow & & \rightarrow & & \rightarrow & & \rightarrow & & \\
0 & \rightarrow & (Z, \_ \_ ) & \rightarrow & (Y, \_ \_ ) & \rightarrow & (N, \_ \_ ) & \rightarrow & S_N & \rightarrow & 0 \\
& & & & & & & & \downarrow \quad G & & \\
& & & & & & & & \downarrow & & \\
& & & & & & & & \downarrow & & \\
& & & & & & & & 0 & & 
\end{array}
$$
Two Possibilities for $u$

\[
\begin{array}{ccccccc}
0 & \longrightarrow & (Z, _) & \longrightarrow & (E, _) & \longrightarrow & (K, _) & \longrightarrow & F & \longrightarrow & 0 \\
1 & \downarrow &  & \downarrow &  & \downarrow & (u, _) & \downarrow & \alpha \\
0 & \longrightarrow & (Z, _) & \longrightarrow & (Y, _) & \longrightarrow & (N, _) & \longrightarrow & S_N & \longrightarrow & 0 \\
& & & & & & & \downarrow G & & \\
& & & & & & & & \downarrow & \\
& & & & & & & & & 0 \\
\end{array}
\]
Case 1: $\alpha = 0$

\[0 \rightarrow (Z, _) \rightarrow (E, _) \rightarrow (K, _) \rightarrow F \rightarrow 0\]
\[\downarrow \quad 1 \quad \downarrow \quad \quad \downarrow (u, _) \quad \alpha = 0\]
\[0 \rightarrow (Z, _) \rightarrow (Y, _) \rightarrow (N, _) \rightarrow S_N \rightarrow 0\]
\[\downarrow (f, _) \quad \downarrow \quad \downarrow G \quad \downarrow \quad \downarrow 0\]
$(u, \_)$ factors through $(f, \_)$

\[
\begin{array}{ccccccc}
0 & \to & (Z, \_ ) & \to & (E, \_ ) & \to & (K, \_ ) & \to & F & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & (u, \_ ) & & \alpha = 0 \\
0 & \to & (Z, \_ ) & \to & (Y, \_ ) & \xrightarrow{(f, \_ )} & (N, \_ ) & \to & S_N & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & G & & \downarrow & \downarrow \\
& & & & & 0 & & & & \\
\end{array}
\]
$\alpha = 0$
Case 2: $\alpha = \text{epimorphism}$

\[
\begin{array}{ccccccc}
0 & \longrightarrow & (Z, \_ ) & \longrightarrow & (E, \_ ) & \longrightarrow & (K, \_ ) & \longrightarrow & F & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \alpha \\
0 & \longrightarrow & (Z, \_ ) & \longrightarrow & (Y, \_ ) & \rightarrow & (N, \_ ) & \longrightarrow & S_N & \longrightarrow & 0 \\
(f, \_ ) & & & & & \downarrow & & & & \downarrow & & \text{G} & \downarrow & 0 \\
& & & & & & & & & & 0
\end{array}
\]
\((u, \_ ) = \text{epimorphism and hence } u = \text{section.}\)

\[
\begin{array}{cccccc}
0 & \rightarrow & (Z, \_ ) & \rightarrow & (E, \_ ) & \rightarrow & (K, \_ ) & \rightarrow & F & \rightarrow & 0 \\
\downarrow 1 & & \downarrow & & \downarrow & & \downarrow (u, \_ ) & & \alpha \\
0 & \rightarrow & (Z, \_ ) & \rightarrow & (Y, \_ ) & \rightarrow & (N, \_ ) & \rightarrow & S_N & \rightarrow & 0 \\
\downarrow (f, \_ ) & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
0 & & 0 & & 0 & & 0 & & \\
\end{array}
\]

In this case \(u\) must be a section.
The fact that $f$ is left minimal follows from the minimality of the presentation

$$(Y, \_ ) \rightarrow (N, \_ ) \rightarrow S_N \rightarrow 0$$
Almost Split Sequences

Definition (Auslander, Reiten)

An exact sequence

\[ 0 \longrightarrow N \overset{f}{\longrightarrow} Y \overset{g}{\longrightarrow} Z \longrightarrow 0 \]

is **almost split** if

1. \( f \) is left minimal, so \( hf = f \) implies \( h \) is an isomorphism.
2. If \( u: N \to K \) is not a section then \( u = fu' \).
Almost Split Sequences

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Note: These are simply the properties of the sequences obtained by applying \( w \) to minimal projective resolutions of simple functors \( S_N \).
David Buchsbaum in a personal communication to Robin Hartshorne:

“It was always a little difficult to know just what Maurice had in mind when he started on something. Certainly in the case of coherent functors, the choice of the term “coherent” indicates that he was onto the notion of finite presentation... when he first spoke to me about coherent functors, he didn’t speak about them in any way in connection with the applications he finally came up with. He was playing; his representable functors were his finitely generated projectives, and so his coherent functors generalized existing notions of the time (this is what he told me).”
\( \mathcal{A} \)-modules

\( \mathcal{A} \) - any small pre-additive category.

**Definition**

\[
\text{Mod}(\mathcal{A}^{op}) = (\mathcal{A}, \text{Ab})
\]

A left \( \mathcal{A} \)-module is a functor \( F: \mathcal{A} \rightarrow \text{Ab} \).

In the general approach, functors are viewed as generalizations of modules.
For the ring $R$, one recovers all left $R$-modules as follows:
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- Treat $R$ as a pre-additive category consisting of one object $\ast$. 

\[(\ast, \ast) = R\]
\[(\ast, \mathbb{Ab}) = \text{Mod}(R)\text{op}\] in the usual sense.
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- Treat $R$ as a pre-additive category consisting of one object $\ast$.
- $(\ast, \ast) = R$
- Composition of maps is multiplication in the ring.
Left Modules Over a Ring

For the ring $R$, one recovers all left $R$-modules as follows:

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- $(R^{op}, \text{Ab}) = \text{Mod}(R^{op})$ in the usual sense.
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Left Modules Over a Ring

For the ring $R$, one recovers all left $R$-modules as follows:

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Fisher-Palmquist in her dissertation studied the general tensor product

\[ \otimes_A : \text{Mod}(A) \times \text{Mod}(A^{op}) \to \text{Ab} \]

This bifunctor is completely determined by the following criterion:

\[ (_-, X) \otimes_A G \cong G(X). \]
General Tensor Product

**Definition (Kan, Lawvere, Freyd, Ulmer)**

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- \(F \otimes_A (X, \_) \cong F(X)\).
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### General Tensor Product

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This bifunctor is completely determined by the following criterion:

- $(_ , X) \otimes_{\mathcal{A}} G \cong G(X)$.
- $F \otimes_{\mathcal{A}} (X, _) \cong F(X)$.
- $F \otimes_{\mathcal{A}} _$ is right continuous.
- $_ \otimes_{\mathcal{A}} G$ is right continuous.
What about Bi-modules?

The role of a bi-module is played by a bifunctor

\[ b : \mathcal{A}^{op} \times \mathcal{A} \to \text{Ab} \]

for which

\[ b(A, \_ \_ ) \in \text{Mod}(\mathcal{A}^{op}) \]
\[ b(\_ \_ , A) \in \text{Mod}(\mathcal{A}) \]

**Definition**

\[
\left[ F \otimes_{\mathcal{A}} b \right](A) = F \otimes_{\mathcal{A}} b(A, \_ \_ )
\]
\[
\left[ b \otimes_{\mathcal{A}} G \right](A) = b(\_ \_ , A) \otimes_{\mathcal{A}} G
\]
### Definition (Fisher-Palmquist and Newell)

For $F \in \text{Mod}(\mathcal{A}^{op})$:

\[
\text{Nat}(b, F)(A) = \text{Nat}(b(A, _ ), F)n
\]

\[
\text{Nat}(F, b)(A) = \text{Nat}(F, b(A, _ ))
\]

For $G \in \text{Mod}(\mathcal{A})$:

\[
\text{Nat}(b, G)(A) = \text{Nat}(b(_ , A), G)
\]

\[
\text{Nat}(G, b)(A) = \text{Nat}(G, b(_ , A))
\]
Adjunction of $\otimes_{\mathcal{A}}$ and Nat

**Theorem (Fisher-Palmquist and Newell)**

Let $\mathcal{A}, \mathcal{B}$ be pre-additive categories. For any bifunctor

$$b: \mathcal{A}^{op} \times \mathcal{B}$$

The functor $\_ \otimes b: \text{Mod}(\mathcal{A}) \to \text{Mod}(\mathcal{B}^{op})$ is the left adjoint to the functor $\text{Nat}(b, \_ )$.

**Theorem (Fisher-Palmquist and Newell)**

All adjunctions between $\text{Mod}(\mathcal{A})$ and $\text{Mod}(\mathcal{B}^{op})$ arise in this way.
The role of the ring as a bi-module over itself is played by the bifunctor $\text{Hom}: \mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \text{Ab}$.

**Definition (Fisher-Palmquist and Newell)**

For $F \in \text{Mod}(\mathcal{A}^{\text{op}})$, define $F^* \in \text{Mod}(\mathcal{A})$ by

$$F^* := \text{Nat}(F, \text{Hom})$$

that is on any object $A \in \mathcal{A}$

$$F^*(A) := \text{Nat}(F, \text{Hom}(A, _))$$
Proposition (Fisher-Palmquist and Newell)

For $F \in \text{Mod}(\mathcal{A})$ there is a natural transformation

$\beta: \_ \otimes_{\mathcal{A}} F^* \to (F, \_)$

such that the following are equivalent:

1. $\beta: \_ \otimes_{\mathcal{A}} F^* \to (F, \_)$ is an isomorphism.
2. $\beta$ is an epimorphism.
3. $\beta_F$ is an epimorphism.
4. $F$ is a small projective in $\text{Mod}(\mathcal{A}^{op})$. 
Finitely Presented $A$-modules

**Definition**

An object $F$ of an abelian category is called **finitely presented** if $(F, \_\_)$ commutes with direct limits.
Finitely Presented $A$-modules

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Definition

The category $\text{mod}(A^{op})$ consists of all finitely presented $A$-modules.
Definition

An object $F$ of an abelian category is called **finitely presented** if $\langle F, \_ \rangle$ commutes with direct limits.

Definition

The category $\text{mod}(\mathcal{A}^{op})$ consists of all finitely presented $\mathcal{A}$-modules.

- $\text{mod}(\mathcal{A}^{op})$ is an additive category with cokernels.
Finitely Presented $\mathcal{A}$-modules

Definition

An object $F$ of an abelian category is called \textbf{finitely presented} if $(F, \_\_)$ commutes with direct limits.

Definition

The category $\text{mod}(\mathcal{A}^{op})$ consists of all finitely presented $\mathcal{A}$-modules.

- $\text{mod}(\mathcal{A}^{op})$ is an additive category with cokernels.
- $\text{mod}(\mathcal{A}^{op})$ will not be abelian in general.
## Finitely Presented $\mathcal{A}$-modules

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- $\text{mod}(\mathcal{A}^{op})$ is an additive category with cokernels.
- $\text{mod}(\mathcal{A}^{op})$ will not be abelian in general.
- $\text{fp}(\text{mod}(\mathcal{A}^{op}), \text{Ab})$ is abelian and satisfies some very nice properties.
Examples of Finitely Presented Functors

Proposition

For a small pre-additive category $\mathcal{A}$, the functor

$$\_ \otimes F : \text{Mod}(\mathcal{A}) \to \text{Ab}$$

is finitely presented if and only if $F \in \text{mod}(\mathcal{A}^{op})$. In this case $\_ \otimes F \in \text{fp}(\text{mod}(\mathcal{A}^{op}), \text{Ab})$. 
## Summary

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Auslander-Gruson-Jensen Duality

Theorem (Auslander)

Let $R$ be a Noetherian ring. There is a duality

$$D \colon \text{fp}(\text{mod}(R), \text{Ab}) \to \text{fp}(\text{mod}(R^{op}), \text{Ab})$$

given by

$$DF(X) := \text{Nat}(F, _ \otimes X)$$

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Properties of D

**Proposition (Auslander)**

The duality $D$ satisfies the following. For $X \in \text{mod}(R^{op})$

1. $D(X, _) = _ \otimes X$
2. $D(_, \otimes X) = (X, _)$
Appearances of $D$

- It was first discovered by Auslander.
- It was independently discovered by Gruson and Jensen.
- Hartshorne found $D$ using an approach similar to Auslander.
- Krause showed how to obtain $D$ from a universal property.
- It was discovered model theoretically through work of Mike Prest, Ivo Herzog, and Kevin Burke.
- Russell recovered $D$ in a different way while extending the concept of linkage of modules to finitely presented functors.
Let \( \mathcal{A} \) denote any pre-additive category. A **free abelian category on** \( \mathcal{A} \) is an abelian category \( \text{Ab}(\mathcal{A}) \) together with an additive functor \( j: \mathcal{A} \to \text{Ab}(\mathcal{A}) \) satisfying the following universal property:

\[
\mathcal{A} \xrightarrow{j} \text{Ab}(\mathcal{A})
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**Theorem (Gruson, Krause)**

The double Yoneda embedding \( Y^2 : \mathcal{A} \to \text{fp}(\text{mod}(\mathcal{A}^{op}), \text{Ab}) \) is the free abelian category on \( \mathcal{A} \).
Free Abelian Category

The following was stated first by Gruson for rings and then proved for general pre-additive $\mathcal{A}$ by Krause:

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**The Functorial Approach**

The category $\text{fp}(\text{mod}(\mathcal{A}^{op}), \text{Ab})$ is a universal solution to the abelianization problem.
Notation: (Last Time)

\[ R - \text{ring} \]
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\( R \) - ring

\( A, B \) - matrices with entries from \( R \).
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- $\bar{x} - (x_1, x_2, \ldots, x_n)$ sequence of variables.
- $\bar{y} - (y_{n+1}, y_{n+2}, \ldots, y_{n+k})$ sequence of variables.
Definition \( \text{App-formula, abbreviated ppf, is a formula of the form} \)
\[
\phi(x) \iff \exists \bar{y} A x + B \bar{y} = 0
\]

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$\phi(\overline{x}) \iff \exists \overline{y} \ A \overline{x} + B \overline{y} = 0$
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Definition

A **pp-formula**, abbreviated ppf, is a formula of the form

\[ \varphi(\bar{x}) \iff \exists \bar{y} \ A\bar{x} + B\bar{y} = 0 \]
Example

Take $A = r \in \mathbb{R}$. $B = 0$. Then the following annihilator equation

$$rx = 0$$

is a ppf.
Examples

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Example

Take $A$ to be an $n \times n$ matrix with entries in $R$ and again take $B = 0$. Then the matrix equation

$$A\bar{x} = 0$$

is a ppf.
A Very Concrete Example

Example

Take $R = \mathbb{Z}$ and $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

Then the following is a ppf:

$\exists y$ such that

$$2x_1 + 2x_2 + 2y = 0$$

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$$x_1 + 2x_2 + y = 0$$
$$3x_1 + 4x_2 + 2y = 0$$
Let $\phi$ denote a ppf. Define for each left $R$-module $M$ $F_{\phi}(M) = \{x \in M | \exists y \in M \text{ s.t. } Ax + By = 0\}$. Given a morphism $f: M \to N$, $F_{\phi}(f): F_{\phi}(M) \to F_{\psi}(N)$ is defined to be the restriction of $f$ to these subgroups.
Let $\varphi$ denote a ppf.
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Define for each left $R$-module $M$

$$F\varphi(M) = \left\{ \overline{x} \in M^l(\overline{x}) \mid \exists \overline{y} \in M^l(\overline{y}) \ A\overline{x} + B\overline{y} = 0 \right\}$$
The Functor Determined by a ppf

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  \[ F_{\varphi}(f) : F_{\varphi}(M) \rightarrow F_{\psi}(N) \]

  is defined to be the restriction of $f$ to these subgroups.
The Functor Determined by a ppf

Proposition (Prest)

For each ppf $\varphi$, the functor $F_\varphi : \text{mod}(R^{op}) \to \text{Ab}$ is a finitely presented functor.
A pp-pair $\varphi/\psi$ is a pair $(\varphi, \psi)$ of ppf’s in the same number of variables such that $\psi$ implies $\varphi$. 
**Definition**

A **pp-pair** \( \varphi/\psi \) is a pair \((\varphi, \psi)\) of ppf’s in the same number of variables such that \( \psi \) implies \( \varphi \).

**Theorem (Herzog)**

1. The pp-pairs form a category denoted \( \mathbb{L}^\text{eq}{}^{+}_R \)
2. The category \( \mathbb{L}^\text{eq}{}^{+}_R \) is abelian.
Theorem (Burke)
The categories $\mathcal{L}_{eq}^+_{R}$ and $fp(\text{mod}(R^{op}), \text{Ab})$ are equivalent.
An Equivalence of Categories

Theorem (Burke)

The categories $\mathbb{L}_R^{eq+}$ and $fp(mod(R^{op}), Ab)$ are equivalent.

$$\mathbb{L}_R^{eq+} \cong fp(mod(R^{op}), Ab)$$
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Herzog extended $D$ to the category $\mathbb{L}^{eq+}_R$. 
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Herzog extended $D$ to the category $\mathbb{L}_{\mathbb{R}}^{eq+}$.

Auslander in a conversation with Prest suggested that the duality $D$ on $\mathbb{L}_{\mathbb{R}}^{eq+}$ must be the same as the Auslander-Gruson-Jensen duality.
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**Theorem**

The duality $D$ on $\mathbb{L}_R^{\text{eq}+}$ is indeed the Auslander-Gruson-Jensen duality.
I. The Functorial Approach has applications to representation theory. (e.g. Almost Split Sequences.)

II. The Functorial Approach generalizes module theoretic concepts.

III. The Functorial Approach produces a universal solution to the abelianization problem.

IV. The Functorial Approach establishes connections with other fields which are not at all obvious. (e.g. Model Theory)
Thank You!